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SHEARING VISCOELASTICITY IN PARTIALLY DISSIPATIVE TIMOSHENKO–BOLTZMANN SYSTEMS*

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Abstract. We investigate both the mathematical modeling and stability methods for a new integro-differential system referred to as the *viscoelastic Timoshenko–Boltzmann model.* The modeling is developed for materials with hereditary memory under the creation time scenario whose foundation goes back to Boltzmann's superposition principle in linear viscoelasticity, complemented by Timoshenko's assumptions concerning shearing in certain beam vibrations. The mathematical methodology provides a comprehensive characterization of the uniform stability for the partially damped Timoshenko–Boltzmann system through the identity of wave speeds on the structural coefficients and a pointwise dissipative condition on the memory kernel that does not require differential inequalities.

Key words. beam system, shear force, stability, viscoelasticity

MSC codes. 35B35, 35L51, 35Q74, 45K05

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1. Introduction. In the modern approach by Fabrizio, Giorgi, and Pata [20], a very interesting overview on materials with hereditary memory has been placed by going back to the legacy of Boltzmann and Volterra theories in order to lay down mathematical models in PDE with memory effects. The resulting discussion in [20, sect. 2] leads us to a viscoelastic stress-strain constitutive law where the past history vanishes before some nonpositive time, hereafter called *creation time* and denoted by $\alpha \leq 0$. As a consequence, the following (normalized) integro-differential wavelike motion equation can be deduced,

$$u_{tt} - \Delta u + \int_{\alpha}^{t} g(t-s)\Delta u(\cdot,s)\,ds = 0 \tag{1.1}$$

with proper initial-boundary conditions for the unknown axial displacement $u = u(x,t)$ related to a reference domain Ω and time t. As usual, g is referred to as the *memory kernel.* With respect to the creation time parameter α, the standard literature in linear viscoelasticity is mainly focused in the *limit* situations "$\alpha = -\infty$" (say, Boltzmann's case) and "$\alpha = 0$" (regarding a special Volterra's case), where a meaningful amount of

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work dealing with well-posedness and stability results can be found. Among them, we highlight the article by Conti, Gatti, and Pata [12], where uniform decay properties for Volterra's instance $\alpha = 0$ is addressed without invoking differential inequality assumptions for g, which constitutes the groundwork of our coming purposes. Also, this paper enlightens the insight of the lack of the semigroup structure for $(1.1)_{\alpha=0}$. On the opposite side, as in [20] (see also [25] for precise arguments), when dealing with $(1.1)_{\alpha=-\infty}$, the semigroup feature comes into play by means of a definition of a new variable known as *relative displacement history,* which is given in terms of the vertical displacement u as follows:

$$w^t(\cdot, s) := u(\cdot, t) - u(\cdot, t-s), \quad t \geq 0,\, s > 0. \tag{1.2}$$

This allows us to rewrite $(1.1)_{\alpha=-\infty}$ as an autonomous system in terms of the couple (u, w) and then employ powerful tools in linear semigroup theory for well-posedness and stability results (cf. [9, 38]).

On the one hand, motivated by the inspiring scenario promoted by [20], we supplement the present work with the derivation of two (wavelike) integro-differential beam systems by taking into account the time creation perspective. More precisely, we derive the two classes of viscoelastic evolution models

$$\begin{cases} \rho_1 \phi_{tt} - \kappa(\phi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\,\psi_{xx} + b \displaystyle\int_\alpha^t g(t-s)\psi_{xx}(\cdot, s)\, ds + \kappa(\phi_x + \psi) = 0 \end{cases} \tag{1.3}$$

and

$$\begin{cases} \rho_1 \phi_{tt} - \kappa(\phi_x + \psi)_x + \kappa \displaystyle\int_\alpha^t g(t-s)(\phi_x + \psi)_x(\cdot, s) ds = 0, \\ \rho_2 \psi_{tt} - b\,\psi_{xx} + \kappa(\phi_x + \psi) - \kappa \displaystyle\int_\alpha^t g(t-s)(\phi_x + \psi)(\cdot, s) ds = 0, \end{cases} \tag{1.4}$$

both placed for unknown variables as the vertical displacement $\phi = \phi(x, t)$ and rotation angle $\psi = \psi(x, t)$, where x varies along the beam with length L and time $t \geq \alpha$. The structural coefficients ρ_1, ρ_2, κ, b are all positive constants and depend upon the structure of the beam material. The precise physical (and mathematical) modeling of both systems (1.3) and (1.4) is provided in Appendix A by regarding linear viscoelasticity not only on the bending moment (referring to (1.3)) but also for shear stress (leading to (1.4)) in certain beam vibrations.

On the other hand, still motivated by the characterization of uniform stability in linear viscoelasticity as discussed in [9, 38], we can observe that the special limit situations mentioned earlier ($\alpha = 0$ and $\alpha = -\infty$) give rise to distinct cases regarding the integro-differential systems (1.3) and (1.4), which will be discussed in detail as follows. In this context, we remark that system (1.3) has been extensively studied in the literature, while (1.4) has not been considered so far in the history framework (i.e., in the case $\alpha = -\infty$), which is exactly the unexplored scenario we are focused in the present article. We also observe that due to the partially damped feature of both systems (1.3) and (1.4), their stability results shall depend upon the assumptions:

- *the behavior of the memory kernel g (to be set later on*[1]*);*
- *equal wave speeds (EWS) condition, namely,*

$$\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}. \tag{1.5}$$

[1]It will be clarified during the introduction and precisely given in sections 2 and 3.

1.1. Viscoelasticity on the bending moment. *Null history case:* $\alpha = 0$. With reference to $(1.3)_{\alpha=0}$, we have the following Timoshenko–Volterra integro-differential system

$$\begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\,\psi_{xx} + b\displaystyle\int_0^t g(t-s)\psi_{xx}(s)\,ds + \kappa(\phi_x + \psi) = 0. \end{cases} \tag{1.6}$$

A pioneering work dealing with (1.6), besides initial and Dirichlet-boundary conditions, has been done by Ammar-Khodja et al. [3]. Summarizing what is of importance in the present work, the essential core of the assumptions and stability results therein (see [3, Thms. 2.7, 3.5, and 4.1]) can be stated as follows, where we consider exponential memory kernels $g > 0$ via the differential inequality assumption and then state the referred stability result.

- *Exponential condition:* there exists $\delta > 0$ such that

$$g'(s) + \delta g(s) \leq 0, \quad s > 0, \tag{1.7}$$

  where g is a nonincreasing differentiable function with $g(0) > 0$ and $\int_0^\infty g(s)ds < \infty$.

  Statement I. *Under the exponential assumption on g given by* (1.7)*, the EWS condition* (1.5) *is a necessary and sufficient condition to prove the exponential stability of the energy functional corresponding to* (1.6).

Since then, several stability results related to the viscoelastic Timoshenko–Volterra system (1.6) emerged in the literature aiming to generalize the hypothesis (1.7), by requesting (1.5) or not, with the primary objective of providing a variety of stability properties for (1.6), as general as possible. However, it is worth pointing out that all such *general* decays only span from a very slow decay to the exponential one. For instance, we quote [3, 28, 29, 30, 32, 36, 35, 37, 42] in what concerns stability results for (1.6) placed on bounded and unbounded domains, where some distinct types of generalization for (1.7) are taken into account and are mainly derived from studies carried out to second-order evolution models in viscoelasticity. Among the spectrum of general assumptions on the kernel g, we draw attention to the recent paper by Conti and Pata [13] (see also the references therein) where a robust assumption generalizing (1.7) is considered when dealing with Volterra integro-differential equations of hyperbolic type.

In the authors' opinion, the approach of [13, sect. 4] can be reapplied to other linear wavelike models with finite memory (null history) where viscoelasticity is regarded solely on bending displacements as, for example, in (1.6), and still for viscoelasticity on shearing as in the case $(1.4)_{\alpha=0}$, whenever we fix the EWS assumption (1.5). Nonetheless, here our goal is to address the viscoelasticity in the history framework (namely, $(1.4)_{\alpha=-\infty}$) and so, instead of taking the aforementioned general assumptions on g, we are going to put our efforts towards general assumptions on the memory kernel when the semigroup structure comes into the picture. This is the subject of the next case $(1.3)_{\alpha=-\infty}$, where we can go further and explore even more the assumptions in Statement I by asking, for example, the following.

*Q*1. *By fixing the EWS condition* (1.5)*, is it possible to evaluate in which case the assumption* (1.7) *is a necessary and sufficient condition for exponential stability?*

History case: $\alpha=-\infty$. Now, with respect to $(1.3)_{\alpha=-\infty}$, the following Timoshenko–Boltzmann integro-differential with past history system comes into play:

$$\begin{cases} \rho_1\phi_{tt}-\kappa(\phi_x+\psi)_x=0, \\ \rho_2\psi_{tt}-b\,\psi_{xx}+b\displaystyle\int_{-\infty}^{t} g(t-s)\psi_{xx}(s)\,ds+\kappa(\phi_x+\psi)=0. \end{cases} \tag{1.8}$$

Unlike (1.6), the variable $\psi(\cdot,s)$ is now supposed to be defined for $s\leq 0$ as an initial datum, which invariably requires the definition of a new variable in order to convert (1.8) into an autonomous scenario for all (positive) time. In this way, and based on the influential works by Dafermos [14, 15] (see also [25, sect. 2]), the authors in [21] set the relative displacement history variable as for vertical displacements u in wavelike models (see (1.1)–(1.2)) but now in terms of the rotation angle

$$\zeta^t(x,s):=\psi(x,t)-\psi(x,t-s),\quad x\in(0,L),\, t\geq 0,\, s>0, \tag{1.9}$$

and then provided for the first time the stability results for the related equivalent problem (besides proper initial-boundary conditions) on a suitable extended phase space through the associated semigroup (solution) $T(t)$ and the corresponding energy $E(t)$. Summarizing the main results in [21, Thms. 3.7 and 4.2], we can slightly rephrase Statement I as the following:

Statement II. *Let g be a kernel satisfying the exponential assumption* (1.7). *Then, the EWS condition* (1.5) *holds if and only if the associated semigroup $T(t)$ is exponentially stable.*

As in the previous case, several papers have appeared in the literature concerning the general stability of energy (also in terms of the semigroup) related to the viscoelastic Timoshenko–Boltzmann (1.8), most of them with the same goal in generalizing the exponential assumption on g given by (1.7); see, for instance, [2, 11, 18, 21, 27, 34]. In order to consider an assumption as general as possible related to characterization of stability, and looking for the answer of Q1, among these latter works we are going to highlight the assumptions of [11], which have a foundation in linear viscoelasticity as first introduced by [9] and which today is known as the *δ-condition* (cf. [10]), and still receives the notion of *admissible memory kernel.*

According to [9, 11], we first observe that (1.7) is a particular case (with $C=1$) of the following assumption:

- **δ-condition:** *there exist $C\geq 1$ and $\delta>0$ such that*

$$g(t+s)\leq Ce^{-\delta t}g(s),\quad t>0,\ a.e.\ s>0, \tag{1.10}$$

where g is assumed to be a nonincreasing absolutely continuous summable function with finite total mass.

Therefore, rephrasing the stability result in [11, Thm. 1], we can precisely state the answer for Q1 by giving the characterization of uniform stability in terms of the memory kernel, instead of (1.5) as in Statements I and II.

Statement III. *Under the EWS assumption* (1.5)*, the semigroup $T(t)$ associated with* (1.8) *is exponentially stable if and only if g satisfies the δ-condition* (1.10).[2]

In Statement III, once we have fixed the EWS condition (1.5), the equivalence stated therein is not true if we change (1.10) by (1.7), which shows the δ-condition is the

[2]As stressed on p. 368 of [11] (see remark therein), there is no need of (1.5) in necessity's proof.

exact assumption to answer Q1, besides elucidating its strength in the characterization of uniform stability via the memory kernel. In this direction, we point out that either assumption (1.5) or (1.10) (resp. (1.7)) is always fixed and the other one is used in the equivalence of the exponential stabilization. Hence, we name Statements I, II, and III as a *partial characterization* via either condition (1.5) or (1.10). This fact leads us to the question of whether it is possible to use both assumptions for the uniform stability characterization, or more specifically, the following.

> *Q2. Is it possible to prove that both* (1.5) *and* (1.10) *are (at the same time) necessary and sufficient for exponential stability of the corresponding semigroup (energy) when dealing within the history scenario?*

As far as the authors know, this issue has not been explored until now with respect to system (1.8). A positive answer to the question will be called a *complete characterization* via (1.5) and (1.10). This is precisely the content of the next subsection where our main goal is to address and clarify it for the case $(1.4)_{\alpha=-\infty}$, which can be properly brought to the present case.

1.2. Viscoelasticity on the shear stress. *Null history case:* $\alpha = 0$. The Timoshenko–Volterra integro-differential system $(1.4)_{\alpha=0}$ is written down as

$$\begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x+\psi)_x + \kappa\displaystyle\int_0^t g(t-s)(\phi_x+\psi)_x(s)ds = 0, \\ \rho_2\psi_{tt} - b\,\psi_{xx} + \kappa(\phi_x+\psi) - \kappa\displaystyle\int_0^t g(t-s)(\phi_x+\psi)(s)ds = 0. \end{cases} \tag{1.11}$$

The development of the physical modeling and mathematical results for (1.11) has been devised only recently by Alves et al. [1]. Uniform and nonuniform stability results are provided for solutions of (1.11) with proper initial-boundary conditions, by taking into account general assumptions on g that include (1.7) as a particular case and also assuming (or not) the EWS condition (1.5), besides a nonstandard assumption on the memory kernel g in terms of ρ_1, ρ_2, and the length L of the beam; cf. [1, Thms. 3.10, 3.12, and 4.1]. Just to be clearer, we note that such a peculiar assumption on g is presented in Assumption 3.5 therein, being given by

$$\int_0^\infty g(s)\,ds > \left\{\frac{31}{32}, \frac{64\rho_1 L^2}{64\rho_1 L^2+\rho_2}\right\} \in (0,1), \tag{1.12}$$

which proved to be effective to reach a new observability inequality by means of new multipliers (cf. [1, Prop. 3.8]), since it gives a proper balance to weight the perturbed functionals. Despite general results, when speaking in terms of characterization of uniform exponential stability, the summary of the main results in [1] reads precisely as in Statement I, which is somehow expected (unless requesting (1.12)). So we turn our attention back to the presence of history as follows.

History case: $\alpha = -\infty$. We finally approach the instance that expresses the main object of study in the present article, namely, the viscoelastic Timoshenko–Boltzmann integro-differential system $(1.4)_{\alpha=-\infty}$

$$\begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x+\psi)_x + \kappa\displaystyle\int_{-\infty}^t g(t-s)(\phi_x+\psi)_x(s)ds = 0, \\ \rho_2\psi_{tt} - b\,\psi_{xx} + \kappa(\phi_x+\psi) - \kappa\displaystyle\int_{-\infty}^t g(t-s)(\phi_x+\psi)(s)ds = 0. \end{cases} \tag{1.13}$$

Suitable initial-boundary conditions will be precisely set later in (2.2)–(2.3). Unlike (1.11), we are now facing prescribed initial data for the sum $(\phi_x+\psi)(\cdot,s)$, $s\le 0$, which in turn allows us to define a relative displacement history variable in terms of the shear component $S:=\kappa(\phi_x+\psi)$ in order to convert (1.13) into a tangible equivalent autonomous problem, yet depending on the initial-boundary conditions, as clarified right after and in section 2.

Before giving our main contributions, let us compare (1.13) with (1.11) or (1.8), though it is not completely fair. Indeed, while in (1.11) one has the lack of semigroup characteristic and so the stability has been done via multipliers in energy perturbation, here we guarantee the generation a linear semigroup $S(t)$ associated with (1.13). Hence, a complete characterization of uniform stability is provided for the associated semigroup, by giving a positive answer to Q2 and still removing the drawback property (1.12) as imposed in [1]. Furthermore, concerning (1.8), one sees that it has the viscoelastic feedback through the bending moment and so all propagation of dissipativity as done in [11, 21] is not enough in the present case, where a new path of technical computations must be revealed in terms of the viscoelastic shearing component.

The main contributions and organization of the paper are as follows.

(i) *Well-posedness.* Due to the Neumann boundary condition (2.2) for ψ, the setting of (1.9) alone does not seem to be compatible in the well-posedness statement as it brings trouble in terms of ζ to fulfill all requested properties of the domain arising in the resulting autonomous problem. To overcome this difficulty, we are going to introduce the relative displacement history in terms of shear stress $S:=\kappa(\phi_x+\psi)$ as

$$\eta^t(x,s):=\frac{1}{\kappa}\int_0^x [S(y,t)-S(y,t-s)]\,dy. \tag{1.14}$$

Such a formulation can also be done for (1.9).[3] With (1.14) in hands, we are capable of setting (1.13) and its proper initial-boundary conditions (IBC) into the semigroup framework, which is completely different from [1] in technical aspects and also differs from [11, 26] in terms of IBC and the setting of relative displacement histories. This is the subject of section 2, where the generation of the semigroup (solution) $S(t):=e^{t\mathbb{A}}$ is ensured; see Theorem 2.3.

(ii) *Uniform Stability.* In section 3, we state and prove our main stability results, namely, Theorems 3.1, 3.2, 3.3. Under the forthcoming Assumption 2.1 and also taking $\lim_{s\to 0} g(s)=g_0<+\infty$, our main results state

– $S(t)=e^{t\mathbb{A}}$ is exponentially stable if and only if $\begin{cases} (1.5) \text{ holds true,} \\ (1.10) \text{ is satisfied.} \end{cases}$

– *In particular, when* (1.5) *does not hold, the semigroup* $S(t)=e^{t\mathbb{A}}$ *is only semiuniformly stable with optimal rate* $\sqrt{t}$.

The above statements provide a complete characterization of stability for the associated semigroup, generalizes the partial characterization of Statements I, II, and III to our context, and gives the positive answer to Q2 in our viscoelastic shearing case, besides removing (1.12) as requested in the null history case. Moreover, we also believe it could be replied to viscoelasticity on

[3] For instance, one can rewrite (1.9) formally in terms of the bending moment $M:=b\psi_x$ when dealing with proper boundary conditions, namely, $\zeta^t(x,s)=\frac{1}{b}\int_0^x [M(y,t)-M(y,t-s)]\,dy$.

the bending moment. The proofs are based on the spectral analysis through the resolvent equation where new technical lemmas are original in this work and then the well-known results in linear semigroup theory are invoked.

2. Well-posedness.

To address (1.13) we consider the following IBVP:

$$\begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x+\psi)_x + \kappa\displaystyle\int_{-\infty}^{t} g(t-s)(\phi_x+\psi)_x(s)ds = 0 \ \text{ in } \ (0,L)\times\mathbb{R}^+, \\ \rho_2\psi_{tt} - b\,\psi_{xx} + \kappa(\phi_x+\psi) - \kappa\displaystyle\int_{-\infty}^{t} g(t-s)(\phi_x+\psi)(s)ds = 0 \ \text{ in } \ (0,L)\times\mathbb{R}^+ \end{cases} \tag{2.1}$$

with boundary conditions

$$\phi(0,t) = \phi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0, \quad t\in\mathbb{R}, \tag{2.2}$$

and initial (also *compatible*) data

$$\begin{cases} \phi(x,0) = \phi_0(x),\ \phi_t(0,x) = \phi_1(x), \quad x\in(0,L), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \quad x\in(0,L), \\ \phi(x,t) + \displaystyle\int_0^x \psi(y,t)\,dy = p_0(x,t), \quad (x,t)\in(0,L)\times(-\infty,0). \end{cases} \tag{2.3}$$

Remark 2.1. Let us discuss the initial conditions (2.3) a little bit. The first natural attempt to consider initial data can be found in [26], namely, instead of $(2.3)_3$ we could consider $\phi(\cdot,t) = \phi_0(\cdot,t)$ and $\psi(\cdot,t) = \psi_0(\cdot,t)$, $t\le 0$, separately. Nonetheless, with these latter history data studied separately, our trouble is twofold. Indeed, the first one arises because the possible relative displacement history ζ in (1.9) will not completely fulfill the semigroup formulation in terms of Neumann boundary conditions inherited by ψ. The second one is that by means of (1.14) (see also (2.4) and the supplementary equation in (2.14)) we will never be able to recover the initial data $\phi_0(\cdot,t)$ and $\psi_0(\cdot,t), t\le 0$, separately, but also the (not so usual) prescribed history data $(2.3)_3$, which in turn arises due to the viscoelastic coupling on the shear force, being called herein as a kind of compatibility condition. As we shall see later, it is intrinsically necessary to transit equivalently between an autonomous problem and the original system (2.1)–(2.3).

Since all structural constants are positive (ρ_1, ρ_2, κ, $b>0$), we only need to assume conditions on the memory kernel g as follows.

Assumption 2.1. The kernel $g:\mathbb{R}^+\to\mathbb{R}^+$ is absolutely continuous, nonincreasing, and summable, with total mass

$$\ell := \int_0^\infty g(s)\,ds \in (0,1).$$

Remark 2.2. As a matter of fact, to approach the well-posedness of (2.1)–(2.3) (and still its asymptotic behavior) via semigroup theory, one could consider a more general class of kernels absolutely continuous by parts containing a finite number of jumps or even a countable number of increasing jumps; see for instance [9, 23]. However, to our purposes in the present paper, we will not consider such classes of kernels.

From (1.14), we rewrite the relative displacement history variable as

$$\eta^t(x,s) = \left(\phi+\widetilde{\psi}\right)(x,t) - \left(\phi+\widetilde{\psi}\right)(x,t-s) \tag{2.4}$$

for $x\in(0,L)$, $t\geq 0$, $s>0$, where we set the notation for the sake of simplicity

$$\widetilde{\psi}(x,t)=\int_0^x \psi(y,t)\,dy.$$

By means of (2.4) and setting $\omega:=1-\ell$, we convert (2.1)–(2.3) into the following autonomous system

$$\begin{cases} \rho_1\phi_{tt}-\kappa\left[\omega(\phi_x+\psi)+\displaystyle\int_0^\infty g(s)\eta_x(s)\,ds\right]_x=0 & \text{in} \quad (0,L)\times\mathbb{R}^+, \\ \rho_2\psi_{tt}-b\psi_{xx}+\kappa\left[\omega(\phi_x+\psi)+\displaystyle\int_0^\infty g(s)\eta_x(s)\,ds\right]=0 & \text{in} \quad (0,L)\times\mathbb{R}^+, \\ \eta_t+\eta_s-\left(\phi+\widetilde{\psi}\right)_t=0 & \text{in} \quad (0,L)\times\mathbb{R}^+\times\mathbb{R}^+ \end{cases} \tag{2.5}$$

with boundary conditions

$$\begin{cases} \phi(0,t)=\phi(L,t)=\psi_x(0,t)=\psi_x(L,t)=0, \quad t\geq 0, \\ \eta^t(0,s)=\eta^t(L,s)=0, \qquad t\geq 0,\ s>0, \\ \eta^t(x,0)=0, \quad (x,t)\in(0,L)\times[0,\infty), \end{cases} \tag{2.6}$$

and initial data

$$\begin{cases} \phi(x,0)=\phi_0(x),\ \phi_t(0,x)=\phi_1(x), \quad x\in(0,L), \\ \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x), \quad x\in(0,L), \\ \eta^0(x,s)=\phi_0+\widetilde{\psi}_0-p_0(-s):=\eta_0(x,s), \quad (x,s)\in(0,L)\times\mathbb{R}^+. \end{cases} \tag{2.7}$$

By using an approach similar to [25], we can state that (2.5)–(2.7) is equivalent somehow to (2.1)–(2.3). A brief discussion on this subject shall be given in subsection 2.2. In this way, hereafter, we focus our mathematical results on the equivalent autonomous system (2.5)–(2.7).

2.1. The semigroup solution. We first introduce some notations as follows. Let us consider $L^2(0,L)$, the standard complex L^2-space with inner product and norm

$$(u,v)=\int_0^L u(x)\overline{v(x)}\,dx, \quad \|u\|=\left(\int_0^L |u(x)|^2\,dx\right)^{1/2}.$$

The space $H_0^1(0,L)$ stands for the usual Sobolev space, and

$$L_*^2(0,L)=\left\{u\in L^2(0,L),\ \frac{1}{L}\int_0^L u(x)dx=0\right\}, \quad H_*^1(0,L)=H^1(0,L)\cap L_*^2(0,L),$$

equipped with the norms

$$\|u\|_{L_*^2(0,L)}=\|u\|, \quad \|u\|_{H_0^1(0,L)}=\|u\|_{H_*^1(0,L)}=\|u_x\|.$$

For any $h\in L^1(\mathbb{R}^+)$, we consider the Hilbert memory space

$$\mathcal{M}_h:=\left\{\eta:\mathbb{R}^+\to H_0^1(0,L);\ \int_0^\infty h(s)\|\eta_x(s)\|^2\,ds<\infty\right\}$$

with inner product and norm

$$(\eta,\xi)_{\mathcal{M}_h}=\int_0^\infty h(s)(\eta_x(s),\xi_x(s))\,ds,\quad \|\eta\|^2_{\mathcal{M}_h}:=\int_0^\infty h(s)\|\eta_x(s)\|^2\,ds.$$

Under the above notations we consider the extended phase space

$$\mathcal{H}=H^1_0(0,L)\times L^2(0,L)\times H^1_*(0,L)\times L^2_*(0,L)\times\mathcal{M}_g,$$

equipped with inner product

$$(z^1,z^2)_{\mathcal{H}}=\rho_1(\Phi^1,\Phi^2)+\rho_2(\Psi^1,\Psi^2)+\kappa\omega(\phi^1_x+\psi^1,\phi^2_x+\psi^2)+b(\psi^1_x,\psi^2_x)+\kappa(\eta^1,\eta^2)_{\mathcal{M}_g}$$

and norm

$$\|z\|^2_{\mathcal{H}}=\rho_1\|\Phi\|^2+\rho_2\|\Psi\|^2+\kappa\omega\|\phi_x+\psi\|^2+b\|\psi_x\|^2+\kappa\|\eta\|^2_{\mathcal{M}_g},$$

where $z_i=(\phi^i,\Phi^i,\psi^i,\Psi^i,\eta^i)$, $z=(\phi,\Phi,\psi,\Psi,\eta)\in\mathcal{H}$, $i=1,2$.

We also consider the operator $\mathbb{L}:D(\mathbb{L})\subset\mathcal{M}_g\to\mathcal{M}_g$ given by

$$D(\mathbb{L}):=\{\eta\in\mathcal{M}_g,\,\mathbb{L}\eta\in\mathcal{M}_g\text{ and }\eta(0)=0\},\quad \mathbb{L}\eta:=-\partial_s\eta,$$

which is the infinitesimal generator of the right-translation semigroup $R(t):\mathcal{M}_g\to\mathcal{M}_g$ given by

$$[R(t)\eta](s):=\begin{cases}\eta(s-t), & s>t,\\ 0, & 0<s\le t.\end{cases}$$

Moreover, by setting $\Phi=\phi_t$, $\Psi=\psi_t$, and $z_0=(\phi_0,\phi_1,\psi_0,\psi_1,\theta_0,\eta^0)$, we can now rewrite (2.5)–(2.7) as the following Cauchy problem

$$\begin{cases}z_t=\mathbb{A}z, & t>0,\\ z(0)=z_0,\end{cases}\tag{2.8}$$

where the linear operator $\mathbb{A}:D(\mathbb{A})\subset\mathcal{H}\to\mathcal{H}$ is given by

$$\mathbb{A}z=\begin{bmatrix}\Phi\\ \dfrac{\kappa}{\rho_1}\left[\omega(\phi_x+\psi)+\displaystyle\int_0^\infty g(s)\eta_x(s)\,ds\right]_x\\ \Psi\\ \dfrac{b}{\rho_2}\psi_{xx}-\dfrac{\kappa}{\rho_2}\left[\omega(\phi_x+\psi)+\displaystyle\int_0^\infty g(s)\eta_x(s)\,ds\right]\\ \mathbb{L}\eta+(\Phi+\widetilde{\Psi})\end{bmatrix}\tag{2.9}$$

and its domain $D(\mathbb{A})$ consists of all functions

$$z=(\phi,\Phi,\psi,\Psi,\eta)\in H^1_0(0,L)\times H^1_0(0,L)\times\left[H^2(0,L)\cap H^1_*(0,L)\right]\times H^1_*(0,L)\times D(\mathbb{L})$$

such that

$$\psi_x\in H^1_0(0,L),\qquad \omega\phi+\int_0^\infty g(s)\eta(s)\,ds\in H^2(0,L).$$

We are finally in condition to state and prove the existence and uniqueness result for (2.5)–(2.7) by means of (2.8). In other words, we are going to show that the $\mathbb{A}$ set in (2.9) is the infinitesimal generator of a C_0-semigroup of contractions $S(t):=e^{\mathbb{A}t}$, which in turn is called herein by a solution semigroup. More precisely, we have the following.

THEOREM 2.3. *Let g be a kernel satisfying Assumption* 2.1. *Then, for every $z_0 \in \mathcal{H}$, problem* (2.8) *has a unique mild solution $z \in C(0,\infty;\mathcal{H})$ given by*

$$z(t) = S(t)z_0, \quad t \geq 0. \tag{2.10}$$

In addition, if $z_0 \in D(\mathbb{A})$, then z is the regular solution of (2.8) *with*

$$z \in C^1(0,\infty;\mathcal{H}) \cap C(0,\infty;D(\mathbb{A})).$$

Proof. From the Lumer–Phillips theorem (cf. [33, Thm. 1.2.4]) we need to prove that $\mathbb{A}$ is dissipative and onto. This is enough to conclude that $\mathbb{A}$ generates a C_0-semigroup of contractions $S(t) = e^{t\mathbb{A}}$ as desired. Let us sketch the proof as follows. *Dissipativity.* For any $z = (\phi, \Phi, \psi, \Psi, \eta) \in D(\mathbb{A})$, we have

$$\begin{aligned}
\operatorname{Re}(\mathbb{A}z, z)_{\mathcal{H}} &= \kappa \operatorname{Re}(\mathbb{L}z, z)_{\mathcal{M}_g} \\
&= -\frac{\kappa}{2}\int_0^\infty g(s)\frac{d}{ds}\|\eta_x(s)\|^2\,ds \\
&= -\frac{\kappa}{2}\lim_{y\to 0^+}\left[-g(y)\|\eta_x(y)\|^2 + g\left(\frac{1}{y}\right)\left\|\eta_x\left(\frac{1}{y}\right)\right\|^2 - \int_y^{1/y} g'(s)\|\eta_x(s)\|^2\,ds\right].
\end{aligned}$$

Proceeding as in [25] (see also [31, sect. 2] for computations with a little more detail), we deduce

$$\lim_{y\to 0^+} g(y)\|\eta_x(y)\|^2 = \lim_{y\to 0^+} g\left(\frac{1}{y}\right)\left\|\eta_x\left(\frac{1}{y}\right)\right\|^2 = 0. \tag{2.11}$$

Then, since $g'(s) \leq 0$ for almost every $s \in \mathbb{R}^+$ we get

$$\operatorname{Re}(\mathbb{A}z, z)_{\mathcal{H}} = \frac{\kappa}{2}\int_0^\infty g'(s)\|\eta_x(s)\|^2\,ds \leq 0,$$

which is enough to conclude that $\mathbb{A}$ is dissipative.
$I - \mathbb{A}$ is onto. Given $f = (f^1, f^2, f^3, f^4, f^5) \in \mathcal{H}$, we need to find $z = (\phi, \Phi, \psi, \Psi, \eta) \in D(\mathbb{A})$ such that

$$\begin{cases}
\phi - \Phi = f^1, \\
\rho_1\Phi - \kappa\left[\omega(\phi_x + \psi) + \displaystyle\int_0^\infty g(s)\eta_x(s)\,ds\right]_x = \rho_1 f^2, \\
\psi - \Psi = f^3, \\
\rho_2\Psi - b\psi_{xx} + \kappa\left[\omega(\phi_x + \psi) + \displaystyle\int_0^\infty g(s)\eta_x(s)\,ds\right] = \rho_2 f^4, \\
\eta + \eta_s - (\Phi + \widetilde{\Psi}) = f^5.
\end{cases} \tag{2.12}$$

To do so, we consider the following continuous coercive sesquilinear form in $H_0^1(0,L) \times H_*^1(0,L)$,

$$\Lambda((\phi,\psi),(\vartheta,\zeta)) = \rho_1(\phi,\vartheta) + \rho_2(\psi,\zeta) + b(\psi_x,\zeta_x) + \kappa\left[1 - \int_0^\infty g(s)e^{-s}\,ds\right](\phi_x + \psi, \vartheta_x + \zeta)$$

and the antilinear functional $F : H_0^1(0,L) \times H_*^1(0,L) \to \mathbb{C}$ set as

$$\begin{aligned}
F(\vartheta,\zeta) &= \rho_1(f^1 + f^2, \vartheta) + \rho_2(f^3 + f^4, \zeta) + \kappa\left[\ell - \int_0^\infty g(s)e^{-s}\,ds\right](f_x^1 + f^3, \vartheta_x + \zeta) \\
&\quad - \kappa\int_0^\infty g(s)\int_0^s e^{-(s-\tau)}(f_x^5(\tau), \vartheta_x + \zeta)\,d\tau ds.
\end{aligned}$$

From the Lax–Milgram theorem, there exists $(\phi,\psi)\in H_0^1(0,L)\times H_*^1(0,L)$ such that

$$\Lambda((\phi,\psi),(\vartheta,\zeta)) = F(\vartheta,\zeta) \quad \forall\, (\vartheta,\zeta)\in H_0^1(0,L)\times H_*^1(0,L). \tag{2.13}$$

Them, setting $\Phi=\phi-f^1$, $\Psi=\psi-f^3$, and

$$\eta(s) = (1-e^{-s})(\Phi+\widetilde{\Psi}) + \int_0^s f^5(\tau)e^{-(s-\tau)}\,d\tau,$$

and using (2.13), it is possible to prove that the resulting vector $z=(\phi,\Phi,\psi,\Psi,\eta)$ belongs to $D(\mathbb{A})$ and satisfies (2.12). ☐

2.2. Recovering the original system. To end this section, we provide a short discussion of how to go back from (2.5)–(2.7) to (2.1)–(2.3) in some sense.

First, motivated by the third identities in (2.5)–(2.7) and by the aforementioned notations, we consider the supplementary Cauchy problem

$$\begin{cases} \eta_t = \mathbb{L}\eta + (\phi+\widetilde{\psi})_t, & t>0,\\ \eta^0 = \eta_0. \end{cases} \tag{2.14}$$

By means of a rigorous study on the memory space $\mathcal{M}_g$, we can prove analogously to [25, sect. 3] that $\mathbb{L}$ is an infinitesimal generator of a right-translation semigroup $R(t):\mathcal{M}_g\to\mathcal{M}_g$ given by

$$[R(t)\eta](s) := \begin{cases} \eta(s-t), & s>t,\\ 0, & 0<s\le t, \end{cases}$$

which in turn leads us to an explicit representation formula for η (see. (3.6) therein) in terms of $v := (\phi+\widetilde{\psi})_t$. Therefore, bringing the picture to our system (2.14), we have the following result: *for any $\eta_0\in\mathcal{M}_g$, problem* (2.14) *has a unique mild solution $\eta\in C([0,\infty);\mathcal{M}_g)$, which has the explicit form*

$$\eta^t(s) = \begin{cases} \eta^0(s-t) + \left(\phi+\widetilde{\psi}\right)(t) - \left(\phi_0+\widetilde{\psi}_0\right), & s>t,\\ \left(\phi+\widetilde{\psi}\right)(t) - \left(\phi+\widetilde{\psi}\right)(t-s), & 0<s\le t. \end{cases} \tag{2.15}$$

Combining the expression (2.15) with similar arguments given in [25, sect. 4], we can recover the original system (2.1)–(2.3) in a *variational sense.* More precisely, if (ϕ,ψ,η) is a mild (one reads variational) solution of (2.5)–(2.7) with initial data $(\phi_0,\phi_1,\psi_0,\psi_1,\eta_0)$, where

$$\eta_0(s) = \phi_0+\widetilde{\psi}_0 - p_0(-s), \quad s>0,$$

then (ϕ,ψ) is a variational solution of (2.1)–(2.3), paying attention to that we can only recover the coupled history data $(2.3)_3$, but not the prescribed history data for ϕ and ψ separately.

3. Stability analysis. In this section, we are going to provide qualitative stability results for the semigroup $S(t)=e^{t\mathbb{A}}$ and, consequently, for the semigroup solution (2.10).

We start by recalling the following stability notions:

- we say that $S(t)$ is *semiuniformly stable* with rate $\sqrt{t}$ when $0\in\rho(\mathbb{A})$ (resolvent set) and there exists a constant $K>0$ such that

$$\|S(t)\mathbb{A}^{-1}\|_{\mathcal{L}(\mathcal{H})} \le \frac{K}{\sqrt{t}}, \quad t>0;$$

- we say that $S(t)$ is *uniformly exponentially stable* when there exist constants $M \geq 1$ and $\gamma > 0$ such that

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{-\gamma}, \quad t > 0.$$

To reach such stability results for $S(t) = e^{t\mathbb{A}}$, we need to consider an extra dissipative condition on the memory kernel (besides Assumption 2.1, which in turn is already understood to be assumed). Here, in order to take an assumption as general as possible to give a complete characterization of stability, we are going to pay attention to the δ-condition (1.10) that has been introduced for (wavelike) models in linear viscoelasticity; see for instance [9, 10].

Assumption 3.1. There exist constants $\delta > 0$ and $C \geq 1$ such that the kernel $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies

$$g(t+s) \leq Ce^{-\delta t} g(s) \tag{3.1}$$

for every $t > 0$ and for almost every $s > 0$.

Moreover, as we have seen in the introduction, problem (2.5)–(2.7) is only partially damped and its asymptotic stability also depends upon a relationship of its structural coefficients. Thus, we also consider the following parameter of stability

$$\chi_0 := \frac{\kappa}{\rho_1} - \frac{b}{\rho_2}. \tag{3.2}$$

3.1. Statement of the main results. Our first main result guarantees that $S(t) = e^{t\mathbb{A}}$ is (in general) semiuniformly stable, regardless of the value of χ_0.

THEOREM 3.1. *Under the additional Assumption* 3.1*, the semigroup* $S(t) = e^{t\mathbb{A}}$ *is semiuniformly stable with rate* $\sqrt{t}$*, independently of* χ_0.

Our second main stability result ensures that the previous semiuniform *"polynomial"* rate given in Theorem 3.1 is optimal whenever $\chi_0 \neq 0$.

THEOREM 3.2. *Let us additionally suppose that* $\lim_{s\to 0} g(s) = g_0 < +\infty$. *Under the assumptions of Theorem* 3.1 *and taking* $\chi_0 \neq 0$*, the rate* $\sqrt{t}$ *is optimal.*

Last, but not least, our third main stability result provides a full characterization of uniform exponential stability in terms of Assumption 3.1 and the stability parameter χ_0.

THEOREM 3.3. *Let us additionally suppose that* $\lim_{s\to 0} g(s) = g_0 < +\infty$. *Then, we have*

$S(t) = e^{t\mathbb{A}}$ *is exponentially stable if and only if* g *satisfies Assumption* 3.1 *and* $\chi_0 = 0$.

The proofs of the main results are done at the end of this section as a consequence of a deep technical analysis of the resolvent equation.

3.2. Auxiliary results via resolvent equation.

3.2.1. Preliminary tools. To make this work as self-contained as possible, we start by listing some known results that will help us to show our main results.

LEMMA 3.4. *Under Assumptions* 2.1 *and* 3.1*,*

$$\int_0^\infty g(s) \left(\int_0^s \|\eta_x(\tau)\| \, d\tau \right)^2 ds \leq \frac{4C}{\delta^2} \|\eta\|^2_{\mathcal{M}_g} \tag{3.3}$$

for every $\eta \in \mathcal{M}_g$.

Proof. The proof follows the arguments of [8, Lem. 3.2] with $p=2$ and $r=0$. ∎

LEMMA 3.5. *Under Assumption* 2.1, *let us also denote*

$$\widehat{g}(\lambda)=\int_0^\infty g(s)e^{-i\lambda s}\,ds,$$

named the (half-) Fourier transform of g. Then,

$$\lim_{|\lambda|\to+\infty}\widehat{g}(\lambda)=0.$$

In addition, if $\lim_{s\to 0}g(s)=g_0<+\infty$, then

$$\lim_{\lambda\to+\infty}\lambda\widehat{g}(\lambda)=-ig_0.$$

Proof. The first statement follows from the Riemann–Lebesgue lemma (cf. [22, Thm. 8.22]) and the second one is a consequence of [17, Thm. 1] with $p=0$ and $f=g$. ∎

LEMMA 3.6 ([4, Prop. 2.2]). *Let $T(t):X\to X$ be a bounded semigroup on a Banach space X and A its infinitesimal generator. If X is reflexive, then*

$$\sigma(A)\cap i\mathbb{R}=\sigma_{ap}(A)\cap i\mathbb{R},$$

where $\sigma(A)$ is the spectrum of A and

$$\sigma_{ap}(A)=\left\{\lambda\in\mathbb{C},\ \exists\, x_n\in D(A)\ \text{with}\ \|x_n\|_X=1\,\forall\, n\in\mathbb{N}\ \text{and}\ \lim_{n\to\infty}\|\lambda x_n-Ax_n\|_X=0\right\},$$

is the approximated point spectrum of A.

Remark 3.7. (i) As pointed out in [16], the main result of [17] is more general and provides appropriate asymptotic controls on $\widehat{g}$ for a wide class of functions g. (ii) Moreover, since the embedding $D(\mathbb{A})\subset\mathcal{H}$ is not compact (cf. [39]), Lemma 3.6 plays an important role in the study of problems with the memory term in the history context, which in turn requires the $\mathcal{M}_g$-weighted spaces in the composition of the extended phase spaces.

Along this subsection, we denote by $c>0$ all positive (global) constants appearing in computations, namely, those ones depending only on the structural parameters ρ_1, ρ_2, κ, b, L, and the memory kernel g.

Let $\lambda\in\mathbb{R}\backslash\{0\}$ and $f=(f^1,f^2,f^3,f^4,f^5)\in\mathcal{H}$. Initially, let us suppose that $i\lambda\in\rho(\mathbb{A})$, namely, there exists $z=(\phi,\Phi,\psi,\Psi,\eta)\in D(\mathbb{A})$ such that

$$i\lambda z-\mathbb{A}z=f \tag{3.4}$$

or, equivalently,

$$\begin{cases} i\lambda\phi-\Phi=f^1,\\ i\lambda\rho_1\Phi-\kappa\left[\omega(\phi_x+\psi)+\displaystyle\int_0^\infty g(s)\eta_x(s)\,ds\right]_x=f^2,\\ i\lambda\psi-\Psi=f^3,\\ i\lambda\rho_2\Psi-b\psi_{xx}+\kappa\left[\omega(\phi_x+\psi)+\displaystyle\int_0^\infty g(s)\eta_x(s)\,ds\right]=f^4,\\ i\lambda\eta+\eta_s-(\Phi+\widetilde{\Psi})=f^5. \end{cases} \tag{3.5}$$

In the following, we take advantage of the structure of system (3.5) to estimate the components of $\|z\|_{\mathcal{H}}$ in terms of $\|f\|_{\mathcal{H}}$. To do so, let us consider $\alpha>0$ such that the set

$$\mathcal{J}=\{s\in\mathbb{R}^+,\alpha g'(s)+g(s)<0\}$$

has positive Lebesgue measure and define $\tilde{g}(s):=g(s)\chi_{\mathcal{J}}(s)$. The existence of such an α is ensured by Assumption 2.1. Also, some of the forthcoming computations must (first) be done in the space $\mathcal{M}_{\tilde{g}}$.

LEMMA 3.8. *Under the above setting, we have*

$$I_g(\lambda):=\int_0^\infty \tilde{g}(s)(1-\cos(\lambda s))\,ds>0.$$

Proof. Considering the countable set

$$\mathcal{P}:=\left\{s\in\mathcal{J},s=\frac{2j\pi}{\lambda},j\in\mathbb{N}\right\},$$

we observe that $\mathcal{J}\backslash\mathcal{P}$ has positive Lebesgue measure and then

$$\int_0^\infty \tilde{g}(s)(1-\cos(\lambda s))\,ds=\int_{\mathcal{J}\backslash\mathcal{P}} g(s)(1-\cos(\lambda s))\,ds>0,$$

which shows the desired assertion. ∎

3.2.2. Estimating the damping term.

LEMMA 3.9. *There exists $c>0$ such that*

$$\|\eta\|^2_{\mathcal{M}_{\tilde{g}}}-\int_0^\infty g'(s)\|\eta_x(s)\|^2\,ds\le c\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}.$$

Proof. Taking the inner product of (3.4) with z in $\mathcal{H}$, and taking into account (2.11) and the definition of $\tilde{g}$, we get

$$\frac{1}{2\alpha}\|\eta\|^2_{\mathcal{M}_{\tilde{g}}}\le-\frac{\kappa}{2}\int_0^\infty g'(s)\|\eta_x(s)\|^2\,ds\le\operatorname{Re}\left(i\lambda z-\mathbb{A}z,z\right)_{\mathcal{H}}\le\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}},$$

which yields the desired conclusion. ∎

3.2.3. Action of viscoelasticity on the shear force.

LEMMA 3.10. *Under Assumption* 3.1, *there exists $c>0$ such that*

$$\|\phi_x+\psi\|^2\le\frac{c}{I_g(\lambda)}\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}.$$

Proof. Solving the ODE $(3.5)_5$ and using the expressions $(3.5)_1$ and $(3.5)_3$ we obtain

$$\eta(s)=\left(1-e^{-i\lambda s}\right)\left(\phi+\widetilde{\psi}\right)-\frac{1}{i\lambda}\left(1-e^{-i\lambda s}\right)\left(f^1+\widetilde{f^3}\right)+\int_0^s e^{-i\lambda(s-\tau)}f^5(\tau)d\tau. \tag{3.6}$$

Taking the inner product of (3.6) with $\phi+\widetilde{\psi}$ in $\mathcal{M}_{\tilde{g}}$ and extracting the real part of the result, we get

$$I_g(\lambda)\|\phi_x+\psi\|^2=a_1, \tag{3.7}$$

where

$$a_1 := \mathrm{Re}\left[(\eta, \phi + \widetilde{\psi})_{\mathcal{M}_{\tilde{g}}} + \frac{1}{i\lambda}\left(\int_0^\infty \tilde{g}(s)(1 - e^{-i\lambda s})\, ds\right)(f_x^1 + f^3, \phi_x + \psi)\right] - \mathrm{Re}\left[\int_0^\infty \tilde{g}(s) \int_0^s e^{-i\lambda(s-\tau)}(f_x^5(\tau), \phi_x + \psi)\, d\tau ds\right].$$

Now, applying Lemmas 3.4 and 3.9, we deduce

$$\begin{aligned} |a_1| &\le c\|\phi_x + \psi\| \left[\|\eta\|_{\mathcal{M}_{\tilde{g}}} + \frac{1}{|\lambda|}\|f_x^1 + f^3\| + \int_0^\infty g(s) \int_0^s \|f_x^5(\tau)\|\, d\tau ds\right] \\ &\le \frac{I_g(\lambda)}{2}\|\phi_x + \psi\|^2 + c\left[\frac{1}{I_g(\lambda)}\|\eta\|^2_{\mathcal{M}_{\tilde{g}}} + \left(\frac{1}{|\lambda|} + 1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}\right] \\ &\le \frac{I_g(\lambda)}{2}\|\phi_x + \psi\|^2 + c\left(\frac{1}{I_g(\lambda)} + \frac{1}{|\lambda|} + 1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}. \end{aligned}$$

Plugging the above estimate into (3.7), we arrive at the desired result. ∎

3.2.4. Propagating viscoelastic damping effects to other terms.

LEMMA 3.11. *Under Assumption* 3.1*, there exists* $c > 0$ *such that*

$$\|\eta\|^2_{\mathcal{M}_g} \le c\left(\frac{1}{I_g(\lambda)} + 1\right)\left(\frac{1}{I_g(\lambda)} + \frac{1}{|\lambda|} + 1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}.$$

Proof. Taking the inner product of (3.6) with η in $\mathcal{M}_g$, we have

$$\|\eta\|^2_{\mathcal{M}_g} = a_2, \tag{3.8}$$

where

$$a_2 := \int_0^\infty g(s) \int_0^s e^{-i\lambda(s-\tau)}(f_x^5(\tau), \eta_x(s))\, d\tau ds + \int_0^\infty g(s)(1 - e^{-i\lambda s})(\phi_x + \psi, \eta_x(s))\, ds - \frac{1}{i\lambda}\int_0^\infty g(s)(1 - e^{-i\lambda s})(f_x^1 + f^3, \eta_x(s)).$$

Using the Hölder inequality and using Lemmas 3.4, 3.9, and 3.10, we arrive at

$$\begin{aligned} |a_2| &\le c\|\eta\|_{\mathcal{M}_g}\left[\|\phi_x + \psi\| + \frac{1}{|\lambda|}\|f_x^1 + f^3\| + \left(\int_0^\infty g(s)\left(\int_0^s \|f_x^5(\tau)\|\, d\tau\right)^2 ds\right)^{1/2}\right] \\ &\le \frac{1}{2}\|\eta\|^2_{\mathcal{M}_g} + c\left[\|\phi_x + \psi\|^2 + \left(\frac{1}{|\lambda|} + 1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}\right] \\ &\le \frac{1}{2}\|\eta\|^2_{\mathcal{M}_g} + c\left(\frac{1}{I_g(\lambda)} + 1\right)\left(\frac{1}{I_g(\lambda)} + \frac{1}{|\lambda|} + 1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}. \end{aligned}$$

Replacing the above estimate in (3.8), we obtain the desired estimate. ∎

LEMMA 3.12. *Under Assumption* 3.1*, we have for every* $\varepsilon \in (0, 1)$*:*

$$\rho_1\|\Phi\|^2 \le \frac{c}{\varepsilon}\left(\frac{1}{I_g(\lambda)} + 1\right)\left(\frac{1}{I_g(\lambda)} + \frac{1}{|\lambda|} + 1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}} + \varepsilon\|\psi_x\|^2.$$

Proof. Taking the inner product of $(3.5)_2$ with ϕ in $L^2(0,L)$ and using $(3.5)_1$, we infer

$$\rho_1\|\Phi\|^2 = a_3, \tag{3.9}$$

where

$$a_3 := -(f^2,\phi) - \rho_1(\Phi, f^1) + \kappa\omega(\phi_x+\psi,\phi_x) + \kappa\int_0^\infty g(s)(\eta_x(s),\phi_x)\,ds.$$

Applying the Hölder inequality, using Lemmas 3.10 and 3.11 and noting that

$$\|\phi_x\| \le \|\phi_x+\psi\| + c\|\psi_x\|,$$

we arrive at

$$\begin{aligned}
|a_3| &\le c\|\phi_x\|\left[\|f^2\| + \|\phi_x+\psi\| + \|\eta\|_{\mathcal{M}_g}\right] + \rho_1\|\Phi\|\|f^1\| \\
&\le \frac{c}{\varepsilon}\left(\|\phi_x+\psi\|^2 + \|\eta\|^2_{\mathcal{M}_g}\right) + \varepsilon\|\psi_x\|^2 + c\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}} \\
&\le \frac{c}{\varepsilon}\left(\frac{1}{I_g(\lambda)}+1\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}} + \varepsilon\|\psi_x\|^2.
\end{aligned}$$

Combining the above estimate with (3.9), we complete the proof. □

3.2.5. Overcoming the lack of damping effect on the bending moment. In the next two lemmas we provide very new estimates. As far as the authors know, the path of the computations is never given in the literature. The insightful reader will note that this is the exact moment where the parameter χ_0 comes into play.

LEMMA 3.13. *Under Assumption* 3.1*, let us also consider* $\lim_{s\to 0} g(s) = g_0 < +\infty$*. Then, there exists* $c>0$ *such that*

$$\begin{aligned}
\|\Psi\|^2 \le\, & c\left(\frac{1}{I_g(\lambda)}+1\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}} \\
& + c|\chi_0|\left(1+|\lambda|^2\right)\left(\frac{1}{I_g(\lambda)}+1\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}.
\end{aligned}$$

Proof. To clarify the whole proof, we proceed with the following algorithm:

Step 1. Take the inner product of f^1_x with Ψ in $L^2(0,L)$ and use $(3.5)_1$. Then, we have

$$(f^1_x,\Psi) = i\lambda(\phi_x,\Psi) - (\Phi_x,\Psi).$$

Step 2. Take the inner product of f^2 with ψ_x in $L^2(0,L)$ and use $(3.5)_2$. Thus, we get

$$\frac{1}{\rho_1}(f^2,\psi_x) = i\lambda(\Phi,\psi_x) - \frac{\kappa}{\rho_1}\left(\omega(\phi_x+\psi)_x + \int_0^\infty g(s)\eta_{xx}(s)\,ds, \psi_x\right).$$

Step 3. Taking the inner product of f^3_x with Φ in $L^2(0,L)$ and using $(3.5)_3$, we obtain

$$(f^3_x,\Phi) = i\lambda(\psi_x,\Phi) - (\Psi_x,\Phi).$$

Step 4. Taking the inner product of f^4 with $\omega(\phi_x+\psi)+\int_0^\infty g(s)\eta_x(s)\,ds$ in $L^2(0,L)$ and regarding $(3.5)_4$, $(3.5)_3$, and $(3.5)_1$, we find

$$\begin{aligned}\omega\|\Psi\|^2 = {} & i\lambda(\Psi,\phi_x)+\ell(\Psi,\Phi_x)+\ell(\Psi,f^1_x)-\omega(\Psi,f^3)\\ & -\frac{1}{\rho_2}\left(f^4,\omega(\phi_x+\psi)+\int_0^\infty g(s)\eta_x(s)\,ds\right)\\ & +i\lambda\int_0^\infty g(s)(\Psi,\eta_x(s))\,ds+\frac{\kappa}{\rho_2}\left\|\omega(\phi_x+\psi)+\int_0^\infty g(s)\eta_x(s)\,ds\right\|^2\\ & +\frac{b}{\rho_2}\left(\psi_x,\omega(\phi_x+\psi)_x+\int_0^\infty g(s)\eta_{xx}(s)\,ds\right).\end{aligned}$$

Step 5. Taking the inner product of f^5 with $\widetilde{\Psi}$ in $\mathcal{M}_g$ and applying $(3.5)_5$, we infer

$$\begin{aligned}\ell\|\Psi\|^2 = {} & i\lambda\int_0^\infty g(s)(\eta_x(s),\Psi)\,ds+\int_0^\infty g(s)(\eta_{sx}(s),\Psi)\,ds-\ell(\Phi_x,\Psi)\\ & -\int_0^\infty g(s)(f^6_x(s),\Psi)\,ds.\end{aligned}$$

Step 6. Adding all the results obtained in the five previous steps and extracting the real part on both sides of the resulting identity, we arrive at

$$\|\Psi\|^2 = \mathrm{Re}\left[a_4+a_5+\chi_0 a_6+a_7\right], \tag{3.10}$$

where

$$\begin{aligned}a_4 &:= \omega(\Psi,f^3)-(f^1_x,\Psi)-\frac{1}{\rho_1}(f_2,\psi_x)-(f^3_x,\Phi)+\ell(\Psi,f^1_x)-\int_0^\infty g(s)(f^5_x(s),\Psi)\,ds\\ &\quad -\frac{1}{\rho_2}\left(f^4,\omega(\phi_x+\psi)+\int_0^\infty g(s)\eta_x(s)\,ds\right),\\ a_5 &:= \frac{\kappa}{\rho_2}\left\|\omega(\phi_x+\psi)+\int_0^\infty g(s)\eta_x(s)\,ds\right\|^2,\\ a_6 &:= \left(\omega(\phi_x+\psi)+\int_0^\infty g(s)\eta_x\,ds,\psi_{xx}\right),\\ a_7 &:= \int_0^\infty g(s)(\eta_{sx}(s),\Psi)\,ds.\end{aligned}$$

Step 7. (conclusion). Let us give a proper estimate for the right-hand side of (3.10). Indeed, clearly we have

$$|a_4|\le c\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}.$$

Using Lemmas 3.10 and 3.11, we can estimate a_5 by

$$|a_5|\le c\left(\frac{1}{I_g(\lambda)}+1\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}.$$

Now, we use $(3.5)_4$ to write down

$$a_6=\frac{\rho_2}{b}a_5+\frac{1}{b}\left(\omega(\phi_x+\psi)+\int_0^\infty g(s)\eta_x(s)\,ds,i\lambda\rho_2\Psi-f^4\right).$$

From Lemmas 3.10 and 3.11, we get

$$\begin{aligned}|a_6| &\le c\left(\frac{1}{I_g(\lambda)}+1\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}+c|\lambda|\|\Psi\|\left(\|\phi_x+\psi\|+\|\eta\|_{\mathcal{M}_g}\right)\\ &\le c\left(1+|\lambda|^2\right)\left(\frac{1}{I_g(\lambda)}+1\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}+\frac{1}{4}\|\Psi\|^2.\end{aligned}$$

Finally, we estimate the term a_7. Using (2.11), we have

$$a_7=\int_0^\infty g(s)\frac{d}{ds}(\eta_x(s),\Psi)\,ds=-\int_0^\infty g'(s)(\eta_x(s),\Psi)\,ds.$$

At this moment, we use the assumption $\lim_{s\to 0} g(s)=g_0<+\infty$ and Lemma 3.9 to deduce

$$|a_7|\le g_0\left(-\int_0^\infty g'(s)\|\eta_x(s)\|^2\,ds\right)+\frac{1}{4}\|\Psi\|^2\le c\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}+\frac{1}{4}\|\Psi\|^2.$$

Collecting the above estimates and plugging the results into (3.10), we obtain the desired estimate of Lemma 3.13. ∎

LEMMA 3.14. *Under the assumptions of Lemma* 3.13*, there exists* $c>0$ *such that*

$$\begin{aligned}\|\psi_x\|^2 \le\,& c\left(\frac{1}{I_g(\lambda)}+1\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}\\ &+c|\chi_0|\left(1+|\lambda|^2\right)\left(\frac{1}{I_g(\lambda)}+1\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}.\end{aligned}$$

Proof. Taking the inner product of $(3.5)_4$ with ψ in $L^2(0,L)$ and using $(3.5)_3$, we have

$$\|\psi_x\|^2=a_8, \tag{3.11}$$

where

$$a_8:=\frac{1}{b}\left[\left(f^4+\kappa\omega(\phi_x+\psi)+\kappa\int_0^\infty g(s)\eta_x(s)\,ds,\psi\right)+\rho_2(\Psi,f^3+\Psi)\right].$$

From Lemmas 3.10, 3.11, and 3.13, we deduce

$$\begin{aligned}|a_8| &\le c\|\psi_x\|\left(\|f^4\|+\|\phi_x+\psi\|+\|\eta\|_{\mathcal{M}_g}\right)+c\|\Psi\|\left(\|f^3\|+\|\Psi\|\right)\\ &\le \frac{1}{2}\|\psi_x\|^2+c\left(\|\phi_x+\psi\|^2+\|\eta\|^2_{\mathcal{M}_g}+\|\Psi\|^2\right)+c\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}\\ &\le c\left(\frac{1}{I_g(\lambda)}+1\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}+\frac{1}{2}\|\psi_x\|^2\\ &\quad+c|\chi_0|\left(1+|\lambda|^2\right)\left(\frac{1}{I_g(\lambda)}+1\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}}.\end{aligned}$$

Plugging the above estimate into (3.11), we arrive at the desired conclusion. ∎

3.2.6. Conclusion: $\mathcal{H}$-estimate.

LEMMA 3.15. *Under the above scenario, there exists* $c>0$ *such that*

$$\begin{aligned}\|z\|_{\mathcal{H}}\le\,& c\left(\frac{1}{I_g(\lambda)}+1+|\chi_0|\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|f\|_{\mathcal{H}}\\ &+c|\chi_0||\lambda|^2\left(\frac{1}{I_g(\lambda)}+1\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|f\|_{\mathcal{H}}.\end{aligned}$$

Proof. Regarding the estimates from Lemmas 3.10 to 3.14, with $\varepsilon, \tilde{\varepsilon} \in (0,1)$ small enough in Lemmas 3.12 and 3.13, respectively, we arrive at

$$\begin{aligned}\|z\|_{\mathcal{H}}^2 \leq c\left(\frac{1}{I_g(\lambda)}+1+|\chi_0|\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}} \\ +c|\chi_0||\lambda|^2\left(\frac{1}{I_g(\lambda)}+1\right)\left(\frac{1}{I_g(\lambda)}+\frac{1}{|\lambda|}+1\right)\|z\|_{\mathcal{H}}\|f\|_{\mathcal{H}},\end{aligned}$$

and the conclusion follows from Young's inequality. ∎

3.2.7. Existence of solution for the resolvent equation. Now, we use the approximate point spectrum $\sigma_{ap}(\mathbb{A})$ to prove that (3.4) (and consequently (3.5)) has a unique solution for every given $\lambda \in \mathbb{R}$ and $f \in \mathcal{H}$.

LEMMA 3.16. *Let us consider Assumption* 3.1 *and* $\lim_{s\to 0} g(s) = g_0 < +\infty$. *Then,* $i\mathbb{R} \subset \rho(\mathbb{A})$.

Proof. Suppose by contradiction that $i\lambda_* \in \sigma(\mathbb{A})$ for some $\lambda_* \in \mathbb{R}$. Then, we invoke Lemma 3.6 to obtain $\lambda_* \in \sigma_{ap}(\mathbb{A})$. By definition, there exists $z_n = (\phi_n, \Phi_n, \psi_n, \Psi_n, \eta_n) \in D(\mathbb{A})$ such that $\|z_n\|_{\mathcal{H}} = 1$ for all $n \in \mathbb{N}$ and

$$f_n := i\lambda_* z_n - \mathbb{A} z_n \to 0 \text{ in } \mathcal{H}. \tag{3.12}$$

Calling $f_n = (f_n^1, f_n^2, f_n^3, f_n^4, f_n^5)$, we can write (3.12) in terms of its components:

(3.13)
$$\begin{cases} f_n^1 = i\lambda_*\phi_n - \Phi_n \to 0 \text{ in } H_0^1(0,L), \\ f_n^2 = i\lambda_*\rho_1\Phi_n - \kappa\left[\omega(\phi_{nx}+\psi_n) + \displaystyle\int_0^\infty g(s)\eta_{nx}(s)\,ds\right]_x \to 0 \text{ in } L^2(0,L), \\ f_n^3 = i\lambda_*\psi_n - \Psi_n \to 0 \text{ in } H_*^1(0,L), \\ f_n^4 = i\lambda_*\rho_2\Psi_n - b\psi_{nxx} + \kappa\left[\omega(\phi_{nx}+\psi_n) + \displaystyle\int_0^\infty g(s)\eta_{nx}(s)\,ds\right] \to 0 \text{ in } L_*^2(0,L), \\ f_n^5 = i\lambda_*\eta_n + \eta_{ns} - \left(\Phi_n + \widetilde{\Psi}_n\right) \to 0 \text{ in } \mathcal{M}_g. \end{cases}$$

At this point, we will split the proof into two cases as follows.

Case 1*:* $\lambda_* = 0$. In this case, we immediately obtain from $(3.13)_1$, $(3.13)_3$, and $(3.13)_5$ the following convergences:

$$\Phi_n \to 0 \text{ in } H_0^1(0,L), \quad \Psi_n \to 0 \text{ in } H_*^1(0,L), \quad \eta_{ns} \to 0 \text{ in } \mathcal{M}_g. \tag{3.14}$$

Taking into account that $\eta_n \in D(\mathbb{L})$ and the last convergence in (3.14), we can apply Lemma 3.4 to get

$$\|\eta_n\|_{\mathcal{M}_g} \leq \sqrt{\int_0^\infty g(s)\left(\int_0^s \|\eta_{nxs}(\tau)\|\,d\tau\right)^2 ds} \leq c\|\eta_{ns}\|_{\mathcal{M}_g} \to 0. \tag{3.15}$$

On the other hand, taking the inner product of $(3.13)_2$ with ϕ_n in $L^2(0,L)$, the inner product of $(3.13)_4$ with ψ_n in $L_*^2(0,L)$, and adding the results, we deduce

$$\omega\kappa\|\phi_{nx}+\psi_n\|^2 + b\|\psi_{nx}\|^2 = d_n^1, \tag{3.16}$$

where,

$$d_n^1 := (f_n^2, \phi_n) + (f_n^4, \psi_n) - \int_0^\infty g(s)\left(\eta_{nx}(s), \phi_{nx}+\psi_n\right) ds.$$

But, from $(3.13)_2$, $(3.13)_4$, and (3.15), we infer

$$|d_n^1| \le \|f_n^2\|\|\phi_n\| + \|f_n^4\|\|\psi_n\| + \sqrt{\ell}\|\eta_n\|_{\mathcal{M}_g}\|\phi_{nx}+\psi_n\| \le c\|z_n\|_{\mathcal{H}}\left(\|f_n\|_{\mathcal{H}} + \|\eta_n\|_{\mathcal{M}_g}\right) \to 0.$$

Hence, from (3.14), (3.15), and (3.16) we conclude that $\|z_n\|_{\mathcal{H}} \to 0$, which contradicts the equality $\|z_n\|_{\mathcal{H}} = 1$ for all $n \in \mathbb{N}$.

Case 2: $\lambda_* \neq 0$. Since $\lambda_* \in \mathbb{R}\backslash\{0\}$, $f_n \in \mathcal{H}$ and $z_n \in D(\mathbb{A})$ is a solution of (3.12), we are in condition to apply Lemma 3.15 (for z_n) to conclude that

$$\|z_n\|_{\mathcal{H}} \le c\left(|\chi_0||\lambda_*|^2+1\right)\left(\frac{1}{I_g(\lambda_*)}+1+|\chi_0|\right)\left(\frac{1}{I_g(\lambda_*)}+\frac{1}{|\lambda_*|}+1\right)\|f_n\|_{\mathcal{H}} \to 0,$$

which contradicts again the fact $\|z_n\|_{\mathcal{H}} = 1$ for all $n \in \mathbb{N}$.

Since we arrive at a contradiction in both cases, we conclude that $i\mathbb{R} \subset \rho(\mathbb{A})$. ∎

3.2.8. A resolvent estimate from below. In the next result we use the notation

$$a_n \approx c\, b_n \quad \Longleftrightarrow \quad \lim_{n\to\infty}\frac{|a_n|}{|b_n|} = c.$$

LEMMA 3.17. *Suppose that* $\lim_{s\to 0} g(s) = g_0 < +\infty$. *If* $\chi_0 \neq 0$, *then there exist sequences*

$$\lambda_n \approx \sqrt{\frac{b\pi^2}{\rho_2 L^2}}\, n, \qquad B_n \approx \left[\frac{\pi\rho_1^2|\chi_0|^2\sqrt{b}}{\kappa g_0 L\left(\kappa+\rho_1|\chi_0|\right)^2\sqrt{\rho_2}}\right] n,$$

and $f_n \in \mathcal{H}$ *such that*

$$\|\left(i\lambda_n - \mathbb{A}\right)^{-1} f_n\|_{\mathcal{H}} \ge \sqrt{\rho_2}|B_n||\lambda_n|. \tag{3.17}$$

Proof. It is well known that the Dirichlet (or Neumann) 1-dimensional Laplacian operator is positive, self-adjoint, and has a compact inverse. Besides, its eigenvalues are given by

$$\gamma_n = \left(\frac{n\pi}{L}\right)^2, \ n \in \mathbb{N}, \tag{3.18}$$

and the corresponding unitary eigenfunctions are

$$\begin{cases} e_n(x) = \sqrt{\dfrac{2}{L}}\sin(\sqrt{\gamma_n}x) & \text{(Dirichlet boundary condition)},\\ e_n^*(x) = \sqrt{\dfrac{2}{L}}\cos(\sqrt{\gamma_n}x) & \text{(Neumann boundary condition)}. \end{cases} \tag{3.19}$$

Let us consider the sequences $f_n = (0,0,0,-\rho_2^{-1}e_n^*,0) \in \mathcal{H}$. Since $i\mathbb{R} \subset \rho(\mathbb{A})$, there exists $z_n = (\phi_n, \Phi_n, \psi_n, \Psi_n, \eta_n) \in D(\mathbb{A})$ such that

$$i\lambda_n z_n - \mathbb{A}z_n = f_n.$$

Componentwise, we have

$$\begin{cases} i\lambda_n\phi_n - \Phi_n = 0,\\ i\lambda_n\rho_1\Phi_n - \kappa\left[\omega(\phi_{nx}+\psi_n) + \displaystyle\int_0^\infty g(s)\eta_{nx}(s)\,ds\right]_x = 0,\\ i\lambda_n\psi_n - \Psi_n = 0,\\ i\lambda_n\rho_2\Psi_n - b\psi_{nxx} + \kappa\left[\omega(\phi_{nx}+\psi_n) + \displaystyle\int_0^\infty g(s)\eta_{nx}(s)\,ds\right] = -e_n^*,\\ i\lambda_n\eta_n + \eta_{ns} - \left(\Phi_n + \widetilde{\Psi}_n\right) = 0. \end{cases} \tag{3.20}$$

Solving the differential equation $(3.20)_5$ and using $(3.20)_1$ and $(3.20)_3$ in the result, we have

$$\eta(s) = (1 - e^{-i\lambda_n s})\left(\phi_n + \widetilde{\psi}_n\right). \tag{3.21}$$

Now, using $(3.20)_1$, $(3.20)_3$, and (3.21) in $(3.20)_2$ and $(3.20)_4$ we arrive at

$$\begin{cases} \rho_1\lambda_n^2\phi_n + \kappa\left[1 - \widehat{g}(\lambda_n)\right](\phi_{nx} + \psi_n)_x = 0, \\ \rho_2\lambda_n^2\psi_n + b\psi_{nxx} - \kappa\left[1 - \widehat{g}(\lambda_n)\right](\phi_{nx} + \psi_n) = e_n^*. \end{cases} \tag{3.22}$$

We are looking for solutions of (3.22) of the form

$$\phi_n = A_n e_n, \quad \psi_n = B_n e_n^*$$

for some complex sequences A_n and B_n. Replacing these particular choices in (3.22), we obtain the following complex linear system,

$$\begin{cases} \left[p_n(\lambda_n^2) + \kappa\widehat{g}(\lambda_n)\gamma_n\right]A_n - \kappa\left(1 - \widehat{g}(\lambda_n)\right)\sqrt{\gamma_n}B_n = 0, \\ -\kappa\left(1 - \widehat{g}(\lambda_n)\right)\sqrt{\gamma_n}A_n + \left[q_n(\lambda_n^2) + \widehat{g}(\lambda_n)\right]B_n = 1, \end{cases} \tag{3.23}$$

where the polynomials p_n and q_n are given by

$$p_n(s) := \rho_1 s - \kappa\gamma_n, \quad q_n(s) = \rho_2 s - b\gamma_n - \kappa. \tag{3.24}$$

Let

$$M_n = \begin{bmatrix} p_n(\lambda_n^2) + \kappa\widehat{g}(\lambda_n)\gamma_n & -\kappa\left(1 - \widehat{g}(\lambda_n)\right)\sqrt{\gamma_n} \\ -\kappa\left(1 - \widehat{g}(\lambda_n)\right)\sqrt{\gamma_n} & q_n(\lambda_n^2) + \kappa\widehat{g}(\lambda_n) \end{bmatrix}.$$

Note that

$$\det M_n = P_n(\lambda_n^2) + \kappa\widehat{g}(\lambda_n)Q_n(\lambda_n^2),$$

where

$$P_n(s) = p_n(s)q_n(s) - \kappa^2\gamma_n, \quad Q_n(s) = \gamma_n q_n(\lambda_n^2) + p_n(\lambda_n^2) + 2\kappa\gamma_n.$$

In the following we choose a suitable sequence λ_n such that $\det M_n \neq 0$. Actually, we will pick a sequence λ_n satisfying $P_n(\lambda_n^2) = 0$ and $\kappa\widehat{g}(\lambda_n)Q_n(\lambda_n^2) \neq 0$. Indeed, solving the equation $P_n(s) = 0$, we get

$$s_n^{\pm} = \frac{a_1\gamma_n + a_2 \pm \sqrt{(\chi_0\gamma_n)^2 + 2a_1a_2\gamma_n + a_2^2}}{2} > 0,$$

where $a_1 := \frac{\kappa}{\rho_1} + \frac{b}{\rho_2}$, $a_2 := \frac{\kappa}{\rho_2}$. At this moment, considering that $\chi_0 \neq 0$, we choose

$$\lambda_n = \begin{cases} \sqrt{s_n^+} & \text{if } \chi_0 < 0, \\ \sqrt{s_n^-} & \text{if } \chi_0 > 0, \end{cases}$$

and we observe that, up to a subsequence,

$$\det M_n = \kappa\widehat{g}(\lambda_n)Q_n(\lambda_n^2) \neq 0.$$

Returning to system (3.23) and solving it, we get

$$B_n = \frac{p_n(\lambda_n^2) + \gamma_n \kappa \widehat{g}(\lambda_n)}{\kappa \widehat{g}(\lambda_n) Q_n(\lambda_n^2)}. \tag{3.25}$$

Now, we give a properly asymptotic estimate for each term of B_n given by (3.25). First, we note that

$$\lambda_n^2 \approx \frac{b}{\rho_2}\gamma_n. \tag{3.26}$$

From (3.24) we have

$$\frac{1}{\rho_1} p_1(\lambda_n^2) = \frac{1}{\rho_1} p_1(s_n^{\pm}) = \frac{-\chi_0\gamma_n \pm \sqrt{(\chi_0\gamma_n)^2 + 2a_1a_2\gamma_n + a_2^2}}{2} + \frac{a_2}{2}.$$

Then,

$$p_n(\lambda_n^2) \approx \rho_1 |\chi_0| \gamma_n. \tag{3.27}$$

On the other hand, exploring that

$$p_n(\lambda_n^2) q_n(\lambda_n^2) - \kappa^2 \gamma_n = P_n(\lambda_n^2) = 0$$

and using (3.27), we get $q_n(\lambda_n^2) \approx \kappa^2/\rho_1|\chi_0|$. Collecting the above asymptotic estimates, we deduce

$$Q_n(\lambda_n^2) = \gamma_n q_n(\lambda_n^2) + p_n(\lambda_n^2) + 2\kappa\gamma_n \approx \left[\frac{\kappa^2}{\rho_1|\chi_0|} + \rho_1|\chi_0| + 2\kappa\right]\gamma_n. \tag{3.28}$$

Then, from (3.18), (3.25), (3.26), (3.27), and (3.28) and applying Lemma 3.5, we arrive at

$$B_n \approx \left[\frac{\pi \rho_1^2 |\chi_0|^2 \sqrt{b}}{\kappa g_0 L\,(\kappa + \rho_1|\chi_0|)^2 \sqrt{\rho_2}}\right] n. \tag{3.29}$$

Additionally, from $(3.20)_3$, we have

$$\Psi_n = i\lambda_n \psi_n = i B_n \lambda_n \sqrt{\frac{2}{L}} \cos(\sqrt{\gamma_n} x)$$

and, then,

$$\|z_n\|_{\mathcal{H}}^2 \ge \rho_2 \|\Psi_n\|^2 = \frac{2}{L}\rho_2 |B_n|^2 |\lambda_n|^2 \int_0^L \cos^2(\sqrt{\gamma_n} x)\, dx = \rho_2 |B_n|^2 |\lambda_n|^2. \tag{3.30}$$

Hence, combining (3.26), (3.29), and (3.30), the desired conclusion follows. □

3.3. Conclusion of the main results.

3.3.1. Proof of Theorem 3.1. In this case, the proof relies on the Borichev–Tomilov theorem (cf. [7]).

THEOREM 3.18. *Let $T(t) : X \to X$ be a bounded semigroup acting on a Hilbert space X with infinitesimal generator A. If $i\mathbb{R} \subset \rho(A)$, then for every fixed $\beta > 0$, we have*

$$\|(i\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq c|\lambda|^{\beta} \quad \textit{as} \quad |\lambda| \to +\infty \tag{3.31}$$

if and only if

$$\|T(t)A^{-1}\|_{\mathcal{L}(X)} \leq \frac{c}{t^{1/\beta}} \quad \textit{as} \quad t \to +\infty. \tag{3.32}$$

Let $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$ and $f \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} \leq 1$. Since $i\mathbb{R} \subset \rho(\mathbb{A})$, there exists a unique $z \in D(\mathbb{A})$ satisfying (3.4). From Lemma 3.15 there exists $c > 0$ such that

$$\|z\|_{\mathcal{H}} \leq c|\lambda|^2 \left(\frac{1}{I_g(\lambda)} + 1\right)^2 \|f\|_{\mathcal{H}},$$

which implies that

$$\frac{\|(i\lambda - \mathbb{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}}{|\lambda|^2} \leq c\left(\frac{1}{I_g(\lambda)} + 1\right)^2.$$

From Lemma 3.5 and using the expression of $I_g(\lambda)$ defined in Lemma 3.8, we obtain

$$\lim_{|\lambda| \to \infty} \frac{\|(i\lambda - \mathbb{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}}{|\lambda|^2} \leq c\left[\left(\int_0^\infty \tilde{g}(s)\, ds\right)^{-1} + 1\right]^2 < \infty.$$

Hence, applying Theorem 3.18 with $\beta = 2$, the desired conclusion follows.

3.3.2. Proof of Theorem 3.2. Suppose by contradiction that the rate $\sqrt{t}$ can be improved, namely, there exists $\nu \in (0, 2)$ such that

$$\|S(t)\mathbb{A}^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{c}{t^{1/(2-\nu)}} \quad \text{as} \quad t \to +\infty.$$

From Theorem 3.18, we arrive at

$$\frac{1}{|\lambda|^{2-v}}\|(i\lambda - \mathbb{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq c \quad \text{as} \quad |\lambda| \to +\infty. \tag{3.33}$$

However, from Lemma 3.17, there exist $\lambda_n, B_n \in \mathbb{R}$ and $f_n \in \mathcal{H}$ such that

$$\frac{1}{|\lambda_n|^{2-v}}\|\left(i\lambda_n - \mathbb{A}\right)^{-1} f_n\|_{\mathcal{H}} \geq \sqrt{\rho_2}|B_n||\lambda_n|^{v-1} \approx c\, n^{\nu}.$$

Then,

$$\lim_{n \to +\infty} \frac{1}{|\lambda_n|^{2-v}}\|\left(i\lambda_n - \mathbb{A}\right)^{-1} f_n\|_{\mathcal{H}} = +\infty,$$

which contradicts (3.33). Hence, the decay rate $\sqrt{t}$ is optimal.

3.3.3. Proof of Theorem 3.3. To prove our last result, we appeal to the Gearhart–Prüss theorem.

THEOREM 3.19 ([24, 41]). *Let $T(t): X \to X$ be a bounded semigroup acting on a Hilbert space X with infinitesimal generator A. Then, $T(t)$ is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ and*

$$\limsup_{|\lambda|\to+\infty} \|(i\lambda - A)^{-1}\|_{\mathcal{L}(X)} < +\infty. \tag{3.34}$$

Proof of necessity. Suppose that $S(t)$ is exponentially stable. We are going to prove that

(i) g satisfies Assumption 3.1;

(ii) $\chi_0 = 0$.

Proof of (i): We borrow the ideas from [9] somehow. Indeed, let us consider $\eta_0 \in \mathcal{M}_g$ and

$$z(t) = S(t)(0,0,0,0,\eta_0) = (\phi(t), \Phi(t), \psi(t), \Psi(t), \eta^t).$$

Since $S(t)$ is exponentially stable, we have

$$\|z(t)\|^2_{\mathcal{H}} = \|S(t)(0,0,0,0,\eta_0)\|^2_{\mathcal{H}} \le M e^{-\gamma t}\|\eta_0\|^2_{\mathcal{M}_g} \tag{3.35}$$

for some $M \ge 1$, $\gamma > 0$, and for every $t > 0$. Now, by formula (2.15) and using (3.35), we deduce

$$\begin{aligned} \int_t^\infty g(s)\|\eta_{0x}(s-t)\|^2\, ds &\le 2\|\eta^t\|^2_{\mathcal{M}_g} + 2\|\phi_x(t) + \psi(t)\|^2 \\ &\le 2M\left(1 + \frac{1}{\omega\kappa}\right) e^{-\gamma t}\|\eta_0\|^2_{\mathcal{M}_g}. \end{aligned} \tag{3.36}$$

On the other hand, for each $t > 0$ we define

$$\mathcal{N}_t := \left\{ s \in \mathbb{R}^+,\ g(t+s) - 2M\left(1 + \frac{1}{\omega\kappa}\right) e^{-\gamma t} g(s) > 0 \right\}.$$

We claim that $|\mathcal{N}_t| = 0$, for every $t > 0$. Indeed, suppose by contradiction that there exists $t_0 > 0$ such that $|\mathcal{N}_{t_0}| > 0$ (possibly infinite). Then,

$$0 < \int_{\mathcal{N}_{t_0}} \left[g(t_0+s) - 2M\left(1 + \frac{1}{\omega\kappa}\right) e^{-\gamma t_0} g(s) \right] ds < +\infty. \tag{3.37}$$

But, from (3.36),

$$\begin{aligned} 0 &\ge \int_{t_0}^\infty g(s)\|\eta_{0x}(s-t_0)\|^2\, ds - 2M\left(1 + \frac{1}{\omega\kappa}\right) e^{-\gamma t_0} \int_0^\infty g(s)\|\eta_{0x}(s)\|^2\, ds \\ &= \int_0^\infty \left[g(t_0+s) - 2M\left(1 + \frac{1}{\omega\kappa}\right) e^{-\gamma t_0} g(s) \right] \|\eta_{0x}(s)\|^2\, ds. \end{aligned}$$

Now we choose $\eta_0(s) = \chi_{\mathcal{N}_{t_0}}(s)\phi^*$ for some $\phi^* \in H_0^1(0,L)$ such that $\|\phi^*_x\| = 1$. Therefore,

$$\int_{\mathcal{N}_{t_0}} \left[g(t_0+s) - 2M\left(1 + \frac{1}{\omega\kappa}\right) e^{-\gamma t_0} g(s) \right] ds \le 0,$$

which contradicts (3.37). Hence, g satisfies (3.1) with $\delta = \gamma > 0$.

Proof of (ii): Suppose by contradiction that $\chi_0 \neq 0$. From Lemma 3.17 there exists sequences $\lambda_n \to +\infty$ and $f_n \in \mathcal{H}$ such that

$$\lim_{n\to+\infty} \| (i\lambda_n - \mathbb{A})^{-1} f_n \|_{\mathcal{H}} = +\infty.$$

But, it contradicts (3.34). Then, we must have $\chi_0 = 0$.

Proof of sufficiency. Let $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$ and $f \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} \leq 1$. Since $i\mathbb{R} \subset \rho(\mathbb{A})$, there exists $z \in D(\mathbb{A})$ satisfying (3.4). If $\chi_0 = 0$ and g satisfies Assumption 3.1, we can apply Lemma 3.15 to obtain

$$\|z\|_{\mathcal{H}} \leq c \left(\frac{1}{I_g(\lambda)} + 1 \right)^2 \|f\|_{\mathcal{H}},$$

which implies that

$$\|(i\lambda - \mathbb{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq c \left(\frac{1}{I_g(\lambda)} + 1 \right)^2.$$

Hence, from Lemma 3.5 and Theorem 3.19, the desired conclusion follows.

Appendix A. Modeling within the creation time scenario. In this appendix, we proceed with the mathematical modeling for the partially viscoelastic beam models aforementioned in (1.3)–(1.4), whose foundation goes back to the legacy of both Boltzmann [5, 6] and Volterra [45, 46] theories for ageing viscoelastic bodies along with Timoshenko's postulations (see [43, 44]), for shearing stress in beam vibrations. We also take into account some properties on viscoelastic materials inspired by the contributions of Prüss [40], Drozdov and Kolmanovskii [19], and Fabrizio, Giorgi, and Pata [20].

Boltzmann–Volterra Constitutive Law. According to Boltzmann's (and later Volterra's) theory for ageing viscoelastic bodies, the mathematical feature for integro-differential models can be characterized with stress (σ) dependence on time not only via the instantaneous (present) strain (ϵ) but also on the presence of the past strain history of the material.

Under the above scenario, let us start with the following stress-strain constitutive law (cf. [20]),

(A.1)
$$\sigma(\cdot,t) = E\left\{\epsilon(\cdot,t) + \int_\alpha^t \mathbb{G}'(t-s)\epsilon(\cdot,s)ds\right\} := E\left\{\epsilon(\cdot,t) - \int_\alpha^t g(t-s)\epsilon(\cdot,s)ds\right\},$$

where E stands for the Young modulus of elasticity, $\mathbb{G}'$ represents the relaxation measure given by the derivative of the Boltzmann tensor $\mathbb{G}$, $g := -\mathbb{G}' > 0$ is just a proper notation for our coming purposes, and the parameter $\alpha \leq 0$ is known as the creation time whose interpretation reflects on the vanishing of any past history previously to time α. In the modern theory in viscoelasticity, the limit cases $\alpha = -\infty$ and $\alpha = 0$ are proposed by Boltzmann and Volterra (cf. [5, 6, 45, 46]), respectively, and promote distinguished mathematical models.

Timoshenko Assumptions. According to the Timoshenko theory, not only the bending moment but also the shear stress must be taken into account for beams vibrations once, e.g., the rotation angles of a cross section of the beam movement are triggered by both forces. Moreover, we highlight that shearing stresses play a key role in the current development in viscoelasticity. In this way, we consider the classical Timoshenko assumptions as follows (cf. [19, 40]).

Let us consider a 3-dimensional beam $[0,L]\times\Omega$ of length $L>0$ and uniform cross section $\Omega\subset\mathbb{R}^2$ made of homogeneous isotropic viscoelastic material, which can be mathematically written as

$$[0,L]\times\Omega := \{(x,y,z):\ x\in[0,L] \text{ and } (y,z)\in\Omega\}.$$

The Timoshenko assumptions are given by

A1. $(0,0)$ is the center of Ω so that it holds the symmetry $\int_\Omega zdydz = \int_\Omega ydydz = 0$;
A2. very thin beams are in place so that $\operatorname{diam}\Omega << L$;
A3. normal stresses, say in the y-axis, are negligible so that the bending acts only in the (x,z)-plane;
A4. the stress tensor $\sigma=(\sigma_{ij})_{1\le i,j\le 3}$ has only two relevant stresses given by σ_{11} and σ_{13} so that the remaining ones are neglected ($\sigma_{ij}\approx 0$);
A5. the following viscoelastic stress-strain relations (in view of (A.1)) come up,

$$\sigma_{11}(x,z,t) = E\left\{\epsilon_{11}(x,z,t) - \int_\alpha^t g_1(t-s)\epsilon_{11}(x,z,s)ds\right\}, \tag{A.2}$$

$$\sigma_{13}(x,z,t) = 2kG\left\{\epsilon_{13}(x,z,t) - \int_\alpha^t g_2(t-s)\epsilon_{13}(x,z,s)ds\right\}, \tag{A.3}$$

where G is the constant shear modulus, k is a shear correction coefficient, g_1, g_2 are relaxation kernels, and $\alpha\le 0$.

Elastic Strains. Under the above assumptions and looking for notations to the displacements and the rotation angles, we write down

- $u=u(x,t)$: the longitudinal displacement of points lying on the x-axis;
- $\psi=\psi(x,t)$: the angle of rotation for the normal to the x-axis;
- $w_1(x,z,t)=u(x,t)+z\psi(x,t)$: longitudinal displacement;
- $w_2(x,z,t)=\phi(x,t)$: the vertical displacement;

and, consequently, the formulas for the strain tensors (cf. [19, p. 339]) are expressed by

$$\epsilon_{11}(x,z,t) := \frac{\partial w_1}{\partial x} = u_x(x,t) + z\psi_x(x,t), \tag{A.4}$$

$$\epsilon_{13}(x,z,t) := \frac{1}{2}\left(\frac{\partial w_1}{\partial z} + \frac{\partial w_2}{\partial x}\right) = \frac{1}{2}\left(\psi(x,t) + \phi_x(x,t)\right). \tag{A.5}$$

Viscoelastic Bending and Shear Relations. Under the assumptions A1–A4 and regarding the standard identities for the forces in beam vibrations (cf. [40, p. 237]), we consider the bending moment (M) and the shear force (S), respectively, by

$$M(x,t) = \int_\Omega z\sigma_{11}(x,z,t)dydz, \tag{A.6}$$

$$S(x,t) = \int_\Omega \sigma_{13}(x,z,t)dydz, \tag{A.7}$$

where, for the sake of simplicity, we have normalized the formulas by the area $A=\int_\Omega dydz$ and inertial moment $I=\int_\Omega z^2dydz$ of the cross section Ω.

The consequence of the aforementioned formulas is twofold: the first one comes from relations (A.2), (A.4), and (A.6), whose combination provides the classical viscoelastic law for the bending moment,

$$\begin{aligned}M = E&\overbrace{\left(\int_\Omega z dydz\right)}^{=0}\left(u_x - \int_\alpha^t g_1(t-s)u_x(s)ds\right)\\ &+ E\underbrace{\left(\int_\Omega z^2 dydz\right)}_{=I}\left(\psi_x - \int_\alpha^t g_1(t-s)\psi_x(s)ds\right),\end{aligned}$$

or simply

$$M = EI\left(\psi_x - \int_\alpha^t g_1(t-s)\psi_x(s)\,ds\right),\ \alpha \le 0. \tag{A.8}$$

On the other hand, from relations (A.3), (A.5), and (A.7), the following (not so well studied) viscoelastic law for the shear force turns out:

$$S = kGA\left((\phi_x+\psi) - \int_\alpha^t g_2(t-s)(\phi_x+\psi)(s)ds\right),\ \alpha \le 0. \tag{A.9}$$

The new viscoelastic formulation (A.9) for the creation time α is the heart of the present modeling, specially in the case $\alpha = -\infty$, once it is precisely the matter responsible for producing the new and unexplored object of study herein, namely, problem (1.13). See also the further up system (A.14).

In conclusion, the viscoelastic constitutive laws (A.8)–(A.9) provide bending and shear deformations in the context of Timoshenko beams for viscoelastic materials with hereditary (history) properties. In addition, by means of (A.8)–(A.9) one can simply see the case where viscoelastic effects are not taken into account, namely, $g_1, g_2 \equiv 0$. Accordingly, we get the standard (well-known) elastic relations for bending moment and shear force, respectively,

$$M = EI\psi_x, \tag{A.10}$$

$$S = kGA(\phi_x+\psi). \tag{A.11}$$

Motion Equations for Timoshenko Beams. In order to reach distinct partially viscoelastic Timoshenko systems, still including problem (1.13), we are going to consider the following system of partial differential equations for vibrations of thin beams (cf. [43, 44]):

$$\begin{cases}\rho A\phi_{tt} - S_x = 0,\\ \rho I\psi_{tt} - M_x + S = 0\end{cases} \tag{A.12}$$

for $(x,t) \in (0,L)\times\mathbb{R}^+$, $\mathbb{R}^+ = (0,+\infty)$, where ρ represents the mass density per unit area and the remaining notations are previously introduced. Hence, keeping in mind the viscoelastic-elastic constitutive laws (A.8)–(A.11), we can derive four different types of partially viscoelastic systems, two of them being due to Timoshenko–Volterra (say $\alpha = 0$) and the other two possibilities related to Timoshenko–Boltzmann (say $\alpha = -\infty$). As usual, we denote the coefficients hereafter as

$$\rho_1 := \rho A, \quad \rho_2 := \rho I, \quad \kappa := kGA, \quad b := EI.$$

Model with viscoelasticity on the bending moment. Using (A.8) and (A.11), system (A.12) becomes the following new viscoelastic beam under the creation time perspective, say for any $\alpha \le 0$, and memory kernel denoted as $g_1 := g$:

$$\begin{cases}\rho_1\phi_{tt} - \kappa(\phi_x+\psi)_x = 0,\\ \rho_2\psi_{tt} - b\,\psi_{xx} - b\displaystyle\int_\alpha^t g(t-s)\psi_{xx}(s)\,ds + \kappa(\phi_x+\psi) = 0.\end{cases} \tag{A.13}$$

Thus, we specify the distinct cases:

- in the case $\alpha=0$, (A.13) is precisely the well-known problem (1.6), herein called the *viscoelastic Timoshenko-Volterra problem* with null history;
- in the case $\alpha=-\infty$, (A.13) becomes the well-studied problem (1.8), now called the *viscoelastic Timoshenko–Boltzmann problem* in the history context.

Model with viscoelasticity on the shear force. Now, by taking (A.9) and (A.10), system (A.12) turns into the following new viscoelastic beam under the creation time perspective ($\alpha\leq 0$) and memory kernel $g_2:=g$:

$$\text{(A.14)}\qquad \begin{cases} \rho_1\phi_{tt}-\kappa(\phi_x+\psi)_x-\kappa\displaystyle\int_\alpha^t g(t-s)(\phi_x+\psi)_x(s)ds=0,\\ \rho_2\psi_{tt}-b\,\psi_{xx}+\kappa(\phi_x+\psi)-\kappa\displaystyle\int_\alpha^t g(t-s)(\phi_x+\psi)(s)ds=0.\end{cases}$$

Hence, we highlight the following distinguished systems:

- in the case $\alpha=0$, (A.14) represents the recent problem (1.11), herein still called the viscoelastic Timoshenko-Volterra problem with null history;
- in the case $\alpha=-\infty$, (A.14) furnishes the unexplored scenario as in (1.13), still called the viscoelastic Timoshenko-Boltzmann problem with past history.

Remark A.1. Supplementary to the partially viscoelastic systems (A.13) and (A.14), one can obviously consider a full viscoelastic system with viscoelastic coupling on the bending moment and shear stress simultaneously, that is, by replacing the viscoelastic constitutive laws (A.8)–(A.9) in (A.12). This procedure reveals a fully viscoelastic damped problem which is out of scope of this work. A version with past history (one reads the $\alpha=-\infty$ case), and standard nonlinear source and external forces, is addressed in [26] where the asymptotic behavior of solutions is driven by the couple of exponential memory kernels.

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REFERENCES

[1] M. O. Alves, E. H. Gomes Tavares, M. A. Jorge Silva, and J. H. Rodrigues, *On modeling and uniform stability of a partially dissipative viscoelastic Timoshenko system*, SIAM J. Math. Anal., 51 (2019), pp. 4520–4543.

[2] A. M. Al-Mahdi, M. M. Al-Gharabli, A. Guesmia, and S. A. Messaoudi, *New decay results for a viscoelastic-type Timoshenko system with infinite memory*, Z. Angew. Math. Phys., 72 (2021), p. 22.

[3] F. Ammar-Khodja, A. Benabdallah, J. E. Muñoz Rivera, and R. Racke, *Energy decay for Timoshenko systems of memory type*, J. Differential Equation, 194 (2003), pp. 82–115.

[4] W. Arendt and C. J. K. Batty, *Tauberian theorems and stability of one-parameter semigroups*, Trans. Amer. Math. Soc., 306 (1988), pp. 837–852.

[5] L. Boltzmann, *Zur Theorie der elastischen Nachwirkung*, Wien. Ber., 70 (1874), pp. 275–306.

[6] L. Boltzmann, *Zur Theorie der elastischen Nachwirkung*, Wied. Ann., 5 (1878), pp. 430–432.

[7] A. Borichev and Y. Tomilov, *Optimal polynomial decay of functions and operator semigroups*, Math. Ann., 347 (2010), pp. 455–478.

[8] B. M. R. Calsavara, E. H. Gomes Tavares, and M. A. Jorge Silva, *Exponential stability for a thermo-viscoelastic Timoshenko system with fading memory*, J. Math. Anal. Appl., 512 (2022), 126147.

[9] V. V. Chepyzhov and V. Pata, *Some remarks on stability of semigroups arising from linear viscoelasticity*, Asymptot. Anal., 46 (2006), pp. 251–273.

[10] M. Conti, F. Dell'Oro, and V. Pata, *Some unexplored questions arising in linear viscoelasticity*, J. Funct. Anal., 282 (2022), 109422.

[11] M. Conti, F. Dell'Oro, and V. Pata, *Timoshenko systems with fading memory*, Dyn. Partial Differ. Equ., 10 (2013), pp. 367–377.

[12] M. Conti, S. Gatti, and V. Pata, *Uniform decay properties of linear Volterra integro-differential equations*, Math. Models Methods. Appl. Sci., 18 (2008), pp. 21–45.

[13] M. Conti and V. Pata, *General decay properties of abstract linear viscoelasticity*, Z. Angew. Math. Phys., 71 (2020), 6.

[14] C. M. Dafermos, *An abstract Volterra equation with applications to linear viscoelasticity*, J. Differential Equations, 7 (1970), pp. 554–569.

[15] C. M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Ration. Mech. Anal., 37 (1970), pp. 297–308.

[16] F. Dell'Oro, *On the stability of Bresse and Timoshenko systems with hyperbolic heat conduction*, J. Differential Equations, 281 (2021), pp. 148–198.

[17] F. Dell'Oro, E. Laeng, and V. Pata, *A quantitative Riemann-Lebesgue lemma with application to equations with memory*, Proc. Amer. Math. Soc., 145 (2017), pp. 2909–2915.

[18] F. Dell'Oro and V. Pata, *Lack of exponential stability in Timoshenko systems with flat memory kernels*, Appl. Math. Optim., 71 (2015), pp. 79–93.

[19] A. D. Drozdov and V. B. Kolmanovskii, *Stability in Viscoelasticity*, North-Holland, Amsterdam, 1994.

[20] M. Fabrizio, C. Giorgi, and V. Pata, *A new approach to equations with memory*, Arch. Ration. Mech. Anal., 198 (2010), pp. 189–232.

[21] H. D. Fernández Sare and J. E. Muñoz Rivera, *Stability of Timoshenko systems with past history*, J. Math. Anal. Appl., 339 (2008), pp. 482–502.

[22] G. B. Folland, *Real Analysis: Modern Techniques and their Applications*, Wiley, New York, 1999.

[23] S. Gatti, A. Miranville, V. Pata, and S. Zelik, *Attractors for semi-linear equations of viscoelasticity with very low dissipation*, Rocky Mount. J. Math., 38 (2008), pp. 1117–1138.

[24] L. Gearhart, *Spectral theory for contraction semigroups on Hilbert spaces*, Trans. Amer. Math. Soc., 236 (1978), pp. 385–394.

[25] M. Grasselli and V. Pata, *Uniform attractors of nonautonomous dynamical systems with memory*, Progr. Nonlinear Differential Equations Appl., 50 (2002), pp. 155–178.

[26] M. Grasselli, V. Pata, and G. Prouse, *Longtime behavior of a viscoelastic Timoshenko beam*, Discrete Contin. Dyn. Syst., 10 (2004), pp. 337–348.

[27] A. Guesmia and S. A. Messaoudi, *A general stability result in a Timoshenko system with infinite memory: A new approach*, Math. Methods Appl. Sci., 37 (2014), pp. 384–392.

[28] A. Guesmia and S. A. Messaoudi, *On the control of solutions of a viscoelastic equation*, Appl. Math. Comput., 206 (2008), pp. 589–597.

[29] A. Guesmia and S. A. Messaoudi, *On the stabilization of Timoshenko system with memory and different speed of wave propagation*, Appl. Math. Comput., 219 (2013), pp. 9424–9437.

[30] J. H. Hassan, S. A. Messaoudi, and M. Zahri, *Existence and new general decay results for a viscoelastic Timoshenko system*, Z. Anal. Anwend., 39 (2020), pp. 185–222.

[31] M. A. Jorge Silva and R. Racke, *Effects of history and heat models on the stability of thermoelastic Timoshenko systems*, J. Differential Equations, 275 (2021), pp. 167–203.

[32] Y. Liu and S. Kawashima, *Decay property for the Timoshenko system with memory-type dissipation*, Math. Models Methods Appl. Sci., 22 (2012), 1150012.

[33] Z. Liu and S. Zheng, *Semigroups Associated with Dissipative Systems*, Chapman & Hall/CRC Res. Notes in Math. 398, CRC Press/Chapman & Hall, Boca Raton, FL, 1999.

[34] S. A. Messaoudi and B. Said-Houari, *Uniform decay in a Timoshenko-type system with past history*, J. Math. Anal. Appl., 360 (2009), pp. 459–475.

[35] S. A. Messaoudi and M. I. Mustafa, *A general stability result in a memory-type Timoshenko system*, Commun. Pure Appl. Anal., 12 (2013), pp. 957–972.

[36] S. A. Messaoudi and M. I. Mustafa, *A stability result in a memory-type Timoshenko system*, Dynam. Systems Appl., 18 (2009), pp. 457–468.

[37] N. Mori, *Dissipative structure and global existence in critical space for Timoshenko system of memory type*, J. Differential Equations, 265 (2018), pp. 1627–1653.

[38] V. Pata, *Exponential stability in linear viscoelasticity*, Quart. Appl. Math., 64 (2006), pp. 499–513.

[39] V. Pata and A. Zucchi, *Attractors for a damped hyperbolic equation with linear memory*, Adv. Math. Sci. Appl., 11 (2001), pp. 505–529.

[40] J. Prüss, *Evolutionary Integral Equations and Applications*, Monogr. Math. 87, Birkhäuser, Basel, 1993.

[41] J. Prüss, *On the spectrum of C_0-semigroups*, Trans. Amer. Math. Soc., 284 (1984), pp. 847–857.

[42] N.-e. Tatar, *Viscoelastic Timoshenko beams with occasionally constant relaxation functions*, Appl. Math. Optim., 66 (2012), pp. 123–145.

[43] S. P. Timoshenko, *On the correction for shear of the differential equation for transverse vibrations of prismatic bars*, Philos. Mag., 41 (1921), pp. 744–746.

[44] S. P. Timoshenko, *Vibration Problems in Engineering*, Van Nostrand, New York, 1955.

[45] V. Volterra, *Sur les équations intégro-différentielles et leurs applications*, Acta Math., 35 (1912), pp. 295–356.

[46] V. Volterra, *Leçons sur les fonctions de lignes*, Gauthier-Villars, Paris, 1913.