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ORIGINAL PAPER

Dynamics of a Thermoelastic Balakrishnan–Taylor Beam Model with Fractional Operators

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Abstract
This paper contains new results about the well-posedness and the asymptotic dynamics of solutions for a general abstract coupled system that arises in connection with thermoelastic Balakrishnan–Taylor beam models with fractional operators.

Keywords Attractors · Balakrishnan–Taylor · Rotational inertia · Thermoelastic beam

Mathematics Subject Classification 35B40 · 35B41 · 37L30 · 35L75 · 74H40 · 74K99

1 Introduction

1.1 The Model

Let Ω be a bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary $\Gamma = \partial\Omega$. In $\Omega \times \mathbb{R}^+$ we consider the general thermoelastic beam/plate equation with fractional operators associated with rotational inertia and coupling terms

$$\begin{cases} \left[I + (-\Delta)^\beta \right] u_{tt} + \Delta^2 u - M(u, u_t)\Delta u + f(u) - (-\Delta)^\alpha \theta = h, \\ \kappa\, \theta_t - \Delta\theta + (-\Delta)^\alpha u_t = 0, \end{cases} \tag{1.1}$$

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where $(\alpha, \beta, \kappa) \in [\frac{1}{2}, 1] \times [0, 2\alpha - 1] \times [0, 1]$, M is a nonlocal term given by

$$M(u, u_t) := \zeta_0 + \zeta_1 \int_\Omega |\nabla u|^2 dx + \zeta_2 \left| \int_\Omega \nabla u \cdot \nabla u_t dx \right|^{q-2} \int_\Omega \nabla u \cdot \nabla u_t dx,$$

with $\zeta_0 \in \mathbb{R}$, $\zeta_1, \zeta_2 > 0$, $q \geq 2$, f is a nonlinear function that represents an additional source term, and h is an external force in $L^2(\Omega)$. The model (1.1) is a coupled system of semilinear beam/plate and linear heat equations, the parameter $\kappa > 0$ is related to heat/thermal capacity, and $M(u, u_t)$ is the nonlocal term coming from the Balakrishnan–Taylor works [2, 3]. Here, we study (1.1) with hinged boundary condition on the displacement u and Dirichlet boundary condition on the temperature θ given, respectively, by

$$u = \Delta u = 0 \quad \text{and} \quad \theta = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+. \tag{1.2}$$

The initial conditions are given by

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in \Omega. \tag{1.3}$$

As we are going to clarify below and summarize in Table 1, the IBVP (1.1)–(1.3) corresponds to a generalized mathematical model that encompasses various models from the existing literature. Indeed, for particular choices of the parameters α, β, and κ, we can find models with a robust physical foundation. Such particular models motivated us to consider the thermal version (1.1) with fractional terms in a generalized framework. Additionally, the physical interpretation of these fractional operators on rotational inertial and coupling terms may not have a direct physical counterpart, but it offers a way to capture the complex mathematical dynamics of the system, as given in Sects. 2 and 3.

Let us first contextualize some special cases in elasticity and thermoelasticity related to beam models and then highlight the novelty of the present paper.

1.2 Case $\kappa = 0$: Elastic Beam/Plate Models

In the limit case $\kappa = 0$, the Eq. (1.1) can be decoupled. Indeed, if $\kappa = 0$, from $(1.1)_2$ we have $\Delta\theta = (-\Delta)^\alpha u_t$ which implies that $(-\Delta)^\alpha \theta = -(-\Delta)^{2\alpha-1} u_t$. Then, substituting in Eq. $(1.1)_1$ the term $(-\Delta)^\alpha \theta$ obtained in Eq. $(1.1)_2$ we obtain a beam/plate model with fractional structural damping $(-\Delta)^{2\alpha-1} u_t$. The model is then described by

$$\left[I + (-\Delta)^\beta \right] u_{tt} + \Delta^2 u - M(u, u_t)\Delta u + f(u) + (-\Delta)^{2\alpha-1} u_t = h. \tag{1.4}$$

Equation (1.4) is a generalized n-dimensional version of a model for flight structures with viscous and nonlinear nonlocal damping proposed by Balakrishnan–Taylor [2, 3]. Indeed, in the one-dimensional case, it can be seen as follows when $\beta = 0$, $\alpha = 1$,

$f(u) \equiv 0$ and $h \equiv 0$:

$$u_{tt} + \frac{EI}{\varrho} u_{xxxx} - \frac{c}{\varrho} u_{xxt} - \left[\frac{H}{\varrho} + \frac{EA}{2L\varrho} \int_0^L |u_x|^2 dx + \frac{\tau}{\varrho} \left(\int_0^L u_x u_{xt} \, dx \right)^{2(N+\eta)+1} \right] u_{xx} = 0, \tag{1.5}$$

where $u = u(x, t)$ represents the transversal deflection of an extensible beam with length $2L > 0$ in the rest position, $\varrho > 0$ is the mass density, E is the Young's modulus of elasticity, I is the cross-sectional moment of inertia, H is the axial force (either traction or compression), A is the cross-sectional area, $c > 0$ is the coefficient of viscous damping, $\tau > 0$ is the Balakrishnan–Taylor damping coefficient, $0 \le \eta < \frac{1}{2}$ and $N \in \mathbb{N}$. We refer to [3, Sect. 4] for the precise modeling of (1.5). See also Bass and Zes [6, Eqs. (14a)–(14c)]. The Balakrishnan–Taylor model (1.5) has been studied well in recent years in its well-posedness and dynamics. For researches on the related Balakrishnan–Taylor model (1.5), one can refer to literatures [14, 18, 22, 37, 38] and references therein. It is important to note that, the Balakrishnan–Taylor damping $-\frac{\tau}{\varrho} \left(\int_0^L u_x u_{xt} \, dx \right)^{2(N+\eta)+1} u_{xx}$ alone ($c \equiv 0$) is not enough to produce the desired results on stability nor long-time behavior (see [37, Appendix A therein]). When $\tau \equiv 0$ the model is connected to the extensible beam equation of Woinowsky–Krieger [35], which was extensively studied by many authors. See for instance [4, 5, 8, 9, 11, 15, 17, 25–27, 30, 36] and references therein.

When the exponent $\beta = 1$ in (1.4), the rotational inertial momenta of the elements of the beam is taken into account. This case is associated with von Karmam models that represents a purely hyperbolic dynamics with finite speed of propagation. This type of models were well treated in its various aspects by Chueshov and Lasiecka in [12, 13]. In [12, Eq. (7.11)] they proposed the following extensible beam model within the context of rotational inertia term

$$(1 - \omega\Delta) u_{tt} + \Delta^2 u - \omega \Delta u_t + \left(Q - \int_\Omega |\nabla u|^2 dx \right) \Delta u = p(u, u_t), \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.6}$$

where the coefficient $\omega \ge 0$ represents rotational forces, the parameter Q describes in-plane forces applied to the plate and the function p represents transverse loads which may depend on the displacement u and velocity u_t. They studied the well-posedness and longtime dynamics of solutions. Howell–Toundykov–Webster [24] studied Eq. (1.6) with clamped-free boundary condition in the context of the piston-theory. They addressed both the case with rotational inertia ($\omega > 0$) and the case without one ($\omega \equiv 0$). The well-posedness and long-time dynamics of solutions were studied. Equation (1.6) with rotational effect also has been well considered recently in the context of nonlocal damping [16, 31, 33] and references therein.

1.3 Case $\kappa > 0$: Thermoelastic Beam/Plate Models

In what follows, we also mention some important works on thermoelastic beam/plate equations that inspired the generalized model (1.1)–(1.3). First, we refer the work by Chueshov–Lasiecka [10] where the following thermoelastic von Karman plate subjected to an external and internal forcing is considered

$$\begin{cases} (1-\gamma\Delta)u_{tt} + \Delta^2 u + \beta\Delta\theta = B(u), \\ \kappa\theta_t - \eta\Delta\theta - \nu\Delta u_t = 0, \end{cases} \tag{1.7}$$

with $B(u) = [u, v + F_0] + p(x)$, where the Airy's stress function $v = v(u)$ is a solution to the problem

$$\Delta^2 v + [u, u] = 0, \quad v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Gamma,$$

and the von Karman bracket $[u, v]$ is given by

$$[u, v] = \partial^2_{x_1} u \cdot \partial^2_{x_2} v + \partial^2_{x_2} u \cdot \partial^2_{x_1} v - 2 \cdot \partial^2_{x_1 x_2} u \cdot \partial^2_{x_1 x_2} v,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with the boundary $\partial\Omega = \Gamma$, the function F_0 describes in-plane forces acting on the plate and the function p represents external transverse forces applied to the plate, the parameters β and ν are positive, and γ and κ are non-negative. They proved that the ultimate (asymptotic) behavior of the problem (1.7) is described by finite dimensional global attractor and that the obtained estimate for the dimension and the size of the attractor are independent of the rotational inertia parameter γ and heat/thermal capacity κ. Smoothness and upper-semicontinuity with respect to parameters γ and κ are established.

The thermoelastic version of an extensible beam associated with problem (1.7) was treated by Giorgi-Naso [20], where they combined the pioneering ideas of Woinowsky–Krieger [35] with the theory of linear thermoelasticity [7]. Then, they deduced a nonlinear mathematical model for the thermoelastic extensible beam of unitary natural length with hinged ends assuming the Fourier heat conduction law

$$\begin{cases} (1-\partial_{xx})\partial_{tt}u + \partial_{xxxx}u + \partial_{xx}\theta - \left[m + \displaystyle\int_0^1 |\partial_x u|^2 dx\right]\partial_{xx}u = f, \\ \partial_t\theta - \partial_{xx}\theta - \partial_{xxt}u = g, \end{cases} \tag{1.8}$$

where $m \in \mathbb{R}$ and f, g are external forces. This model was then studied by Giorgi et al. in [21]. The authors proved that when the external sources are time-independent, the related semigroup of solutions possess a global attractor of optimal regularity for all parameters $m \in \mathbb{R}$.

Another motivating work has been considered by Fernándes Sare et al. [19], where the stability of the abstract thermoelastic system with fractional powers associated with inertial and couplings terms is approached

$$\begin{cases} (\rho + \mu A^{\gamma})u_{tt} + \sigma A u - m A^{\beta}\theta = 0, \\ c\theta_t + k A^{\alpha}\theta + m A^{\beta} u_t = 0, \end{cases} \tag{1.9}$$

where $\rho, \sigma, c, k, \mu > 0$, $m \neq 0$, $(\beta, \alpha, \gamma) \in [0, 1] \times [0, 1] \times (0, 1]$ and A a self-adjoint, positive definite operator on the Hilbert space. The authors prove that when the parameters β, α, γ vary in the region $S = \{(\beta, \alpha, \gamma) : \frac{1}{2} \leq \beta + \frac{\alpha}{2}, \frac{\gamma}{2} \leq \beta - \frac{\alpha}{2} \leq \frac{1}{2}\}$, the dynamic system associated with the abstract problem (1.9) is exponentially stable, and the region of non-exponential stability for the Fourier model (1.9) is given by the complement of the set S. The polynomial stability in regions of non-exponential stability is also characterized. More recently, Kuang-Liu-Fernándes Sare [28] extended the results of [19], since they decompose the region $E = [0, \frac{\alpha+1}{2}] \times [0, 1] \times [0, 1]$ into three parts where the corresponding semigroups are analytic, of *Gevrey* classes of specific order, and non-smoothing, respectively. Note that the particular case $\beta = \alpha = \gamma = \frac{1}{2}$ is the same context as (1.7) with $A^{1/2}u = -\Delta u$, $D(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega)$, $H = L^2(\Omega)$ and $B(u) \equiv 0$.

Summarizing, one can see from [19, 28] that the fractional powers present in the abstract system (1.9) are capable of showing the complexity and different asymptotic behaviors for the solutions when varying the parameters α, β, and κ; while in [10, 20, 21] it is analyzed the nonlinear dynamics of concrete thermoelastic systems such as (1.7) and (1.8) where the parameters α, β, κ are taken inside the region of uniform stability when looking at the linear problem. It is worth pointing out that problem (1.8) can be seen as a particular case of (1.1) when neglecting the Balakrishnan–Taylor terms given in $M(u, u_t)$. Motivated by these scenarios, our main goal is to explore the dynamics of (1.1), which captures the region of stability given by the parameters α, β, and κ, as well as it keeps the non-locality of the whole Balakrishnan–Taylor term $M(u, u_t)$. Below we present all scenarios approached herein and the novelty of the paper.

1.4 Main Results and Plan of the Paper

The main purpose in the present paper is to study the well-posedness and asymptotic behavior of solutions for (1.1)–(1.3), with $(\alpha, \beta, \kappa) \in \Upsilon$ where

$$\Upsilon := \left[\frac{1}{2}, 1\right] \times [0, 2\alpha - 1] \times [0, 1] .$$

More specifically, we prove the well-posedness of problem (1.1)–(1.3) when $(\alpha, \beta, \kappa) \in \Upsilon$, and if $\alpha > \frac{1}{2}$, we prove the dynamical system $S_{\alpha,\beta,\kappa}$ generated by (1.1)–(1.3) has a compact finite-dimensional global attractor $\{\mathfrak{A}_{\alpha,\beta,\kappa}\}_{(\alpha,\beta,\kappa)\in(\frac{1}{2},1]\times[0,2\alpha-1]\times}$ $[0, 1]$. Furthermore, we prove the upper-semicontinuity of the family of attractors $\{\mathfrak{A}_{\alpha,\beta,\kappa}\}$ with respect to parameters α, β and γ. To our best knowledge, such concepts are not considered for (1.1)–(1.3) in its whole form addressing the Balakrishnan–Taylor framework and the fractional operators associated with rotational inertia and

Table 1 Types of models in relation to parameters $(\alpha, \beta, \kappa) \in \Upsilon$

	$(\alpha = \frac{1}{2}, \beta = 0)$	$\left(\frac{1}{2} < \alpha < 1, 0 \leq \beta \leq 2\alpha - 1 < 1\right)$	$(\alpha = 1, \beta = 1)$
$\kappa = 0$	Balakrishnan–Taylor extensible beam with weak damping	Balakrishnan–Taylor extensible beam with fractional rotational forces ($\beta > 0$) or without rotational forces ($\beta = 0$) and fractional damping	Balakrishnan–Taylor extensible beam with rotational forces and strong damping
$\kappa > 0$	Balakrishnan–Taylor beam equations with thermal effects	Balakrishnan–Taylor extensible beam with thermal effects, fractional rotational forces ($\beta > 0$) or without rotational forces ($\beta = 0$)	Balakrishnan–Taylor extensible beam with thermal effects and rotational forces

coupling terms. Such generalized approach allows us to encompass several models with respect to the powers α, β, and κ. This is clarified in Table 1 as follows.

The remaining paper is organized as follows. In Sect. 2 we fix some notations, present our assumptions, and prove the well-posedness of problem (1.1)–(1.3). In Sect. 3, we show the existence of absorbing set in proper Hilbert spaces $\mathcal{H}_{\beta,\kappa}$. In particular, if $h \equiv 0$ and $f \in C^1(\mathbb{R})$ is such that $f(s)s \geq -\omega s^2$ with $\omega > 0$ sufficiently small, the energy associated with the system is exponentially stable for $(\alpha, \beta, \kappa) \in \Upsilon$. In addition, taking $\alpha > \frac{1}{2}$, we prove that the nonlinear corresponding semigroup $S_{\alpha,\beta,\kappa}$ is quasi-stable and so, in view of the abstract results established in [12, 13], we prove that the dynamical system has a compact global attractor $\{\mathfrak{A}_{\alpha,\beta,\kappa}\}_{(\alpha,\beta,\kappa)\in(\frac{1}{2},1]\times[0,2\alpha-1]\times[0,1]}$. Finally, we prove that the family of global attractors $\{\mathfrak{A}_{\alpha,\beta,\kappa}\}$ is upper-semicontinuous with respect to parameters α, β an κ.

2 Well-Posedness

2.1 Notations and Functional Setting

In what follows, we consider the following function spaces which shall be used throughout this paper. Let $W_0 = L^2(\Omega)$, $W_1 = H_0^1(\Omega)$ and $W_2 = H^2(\Omega) \cap H_0^1(\Omega)$. In the space W_0 we define the biharmonic operator $A = \Delta^2$ defined by

$$(Au, v) = (\Delta u, \Delta v), \quad u, v \in W_2,$$

where $(\cdot, \cdot)$ stands for the inner product of W_0, $\|\cdot\|$ stands for the norm of W_0 and $\|\cdot\|_p$ stands for the norm of $L^p(\Omega)$. Then it follows that the domain of A, $D(A) = \{u \in H^4(\Omega); u = \Delta u = 0 \text{ on } \Gamma\}$ and there exists a complete orthonormal family of W_0, $\{w_j\}_{j\in\mathbb{N}}$, made of eigenvectors of A,

$$Aw_j = \lambda_{jw_j}, \quad j = 1, \ldots, \quad \text{and} \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \lambda_j \to +\infty \quad \text{as} \quad j \to +\infty.$$

We can also define the power A^s of A, for $s \in \mathbb{R}$. The spaces $W_s = D(A^{s/4})$, $s \in \mathbb{R}$, are Hilbert spaces with the scalar products and the norms

$$(u, v)_{W_s} = (A^{s/4}u, A^{s/4}v), \quad \|u\|_{D(A^s)} = \|A^s u\|.$$

In particular, we have $W_0 = D(A^0) = L^2(\Omega)$, $W_1 = D(A^{1/4}) = H_0^1(\Omega)$, $W_2 = D(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega)$ and $D(A^{s_1}) \subset D(A^{s_2})$, $\forall s_1, s_2 \in \mathbb{R}$, $s_1 \geq s_2$, where each space is dense in the following one and the injection is compact. We define $\mathcal{M}_\beta \equiv I + A^{\beta/2}$. We have that $\mathcal{M}_\beta$ also is a positive self-adjoint operator in W_0 and

$$D(\mathcal{M}_\beta^{1/2}) = \begin{cases} W_\beta, & \text{for } \beta > 0, \\ W_0, & \text{for } \beta = 0, \end{cases}$$

with norm

$$\|v\|^2_{D(\mathcal{M}_\beta^{1/2})} = \|v\|^2 + \|A^{\beta/4}v\|^2.$$

For every $(\beta, \kappa) \in [0, 1] \times [0, 1]$ we introduce the Hilbert space

$$\mathcal{H}_{\beta,\kappa} = \begin{cases} W_2 \times D(\mathcal{M}_\beta^{1/2}) \times W_0, & \text{for } \kappa > 0, \\ W_2 \times D(\mathcal{M}_\beta^{1/2}), & \text{for } \kappa = 0, \end{cases} \tag{2.10}$$

with norm

$$||U||^2_{\mathcal{H}_{\beta,\kappa}} = \|A^{1/2}u\|^2 + \|u_t\|^2_{D(\mathcal{M}_\beta^{1/2})} + \kappa\|\theta\|^2, \quad U = (u, u_t, \theta).$$

The energy functional for the model (1.1)–(1.3) is given by

$$\begin{aligned} E^{\beta,\kappa}(U) = \frac{1}{2}\Big[||U(t)||^2_{\mathcal{H}_{\beta,\kappa}} + \zeta_0\|A^{1/4}u(t)\|^2 + \frac{\zeta_1}{2}\|A^{1/4}u(t)\|^4 \Big] \\ + \big(\widehat{f}(u), 1\big) - (h, u), \end{aligned} \tag{2.11}$$

where $\widehat{f}(s) = \int_0^s f(\tau)d\tau$.

Using that $(A^{1/4}u, A^{1/4}u_t) = (A^{1/2}u, u_t)$ and inserted in the framework introduced above, the initial-boundary value problem (1.1)–(1.3) can be put in the form

$$\begin{cases} \mathcal{M}_\beta u_{tt} + Au - A^{\alpha/2}\theta = \mathcal{F}(u, u_t, A^{1/4}u, A^{1/2}u), \\ \kappa\theta_t + A^{1/2}\theta + A^{\alpha/2}u_t = 0. \\ u(0) = u_0, \quad u_t(0) = u_1 \quad \text{and} \quad \theta(0) = \theta_0. \end{cases} \tag{2.12}$$

where

$$\mathcal{F}(u, u_t, A^{1/4}u, A^{1/2}u) = h - \Phi(\|A^{1/4}u\|^2)A^{1/2}u - \Psi(A^{1/2}u, u_t)A^{1/2}u - f(u),$$

with Φ and Ψ given by

$$\Phi(s) = \zeta_0 + \zeta_1 s \quad \text{and} \quad \Psi(s) = \zeta_2 |s|^{q-2} s.$$

In the limit case when $\kappa = 0$, the problem (2.12) is written as follows

$$\begin{cases} \mathcal{M}_\beta u_{tt} + Au + A^{\alpha-1/2} u_t = \mathcal{F}(u, u_t, A^{1/4}u, A^{1/2}u) \\ u(0) = u_0, \quad u_t(0) = u_1. \end{cases} \tag{2.13}$$

2.2 Assumptions

Let λ_1 be the first eigenvalue of A, then

$$\|A^{s_2}u\| \le \lambda_1^{s_2-s_1} \|A^{s_1}u\|, \quad \text{for all } s_1 \ge s_2 \in \mathbb{R} \text{ and } u \in D(A^{s_1}). \tag{2.14}$$

Now we will establish assumptions on the functions f, M and h necessary to analyze the well-posedness of the problem (2.12). Let $K_{f'} > 0$, $K_f \ge 0$ and $\omega \in [0, \lambda_1)$. We assume that f is C^1-function on $\mathbb{R}$ with with polynomial growth such that

$$\begin{cases} |f'(s)| \le K_{f'}(1 + |s|^\rho), \quad \forall\, s \in \mathbb{R}, & (2.15) \\ -K_f - \dfrac{\omega}{4}s^2 \le \widehat{f}(s) \le f(s)s + \dfrac{\omega}{4}s^2, \quad \forall\, s \in \mathbb{R}, & (2.16) \end{cases}$$

where

$$\rho > 0 \text{ if } 1 \le n \le 4 \quad \text{or} \quad 0 < \rho \le \frac{4 + 2\beta}{n-4} \text{ if } n \ge 5.$$

The external force $h \in L^2(\Omega)$.

2.3 Global Solution

Setting

$$U = \begin{cases} (u, u_t, \theta)^\top & \text{for} \quad \kappa > 0, \\ (u, u_t)^\top & \text{for} \quad \kappa = 0. \end{cases} \tag{2.17}$$

We can rewrite the problem (2.12) as the first-order equation

$$\frac{d}{dt}U(t) = \mathcal{L}_{\alpha,\beta,\kappa} U(t) + \mathcal{F}_\kappa(U(t)), \quad U(0) = U_0 \in \mathcal{H}_{\beta,\kappa} \tag{2.18}$$

where $\mathcal{L}_{\alpha,\beta,\kappa} : D(\mathcal{L}_{\alpha,\beta,\kappa}) \subset \mathcal{H}_{\beta,\kappa} \to \mathcal{H}_{\beta,\kappa}$ is a linear operator defined by

$$\underbrace{\mathcal{L}_{\alpha,\beta,\kappa}}_{\kappa>0} = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{M}_\beta^{-1}A & 0 & \mathcal{M}_\beta^{-1}A^{\alpha/2} \\ 0 & -\frac{1}{\kappa}A^{\alpha/2} & -\frac{1}{\kappa}A^{1/2} \end{bmatrix}$$

or

$$\underbrace{\mathcal{L}_{\alpha,\beta,0}}_{\kappa=0} = \begin{bmatrix} 0 & I \\ -\mathcal{M}_\beta^{-1}A & -\mathcal{M}_\beta^{-1}A^{\alpha-1/2} \end{bmatrix}$$

with domain

$$\underbrace{\mathcal{D}(\mathcal{L}_{\alpha,\beta,\kappa})}_{\kappa>0} = \left\{ \begin{bmatrix} u \\ u_t \\ \theta \end{bmatrix} \in \mathcal{H}_{\beta,\kappa} \,\middle|\, \begin{array}{l} u_t \in W_2,\ \theta \in W_1 \\ \mathcal{M}_\beta^{-1}\left[-Au + A^{\alpha/2}\theta\right] \in \mathcal{D}(\mathcal{M}_\beta^{1/2}) \\ -\frac{1}{\kappa}A^{\alpha/2}u_t - \frac{1}{\kappa}A^{1/2}\theta \in W_0 \end{array} \right\} \quad \text{or}$$

$$\underbrace{\mathcal{D}(\mathcal{L}_{\alpha,\beta,0})}_{\kappa=0} = \left\{ \begin{bmatrix} u \\ u_t \end{bmatrix} \in \mathcal{H}_{\beta,0} \,\middle|\, \begin{array}{l} u_t \in W_2,\ \theta \in W_1 \\ \mathcal{M}_\beta^{-1}\left[-Au - A^{\alpha/2-1}u_t\right] \in \mathcal{D}(\mathcal{M}_\beta^{1/2}) \end{array} \right\}$$

and $\mathcal{F}_\kappa : \mathcal{H}_{\beta,\kappa} \to \mathcal{H}_{\beta,\kappa}$ is a nonlinear operator given by

$$\mathcal{F}_\kappa U = \begin{cases} (0, F(U), 0)^\perp & \text{if} \quad \kappa > 0 \\ (0, F(U))^\perp & \text{if} \quad \kappa = 0 \end{cases}, \tag{2.19}$$

where

$$F(U) = \mathcal{M}_\beta^{-1}\left[h - \Phi(\|A^{1/4}u\|^2)A^{1/2}u - \Psi(A^{1/2}u, u_t)A^{1/2}u - f(u)\right]. \tag{2.20}$$

Remark 2.1 We recall that a function $U : [0, t] \to \mathcal{H}_{\beta,\kappa}$ is a *classical solution* of (2.18) on $[0, T)$ if U is continuous on $[0, T)$, continuously differentiable on $(0, T)$, $U(t) \in \mathcal{D}(\mathcal{L}_{\alpha,\beta,\kappa})$ for $t \in (0, T)$ and (2.18) is satisfied on $[0, T)$. A function U is a *mild solution* of (2.18) on $[0, T]$ if $U \in C([0, T]; \mathcal{H}_{\beta,\kappa})$ and satisfies

$$U(t) = e^{\mathcal{L}_{\alpha,\beta,\kappa}t}U_0 + \int_0^t e^{\mathcal{L}_{\alpha,\beta,\kappa}(t-\tau)}\mathcal{F}_\kappa(U(\tau))d\tau, \tag{2.21}$$

where $e^{\mathcal{L}_{\alpha,\beta,\kappa}t}$ is the linear semigroup on $\mathcal{H}_{\beta,\kappa}$ whose infinitesimal operator is $\mathcal{L}_{\alpha,\beta,\kappa}$.

The well-posedness of the problem (2.12) is stated as follows.

Theorem 2.1 (Well-posedness) *We assume Assumptions* 2.1 *with* $q \geq 2$ *and*

$$(\alpha, \beta, \kappa) \in \Upsilon = \left[\frac{1}{2}, 1\right] \times [0, 2\alpha - 1] \times [0, 1].$$

Then for every $T > 0$,

(i) *Case $\kappa > 0$: for all initial data $U_0 = (u_0, u_1, \theta_0)^{\perp} \in \mathcal{H}_{\beta,\kappa}$ problem (2.18) possesses a unique mild solution $U(t) \equiv (u(t), u_t(t), \theta(t))^{\perp} \in C([0, T], \mathcal{H}_{\beta,\kappa})$ which depends continuously on the initial data. And if $U_0 \in \mathcal{D}(\mathcal{L}_{\alpha,\beta,\kappa})$ then the corresponding mild solution $U(t)$ is a classical solution.*

(ii) *Case $\kappa = 0$: for every $U_0 = (u_0, u_1)^{\perp} \in \mathcal{H}_{\beta,0}$ problem (2.18) possesses a unique mild solution $U(t) \equiv (u(t), u_t(t))^{\perp} \in C([0, T], \mathcal{H}_{\beta,0})$ which depends continuously on the initial data. And if $U_0 \in \mathcal{D}(\mathcal{L}_{\alpha,\beta,0})$ then the corresponding mild solution $U(t)$ is a classical solution.*

Proof **(i)** The proof is based on four steps stated bellow. In Step I we will show that the operator $\mathcal{L}_{\alpha,\beta,\kappa}$ is a infinitesimal generator of a C_0 - semigroup of contractions on $\mathcal{H}_{\beta,\kappa}$. This is proved by showing that $\mathcal{L}_{\alpha,\beta,\kappa}$ is dissipative and maximal, and application of the Lumer-Phillips Theorem ([32], Theorem 1.4.3). In Step II we show that the operator $\mathcal{F} : \mathcal{H}_{\beta,\kappa} \to \mathcal{H}_{\beta,\kappa}$ is locally lipschtz. Steps I and II guarantee the existence of local solution ([32, Theorem 6.1.4]). The existence of global solution is guaranteed in Step III.

2.3.1 Step I

It is easy to see that

$$\begin{aligned} \left\langle \mathcal{L}_{\alpha,\beta,\kappa} U(t), U(t) \right\rangle_{\mathcal{H}_{\beta,\kappa}} &= \left\langle \begin{bmatrix} u_t \\ \mathcal{M}_\beta^{-1}\left(-Au + A^{\alpha/2}\theta\right) \\ -\frac{1}{\kappa} A^{\alpha/2} u_t - \frac{1}{\kappa} A^{1/2}\theta \end{bmatrix}, \begin{bmatrix} u \\ u_t \\ \theta \end{bmatrix} \right\rangle \\ &= -\,\|A^{1/4}\theta(t)\|^2 \le 0, \end{aligned} \tag{2.22}$$

for all $U(t) \in \mathcal{D}(\mathcal{L}_{\alpha,\beta,\kappa})$ which shows that $\mathcal{L}_{\alpha,\beta,\kappa}$ is dissipative. To prove maximality of $\mathcal{L}_{\alpha,\beta,\kappa}$ we need to show that $I - \mathcal{L}_{\alpha,\beta,\kappa} : \mathcal{D}(\mathcal{L}_{\alpha,\beta,\kappa}) \to \mathcal{H}_{\beta,\kappa}$ is onto. Indeed, let $U^* = (u^*, w^*, \theta^*) \in \mathcal{H}_{\beta,\kappa}$ and consider the equation

$$(I - \mathcal{L}_{\alpha,\beta,\kappa})U = U^*$$

which can be written as follows

$$\begin{cases} u - u_t & = u* \in W_2, \\ \mathcal{M}_\beta u_t + Au - A^{\alpha/2}\theta & = \mathcal{M}_\beta w^* \in \mathcal{D}(\mathcal{M}_\beta^{1/2}), \\ A^{\alpha/2}u_t + \kappa\theta + A^{1/2}\theta & = \kappa\theta^* \in W_0. \end{cases}$$

Substituting $u = u_t + u^*$ in the second equations of (2.23), we obtain

$$\begin{cases} \mathcal{M}_\beta u_t + Au_t - A^{\alpha/2}\theta = \mathcal{M}_\beta w^* - Au^* := u^{**} \in W_2', \\ A^{\alpha/2}u_t + \kappa\theta + A^{1/2}\theta \;= \kappa\theta^* := w^{**} \in W_1'. \end{cases} \tag{2.23}$$

To solve the elliptic problem (2.23) we apply Lax Millgram Theorem with the bilinear form $a : [W_2 \times W_1] \times [W_2 \times W_1] \to \mathbb{R}$ given by

$$a((u_t,\theta),(z,w)) = (\mathcal{M}_\beta^{1/2}u_t, \mathcal{M}_\beta^{1/2}z) + (A^{1/2}u_t, A^{1/2}z) - (A^{\alpha/4}\theta, A^{\alpha/4}z) \\ + \ (A^{\alpha/4}u_t, A^{\alpha/4}w) + \kappa(\theta, w) + (A^{1/4}\theta, A^{1/4}w).$$

Using that $W_2 \hookrightarrow W_1 \hookrightarrow W_\alpha \hookrightarrow W_\beta \hookrightarrow W_0$ and inequality (2.14), it is easy to see that

$$\begin{aligned}|a((u_t,\theta),(z,w))| \le\ & \|u_t\|_{\mathcal{D}(\mathcal{M}_\beta^{1/2})}\|u_t\|_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} + \|A^{1/2}u_t\|\|A^{1/2}z\| + \|A^{\alpha/4}\theta\|\|A^{\alpha}z\| \\ & + \ \|A^{\alpha/4}u_t\|\|A^{\alpha}w\| + \kappa\|\theta\|\|w\| + \|A^{1/4}\theta\|\|A^{1/4}w\| \\ \le\ & C_{\lambda_1}\|(u,\theta)\|_{W_2\times W_1}\|(z,w)\|_{W_2\times W_1}.\end{aligned}$$

and it is straightforward that

$$\begin{aligned}a((u_t,\theta),(u,\theta)) &= \|u_t\|^2 + \|A^{\beta/4}u_t\|^2 + \|A^{1/2}u_t\|^2 + \kappa\|\theta\|^2 \\ & + \|A^{1/4}\theta\|^2 \ge \|(u_t,\theta)\|^2_{W_2\times W_1}.\end{aligned}$$

This shows that a is continuous and coercive. Thus, using Lax Millgram Theorem with $\psi := (u^{**}, \theta^{**}) \in [W_2 \times W_1]'$ there exists a unique $(u,\theta) \in W_2 \times W_1$ such that

$$a((u_t,\theta),(z,w)) = (\psi,(z,w)), \quad \forall\ (z,w) \in W_2 \times W_1.$$

This implies that $u_t \in W_2$ and $\theta \in W_1$ satisfy (2.23). Hence, the element $U = (u, u_t, \theta)$, where $u = u^* + u_t$ satisfies the system (2.23). Thus, obviously $U \in \mathcal{D}(\mathcal{L}_{\alpha,\beta,\kappa})$. This shows that $\mathcal{L}_{\alpha,\beta,\kappa}$ is a maximal operator and due to Lummer-Phillips Theorem $\mathcal{L}_{\alpha,\beta,\kappa}$ is a infinitesimal generator of a C_0 - semigroup of contractions on $\mathcal{H}_{\beta,\kappa}$.

2.3.2 Step II

The operator $\mathcal{F}_\kappa : \mathcal{H}_{\beta,\kappa} \to \mathcal{H}_{\beta,\kappa}$ given in (2.19) is locally Lipschitz. This follows from local Lipschitz property of (2.20). Indeed, let us first take $R > 0$ and $U = (u, u_t, \theta)$, $V = (v, v_t, \zeta)$ such that $||U||_{\mathcal{H}_{\beta,\kappa}}, ||V||_{\mathcal{H}_{\beta,\kappa}} \le R$. Denoting $w = u - v$ and using that

$$\Phi(s) = \zeta_0 + \zeta_1 s \quad \text{and} \quad \Psi(s) = \zeta_2|s|^{q-2}s,$$

from definition (2.20), we have

$$F(U) - F(V) = -\ \mathcal{M}_\beta^{-1}\left[\mathrm{S}_{(\Phi,\Psi)}A^{1/2}w + \mathrm{D}_{(\Phi,\Psi)}A^{1/2}(u+v) + \mathrm{D}_f\right],$$

where

$$
\begin{aligned}
\mathrm{S}_{(\Phi,\Psi)} &= \frac{1}{2}\Big[\,\Phi(\|A^{1/4}u\|^2)+\Phi(\|A^{1/4}u\|^2)\Big]+\frac{1}{2}\Big[\,\Psi(A^{1/2}u,u_t)+\Psi(A^{1/2}v,v_t)\Big],\\
\mathrm{D}_{(\Phi,\Psi)} &= \frac{1}{2}\Big[\,\Phi(\|A^{1/4}u\|^2)-\Phi(\|A^{1/4}v\|^2)\Big]+\frac{1}{2}\Big[\,\Psi(A^{1/2}u,u_t)-\Psi(A^{1/2}v,v_t)\Big],\\
\mathrm{D}_f &= f(u)-f(v).
\end{aligned}
$$

Using that $\mathcal{M}_\beta^{-1}:\Big[\mathcal{D}(\mathcal{M}_\beta^{1/2})\Big]^{-1}\to\mathcal{D}(\mathcal{M}_\beta^{1/2})$ is a isometrical bijection with respect to the norm $\|u\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})}=\|u\|^2+\|A^{\beta/4}u\|^2$ we have

$$
\|\mathcal{M}_\beta^{-1}w\|_{\mathcal{D}(\mathcal{M}_\beta^{1/2})}=\|w\|_{\left[\mathcal{D}(\mathcal{M}_\beta^{1/2})\right]^{-1}},\quad \forall w\in\Big[\mathcal{D}(\mathcal{M}_\beta^{1/2})\Big]^{-1}.
$$

Thus, from definition of $\mathcal{H}_{\beta,\kappa}-$ norm and using $W_0\hookrightarrow\Big[\mathcal{D}(\mathcal{M}_\beta^{1/2})\Big]^{-1}=W_{-\beta}$, we have

$$
\begin{aligned}
||\mathcal{F}_\kappa(U)-\mathcal{F}_\kappa(V)||_{\mathcal{H}_{\beta,\kappa}} &= ||F(U)-F(V)||_{\mathcal{D}(\mathcal{M}_\beta^{1/2})}\\
&= \Big\|\mathcal{M}_\beta^{-1}\left[\mathrm{S}_{(\Phi,\Psi)}A^{1/2}w+\mathrm{D}_{(\Phi,\Psi)}A^{1/2}(u+v)+\mathrm{D}_f\right]\Big\|_{\mathcal{D}(\mathcal{M}_\beta^{1/2})}\\
&= \big\|\mathrm{S}_{(\Phi,\Psi)}A^{1/2}w+\mathrm{D}_{(\Phi,\Psi)}A^{1/2}(u+v)+\mathrm{D}_f\big\|_{\mathcal{D}(\mathcal{M}_\beta^{1/2})^{-1}} \qquad (2.24)\\
&\le \lambda_1^{-\frac{\beta}{4}}\left|\mathrm{S}_{(\Phi,\Psi)}\right|\|A^{1/2}w\|+\lambda_1^{-\frac{\beta}{4}}\left|\mathrm{D}_{(\Phi,\Psi)}\right|\|A^{1/2}(u+v)\|+\|\mathrm{D}_f\|_{W_{-\beta}}.
\end{aligned}
$$

Now let's estimate the terms on the right side of (2.24). From immersion $W_2\hookrightarrow W_1$, we have

$$
\begin{aligned}
\left|\mathrm{S}_{(\Phi,\Psi)}\right|\|A^{1/2}w\| &\le \zeta_0\|A^{1/2}w\|+\frac{\zeta_1}{2}\Big[\|A^{1/4}u\|^2+\|A^{1/4}v\|^2\Big]\|A^{1/2}w\|\\
&\quad+\frac{\zeta_2}{2}\bigg[\Big(\|A^{1/2}u\|\|u_t\|\Big)^{q-1}+\Big(\|A^{1/2}v\|\|v_t\|\Big)^{q-1}\bigg]\|A^{1/2}w\|\\
&\le\Big[\zeta_0+\zeta_1R^2+\zeta_2R^{2(q-1)}\Big]||U-V||_{\mathcal{H}_{\beta,\kappa}}. \qquad (2.25)
\end{aligned}
$$

Using that $\Psi\in C^1(\mathbb{R})$ with $\Psi'(s)=\zeta_2(q-1)|s|^{q-2}$ and

$$
(A^{1/2}u,u_t)-(A^{1/2}v,v_t)=(A^{1/2}w,u_t)+(A^{1/2}v,w_t),
$$

from the Mean Value Theorem, one can easily prove that

$$
\begin{aligned}
&\left|\Psi(A^{1/2}u, u_t) - \Psi(A^{1/2}v, v_t)\right| \\
&\quad \le \zeta_2 2^{2(q-2)}(q-1)\left[|(A^{1/2}u, u_t)|^{q-2} + |(A^{1/2}v, v_t)|^{q-2}\right]| \\
&\qquad \times (A^{1/2}u, u_t) - (A^{1/2}v, v_t)| \le \zeta_2 2^{2q}(q-1)R^{2q-3}||U - V||_{\mathcal{H}_{\beta,\kappa}}.
\end{aligned}
$$

On the other hand, using that $a^2 - b^2 = (a-b)(a+b)$ and $W_2 \hookrightarrow W_1$, we have

$$
\begin{aligned}
\left|\Phi(\|A^{1/4}u\|^2) - \Phi(\|A^{1/4}v\|^2)\right| &\le \frac{\zeta_1}{2}\left[\,\|A^{1/4}u\| + \|A^{1/4}v\|\,\right]\|A^{1/4}w\| \\
&\le \frac{\zeta_1}{2\lambda_1^{1/2}}\left[\,\|A^{1/2}u\| + \|A^{1/2}v\|\,\right]\|A^{1/2}w\| \\
&\le \frac{\zeta_1 R}{\lambda_1^{1/2}}||U - V||_{\mathcal{H}_{\beta,\kappa}}.
\end{aligned}
$$

From the last two inequalities and the definition of $\mathrm{D}_{(\Phi,\Psi)}$, we have

$$
\left|\mathrm{D}_{(\Phi,\Psi)}\right|\|A^{1/2}(u+v)\| \le \left[\,\zeta_2 2^{2q}(q-1)R^{2(q-1)} + \frac{\zeta_1 R^2}{\lambda_1^{1/2}}\,\right]||U - V||_{\mathcal{H}_{\beta,\kappa}}. \tag{2.26}
$$

When $n \le 4$, using (2.15), Mean Value Theorem, Hölder's inequality with $\frac{\rho}{\rho+1} + \frac{1}{\rho+1} = 1$ and embedding $W_2 \hookrightarrow L^{2(\rho+1)}(\Omega)$, we get

$$
\begin{aligned}
\left\|\mathrm{D}_f\right\|_{W_{-\beta}} &\le K_{f'}\left[\int_\Omega [1 + (|u|+|v|)^\rho]^{\frac{2(\rho+1)}{\rho}}dx\right]^{\frac{\rho}{2(\rho+1)}}\left[\int_\Omega |w|^{2(\rho+1)}dx\right]^{\frac{1}{2(\rho+1)}} \\
&\le 2^{\rho+1}K_{f'}\left[\,|\Omega| + \|u\|_{2(\rho+1)} + \|v\|_{2(\rho+1)}\,\right]^\rho\|w\|_{2(\rho+1)} \\
&\le 2^{\rho+1}K_{f'}C_{|\Omega|}\left[|\Omega| + 2C_{|\Omega|}\left(\|A^{1/2}u\| + \|A^{1/2}v\|\right)\right]^\rho\|A^{1/2}w\| \\
&\le 2^{\rho+1}K_{f'}C_{|\Omega|}\left[|\Omega| + 2C_{|\Omega|}R\right]^\rho||U - V||_{\mathcal{H}_{\beta,\kappa}}.
\end{aligned}
$$

When $n \ge 5$, using (2.15), Mean Value Theorem, Hölder's inequality with $\frac{4+2\beta}{n+2\beta} + \frac{n-4}{n+2\beta} = 1$, embedding $L^{\frac{2n}{n+2\beta}}(\Omega) \hookrightarrow W_{-\beta}$ and $W_2 \hookrightarrow L^{\frac{2n}{n-4}}(\Omega) \hookrightarrow L^{\frac{2n\rho}{4+2\beta}}(\Omega)$, we get

$$
\begin{aligned}
\left\|\mathrm{D}_f\right\|_{-\beta} &\le K_{f'}\left\|\mathrm{D}_f\right\|_{\frac{2n}{n+2\beta}} \\
&\le K_{f'}\left[\int_\Omega [1 + (|u|+|v|)^\rho]^{\frac{2n}{n+2\beta}}|w|^{\frac{2n}{n+2\beta}}dx\right]^{\frac{n+2\beta}{2n}} \\
&\le K_{f'}\left[\int_\Omega [1 + (|u|+|v|)^\rho]^{\frac{2n}{4+2\beta}}dx\right]^{\frac{4+2\beta}{2n}}\left[\int_\Omega |w|^{\frac{2n}{n-4}}dx\right]^{\frac{n-4}{2n}}
\end{aligned}
$$

$$\leq 2^{\rho+1} K_{f'} \left[|\Omega| + \|u\|_{\frac{2n\rho}{4+2\beta}} + \|v\|_{\frac{2n\rho}{4+2\beta}} \right]^{\rho} \|w\|_{\frac{2n}{n-4}}$$
$$\leq 2^{\rho+1} K_{f'} C_{|\Omega|} \left[|\Omega| + 2C_{|\Omega|} \left(\|A^{1/2}u\| + \|A^{1/2}v\| \right) \right]^{\rho} \|A^{1/2}w\|$$
$$\leq 2^{\rho+1} K_{f'} C_{|\Omega|} \left[|\Omega| + 2C_{|\Omega|} R \right]^{\rho} ||U - V||_{\mathcal{H}_{\beta,\kappa}}. \tag{2.27}$$

Substituting (2.25)–(2.27) in (2.24), we obtain

$$||\mathcal{F}_\kappa(U) - \mathcal{F}_\kappa(V)||_{\mathcal{H}_{\beta,\kappa}} \leq K_R ||U - V||_{\mathcal{H}_{\beta,\kappa}}, \tag{2.28}$$

where

$$K_R = \lambda_1^{-\frac{\beta}{4}} \zeta_0 + \lambda_1^{-\frac{\beta}{4}} (1 + \frac{1}{\lambda_1^{1/2}}) \zeta_1 R^2$$
$$+ \lambda_1^{-\frac{\beta}{4}} (1 + 2^{2q}(q-1)) \zeta_2 R^{2(q-1)} + 2^{\rho+1} K_{f'} C_{|\Omega|} \left[|\Omega| + 2C_{|\Omega|} R \right]^{\rho}.$$

Combining the Steps I and II, we obtain from result ([32, Theorem 6.1.4]) that the Cauchy problem (2.18) has a unique mild solution $U \in C([0, T_{\max}); \mathcal{H}_{\beta,\kappa})$ that satisfies (2.21) in a maximal interval $[0, T_{\max})$. Besides, if $T_{\max} < \infty$ then

$$\lim_{t \to T_{\max}} ||U(t)||_{\mathcal{H}_{\beta,\kappa}} = \infty. \tag{2.29}$$

Now, if $U(t)$ is a mild solution of (2.18) with $U_0 \in \mathcal{D}(\mathcal{L}_{\alpha,\beta,\kappa})$. From ([32], Theorem 6.1.5), then the mild solution U is a classical solution of the initial value problem (2.18).

2.3.3 Step III

To show that the mild (or strong) solution is global, let $U(t)$ be a mild solution with initial data $U_0 \in \mathcal{D}(\mathcal{L}_{\alpha,\beta,\kappa})$. Taking the scalar product in W_0 of $(1.1)_1$ by u_t and $(1.1)_2$ by θ we obtain directly

$$\frac{d}{dt} E^{\beta,\kappa}(t) + \zeta_2 \left| (A^{1/2}u, u_t) \right|^q + \|A^{1/4}\theta(t)\|^2 = 0. \tag{2.30}$$

Integrating (2.30) from 0 to t, we obtain

$$E^{\beta,\kappa}(t) + \int_0^t \left[\zeta_2 \left| (A^{1/2}u(s), u_t(s)) \right|^q + \|A^{1/4}\theta(s)\|^2 \right] ds = E^{\beta,\kappa}(0). \tag{2.31}$$

Note that

$$\left| -\frac{\zeta_0}{2} \|A^{1/4}u(t)\|^2 \right| \leq \frac{\zeta_0^2}{4\zeta_1} + \frac{\zeta_1}{4} \|A^{1/4}u(t)\|^4. \tag{2.32}$$

Putting $\varpi = 1 - \frac{\omega}{\lambda_1} > 0$, from Hölder inequality and immersion $W_2 \hookrightarrow W_0$, we have

$$(h, u(t)) \leq \|h\|\|u(t)\| \leq \frac{1}{\varpi\lambda_1}\|h\|^2 + \frac{\varpi}{4}\|A^{1/2}u(t)\|^2.$$

Using assumption (2.16) and immersion $W_2 \hookrightarrow W_0$, we have

$$\left(\widehat{f}(u(t)), 1\right) \geq -\frac{\omega}{4\lambda_1}\|A^{1/2}u(t)\|^2 - K_f|\Omega|. \tag{2.33}$$

Putting $L_0 = K_f|\Omega| + \frac{1}{\varpi\lambda_1}\|h\|^2 + \frac{\zeta_0^2}{4\zeta_1}$ and using definition of $E^{\beta,\kappa}(t)$, substituting (2.32)–(2.33) in (2.31), we obtain

$$\begin{aligned} E^{\beta,\kappa}(t) + L_0 &\geq \frac{1}{2}\|u_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} + \frac{1}{4}\|A^{1/2}u(t)\|^2 + \frac{\kappa}{2}\|\theta(t)\|^2 \\ &\geq \frac{1}{4}||U(t)||^2_{\mathcal{H}_{\beta,\kappa}}. \end{aligned} \tag{2.34}$$

Thus

$$||U(t)||^2_{\mathcal{H}_{\beta,\kappa}} \leq 4|E^{\beta,\kappa}(t)| + 4L_0 \leq 4|E^{\beta,\kappa}(0)| + 4L_0, \quad \forall t \in [0, T_{\max}).$$

Then clearly (2.29) does not hold and therefore $T_{\max} = \infty$. This completes the proof of Theorem 2.1 (**i**).

Proof (**ii**) The limit case $\kappa = 0$ is done in a similar way to case $\kappa > 0$, noting that in this case

$$\begin{aligned} \left\langle \mathcal{L}_{\beta,0}U(t), U(t)\right\rangle_{\mathcal{H}_{\beta,0}} &= \left\langle \begin{bmatrix} u_t \\ \mathcal{M}_\beta^{-1}\left(-Au - A^{\alpha/2-1}u_t\right) \end{bmatrix}, \begin{bmatrix} u \\ u_t \end{bmatrix} \right\rangle \\ &= -\|A^{\frac{1}{2}(\alpha-\frac{1}{2})}u_t(t)\|^2 \leq 0 \end{aligned}$$

and $\mathcal{F}_\kappa U = (0, F(U))$.

This completes the proof of the Theorem 2.1. □

By means of the well-posedness assured by Theorem 2.1 the solutions to problem (2.12)–(2.13) generate a family of dynamical systems $(\mathcal{H}_{\beta,\kappa}, S_{\alpha,\beta,\kappa}(t))$ with the spaces $\mathcal{H}_{\beta,\kappa}$ given in (2.10) and the evolution operator $S_{\alpha,\beta,\kappa}(t)$ given by the formula

$$S_{\alpha,\beta,\kappa}(t)U_0 = \begin{cases} U(t) = (u(t), u_t(t), \theta(t))^\perp & \text{for} \quad \kappa > 0 \\ U(t) \quad = (u(t), u_t(t))^\perp & \text{for} \quad \kappa = 0 \end{cases} \tag{2.35}$$

where $U(t)$ is a mild solution to (2.18).

2.4 Gradient Systems

Proposition 2.2 *The dynamical system $(\mathcal{H}_{\beta,\kappa}, S_{\alpha,\beta,\kappa})$ corresponding to the problem (2.12) is gradient.*

Proof **Case** $\kappa > 0$. We consider $E_{\beta,\kappa}(\cdot)$ the energy functional defined in (2.11). Let $U_0 \in \mathcal{H}_{\beta,\kappa}$, from (2.30), we have

$$E_{\beta,\kappa}(S_{\alpha,\beta,\kappa}(t)U_0) \leq E_{\beta,\kappa}(U_0), \quad t \geq 0,$$

which means that the function $t \mapsto E_{\beta,\kappa}(S_{\alpha,\beta,\kappa}(t)z)$ is a non-increasing function, for any $U_0 \in \mathcal{H}_{\beta,\kappa}$. Now let us suppose $E_{\beta,\kappa}(S_{\alpha,\beta,\kappa}(t)z) = E_{\beta,\kappa}(z)$ for all $t > 0$ and for some $z = (u_0.u_1, \theta_0) \in \mathcal{H}_{\beta,\kappa}$. Then, from (2.31), we have

$$\zeta_2 \int_0^t \left|(A^{1/2}u(s), u_t(s))\right|^q ds + \int_0^t \|A^{1/4}\theta(s)\|^2 ds = 0, \quad t > 0.$$

This implies that

$$\int_0^t \|A^{1/4}\theta(s)\|^2 ds = 0, \quad t > 0,$$

which gives

$$\|A^{1/4}\theta(t)\|^2 = 0, \quad \forall t > 0.$$

Hence, we have $\theta(t) = 0$ a.e. in Ω for all $t > 0$. Substituting this into Eq. $(2.12)_2$, we conclude that $u_t(t) = 0$ a.e. in Ω for all $t > 0$. Thus, $S_{\alpha,\beta,\kappa}(t)z \equiv (u_0, 0, 0) = z$, and the energy functional $E_{\beta,\kappa}(\cdot)$ is a strict Lyapunov functional on $\mathcal{H}_{\beta,\kappa}$.
Case $\kappa = 0$. The limit case $\kappa = 0$ is analogously, noting that from (2.13), we have

$$\frac{d}{dt}E_{\beta,0}(t) + \|A^{\alpha/2-1/2}u_t(s)\|^2 = 0, \quad t \geq 0.$$

□

3 Long-Time Dynamics

3.1 Energy Decay and Absorbing Set

Proposition 3.1 *Under assumptions of Theorem 2.1, if we consider a weak solution $U(t) = S_{\alpha,\beta,\kappa}(t)U_0$ of (2.12) corresponding to initial data $U_0 \in B$, where $B \subset \mathcal{H}_{\beta,\kappa}$ is an arbitrary bounded set, then there exists a constant $\varrho_B > 0$ (depending on the size of B) and a small constant $\varepsilon > 0$ such that*

$$||U(t)||^2_{\mathcal{H}_{\beta,\kappa}} \leq \varrho_B e^{-\frac{2\varepsilon}{3}t} + R,$$

where $R = \frac{12}{\mu}\left[\frac{2}{\mu\lambda_1}\|h\|^2 + K_f|\Omega|\right]$.

Proof Let us begin fixing an arbitrary bounded set $B \subset \mathcal{H}_{\beta,\kappa}$ and consider the solutions $U(t) = S_{\alpha,\beta,\kappa}(t)U_0$ with $U_0 \in B$. Let

$$\widetilde{E}_{\beta,\kappa}(t) := E_{\beta,\kappa}(t) + L_0.$$

From (2.34), we have

$$\widetilde{E}_{\beta,\kappa}(t) \geq \frac{1}{4}||U(t)||^2_{\mathcal{H}_{\beta,\kappa}}, \quad \forall t \geq 0.$$

We define the functional

$$E^{\varepsilon}_{\beta,\kappa}(t) := \widetilde{E}_{\beta,\kappa}(t) + \varepsilon\left(\mathcal{M}_\beta u_t, u + 2\kappa A^{-\alpha/2}\theta\right), \quad \varepsilon > 0. \tag{3.36}$$

The constant ε is a constant sufficiently small that will be conveniently chosen later. From derivation of (3.36) and using that $\frac{d}{dt}\widetilde{E}_{\beta,\kappa}(t) = \frac{d}{dt}E_{\beta,\kappa}(t)$, we have

$$\begin{aligned}\frac{d}{dt}E^{\varepsilon}_{\beta,\kappa}(t) = \frac{d}{dt}E_{\beta,\kappa}(t) &+ \varepsilon\|u_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} + \varepsilon\left(\mathcal{M}_\beta u_{tt}, u\right)\\ &+ 2\varepsilon\kappa\left(\mathcal{M}_\beta u_t, A^{-\alpha/2}\theta_t\right) + 2\varepsilon\kappa\left(\mathcal{M}_\beta u_{tt}, A^{-\alpha/2}\theta\right).\end{aligned} \tag{3.37}$$

Taking the scalar product in W_0 of $(2.12)_1$ with u_t and $(2.12)_2$ with θ, we have

$$\frac{d}{dt}E_{\beta,\kappa}(t) = -\zeta_2\left|(A^{1/2}u, u_t)\right|^q - \|A^{1/4}\theta(t)\|^2. \tag{3.38}$$

Using $(2.12)_1$ the third term on the right of (3.37) can be written as follows

$$\begin{aligned}\varepsilon\left(\mathcal{M}_\beta u_{tt}, u\right) = &-\varepsilon\|A^{1/2}u(t)\|^2 - \varepsilon\zeta_0\|A^{1/4}u(t)\|^2 - \varepsilon\zeta_1\|A^{1/4}u(t)\|^4\\ &- \varepsilon\Psi(A^{1/2}u, u_t)\|A^{1/4}u(t)\|^2 + \varepsilon\left(A^{\alpha/2}\theta, u\right)\\ &-\varepsilon\left(f(u), u\right) + \varepsilon\left(h, u\right).\end{aligned}$$

From $(2.12)_2$, the fourth term can be written as follows

$$2\varepsilon\kappa\left(\mathcal{M}_\beta u_t, A^{-\alpha/2}\theta_t\right) = -2\varepsilon\left(\mathcal{M}_\beta u_t, A^{-\alpha/2+1/2}\theta\right) - 2\varepsilon\|u_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})}.$$

From $(2.12)_1$, the fifth term can be written by

$$\begin{aligned}2\varepsilon\kappa\left(\mathcal{M}_\beta u_{tt}, A^{-\alpha/2}\theta\right) = &-2\varepsilon\kappa\left(A^{1/2}u, A^{-\alpha/2+1/2}\theta\right)\\ &- 2\varepsilon\kappa\left[\zeta_0 + \zeta_1\|A^{1/4}u\|^2 + \Psi(A^{1/2}u, u_t)\right]\left(A^{1/2}u, A^{-\alpha/2}\theta\right)\end{aligned}$$

$$+ 2\varepsilon\kappa\|\theta(t)\|^2 - 2\varepsilon\kappa\left(f(u), A^{-\alpha/2}\theta\right) + 2\varepsilon\kappa\left(h, A^{-\alpha/2}\theta\right). \tag{3.39}$$

Substituting (3.38)–(3.39) in (3.37), we obtain

$$\begin{aligned}\frac{d}{dt}E^{\varepsilon}_{\beta,\kappa}(t) = & - \zeta_2\left|(A^{1/2}u, u_t)\right|^q - \|A^{1/4}\theta(t)\|^2 - \varepsilon\|u_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} - \varepsilon\|A^{1/2}u(t)\|^2 \\ & - \varepsilon\zeta_0\|A^{1/4}u(t)\|^2 - \varepsilon\zeta_1\|A^{1/4}u(t)\|^4 + \varepsilon(h,u) + 2\varepsilon\kappa\|\theta(t)\|^2 + \sum_{i=1}^{9}\mathcal{I}_i,\end{aligned} \tag{3.40}$$

where

$$\begin{aligned}
\mathcal{I}_1 &= -\varepsilon(f(u), u),\\
\mathcal{I}_2 &= -\varepsilon\Psi(A^{1/2}u, u_t)\|A^{1/4}u(t)\|^2,\\
\mathcal{I}_3 &= \varepsilon\left(A^{\alpha/2}\theta, u\right),\\
\mathcal{I}_4 &= -2\varepsilon\left(u_t, A^{-\alpha/2+1/2}\theta\right),\\
\mathcal{I}_5 &= -2\varepsilon\left(A^{\beta/4}u_t, A^{(\beta/4-\alpha/2+1/2)}\theta\right),\\
\mathcal{I}_6 &= -2\varepsilon\kappa\left(A^{1/2}u, A^{-\alpha/2+1/2}\theta\right),\\
\mathcal{I}_7 &= -2\varepsilon\kappa\left[\zeta_0 + \zeta_1\|A^{1/4}u\|^2 + \Psi(A^{1/2}u, u_t)\right]\left(A^{1/2}u, A^{-\alpha/2}\theta\right),\\
\mathcal{I}_8 &= -2\varepsilon\kappa\left(f(u), A^{-\alpha/2}\theta\right),\\
\mathcal{I}_9 &= 2\varepsilon\kappa\left(h, A^{-\alpha/2}\theta\right).
\end{aligned}$$

In what follows we will estimate the terms $\mathcal{I}_1, \ldots, \mathcal{I}_9$ on the right side of the inequality (3.40). First, using assumption (2.16) and immersion $W_2 \hookrightarrow W_0$, we have

$$\mathcal{I}_1 \leq -\varepsilon(\widehat{f}(u), 1) + \frac{\varepsilon\omega}{4}\|u(t)\|^2 \leq -\varepsilon(\widehat{f}(u), 1) + \frac{\varepsilon}{4}\|A^{1/2}u(t)\|^2.$$

Using Young inequality with $\frac{q-1}{q} + \frac{1}{q} = 1$ and immersion $W_2 \hookrightarrow W_1$ there exist $K_{0,B}$ such that

$$\begin{aligned}
\mathcal{I}_2 &= -\varepsilon\zeta_2\left|(A^{1/2}u, u_t)\right|^{q-2}(A^{1/2}u, u_t)\|A^{1/4}u(t)\|^2\\
&\leq \frac{\zeta_2(q-1)}{q}\left|(A^{1/2}u, u_t)\right|^q + \frac{\varepsilon^q\zeta_2}{q}\|A^{1/4}u(t)\|^{2(q-1)}\|A^{1/4}u(t)\|^2\\
&\leq \zeta_2\left|(A^{1/2}u, u_t)\right|^q + \varepsilon^q K_{0,B}\|A^{1/2}u(t)\|^2.
\end{aligned}$$

From immersions $W_2 \hookrightarrow W_1 \hookrightarrow W_\alpha$, we have

$$
\begin{aligned}
\mathcal{I}_3 &= \varepsilon \left(A^{\alpha/4}\theta, A^{\alpha/4}u \right) \le \varepsilon \|A^{\alpha/4}\theta(t)\| \|A^{\alpha/4}u(t)\| \\
&\le \frac{\varepsilon}{\lambda_1^{-\alpha/2+3/4}} \|A^{1/4}\theta(t)\| \|A^{1/2}u(t)\| \le \frac{1}{12}\|A^{1/4}\theta(t)\|^2 + \frac{3\varepsilon^2}{\lambda_1^{-\alpha+3/2}}\|A^{1/2}u(t)\|^2.
\end{aligned}
$$

From $W_1 \hookrightarrow W_{-2\alpha+2} \hookrightarrow W_0$ for $\alpha \in [1/2, 1]$, we have

$$
\begin{aligned}
\mathcal{I}_4 &\le \; 2\varepsilon \|u_t(t)\| \|A^{\frac{1}{4}(-2\alpha+2)}\theta(t)\| \le \frac{2\varepsilon}{\lambda_1^{\alpha/2-1/4}} \|u_t(t)\| \|A^{1/4}\theta(t)\| \\
&\le \; \frac{12\varepsilon^2}{\lambda_1^{\alpha-1/2}} \|u_t(t)\|^2 + \frac{1}{12}\|A^{1/4}\theta(t)\|^2.
\end{aligned}
$$

From $W_1 \hookrightarrow W_{\beta-2\alpha+2} \hookrightarrow W_0$ for $\beta \in [0, 2\alpha - 1]$ with $\alpha \in [1/2, 1]$, we have

$$
\begin{aligned}
\mathcal{I}_5 &\le 2\varepsilon \|A^{\beta/4}u_t(t)\| \|A^{\frac{\beta-2\alpha+2}{4}}\theta(t)\| \le \frac{2\varepsilon}{\lambda_1^{\frac{-\beta+2\alpha-1}{4}}} \|A^{\beta/4}u_t(t)\| \|A^{1/4}\theta(t)\| \\
&\le \frac{12\varepsilon^2}{\lambda_1^{\frac{-\beta+2\alpha-1}{2}}} \|A^{\beta/4}u_t(t)\|^2 + \frac{1}{12}\|A^{1/4}\theta(t)\|^2.
\end{aligned}
$$

Again using that $W_1 \hookrightarrow W_{-2\alpha+2} \hookrightarrow W_0$ for $\alpha \in [1/2, 1]$, we have

$$
\begin{aligned}
\mathcal{I}_6 &\le \; 2\varepsilon\kappa \|A^{1/2}u(t)\| \|A^{\frac{1}{4}(-2\alpha+2)}\theta(t)\| \le \frac{2\varepsilon\kappa}{\lambda_1^{\alpha/2-1/4}} \|A^{1/2}u(t)\| \|A^{1/4}\theta(t)\| \\
&\le \; \frac{12\varepsilon^2\kappa^2}{\lambda_1^{\alpha-1/2}} \|A^{1/2}u(t)\|^2 + \frac{1}{12}\|A^{1/4}\theta(t)\|^2.
\end{aligned}
$$

Now, using that $W_2 \hookrightarrow W_1 \hookrightarrow W_{-2\alpha}$, there exists $K_{1,B} > 0$ such that

$$
\mathcal{I}_7 \le \varepsilon^2 K_{1,B} \|A^{1/2}u(t)\|^2 + \frac{1}{12}\|A^{1/4}\theta(t)\|^2.
$$

When $n \le 4$, from assumption (2.16), immersions $W_2 \hookrightarrow W_1 \hookrightarrow W_0 \hookrightarrow W_{-2\alpha}$ and $W_2 \hookrightarrow L^{2(\rho+1)}$, there exists $K_{2,B}$ such that

$$
\begin{aligned}
\mathcal{I}_8 &\le \frac{2\varepsilon\kappa K_f'}{\lambda_1^{1/4+\alpha/2}} \left[\|u(t)\| + \|u(t)\|_{2(\rho+1)}^{\rho+1} \right] \|A^{1/4}\theta(t)\| \\
&\le \varepsilon^2 K_{2,B} \|A^{1/2}u(t)\|^2 + \frac{1}{12}\|A^{1/4}\theta(t)\|^2.
\end{aligned}
$$

When $n \geq 5$, using (2.16), Hölder's inequality with $\frac{4+2\beta}{2n} + \frac{n-4}{2n} + \frac{n-2\beta}{2n} = 1$, embedding $W_\beta \hookrightarrow L^{\frac{2n}{n-2\beta}}(\Omega)$ and $W_1 \hookrightarrow W_{\beta-2\alpha}$ for $\beta - 2\alpha \leq -1$, we get

$$\begin{aligned}
\mathcal{I}_8 &\leq 2\varepsilon\kappa K_f' \int_\Omega (1+|u|^\rho)|u||A^{-\alpha/2}\theta|dx \\
&\leq 2\varepsilon\kappa K_f' \left[\int_\Omega (1+|u|^\rho)^{\frac{2n}{4+2\beta}} dx\right]^{\frac{4+2\beta}{2n}} \|u\|_{\frac{n-4}{2n}} \|A^{-\alpha/2}\theta\|_{\frac{2n}{n-2\beta}} \\
&\leq 2\varepsilon\kappa K_f' C_{|\Omega|}(1+\|u\|^\rho_{\frac{2n\rho}{4+2\beta}})\|u\|_{\frac{n-4}{2n}}\|A^{\beta/4-\alpha/2}\theta\| \\
&\leq 2\varepsilon\kappa K_f' C_{|\Omega|}\lambda_1^{\frac{\beta-2\alpha-1}{4}}(1+\|A^{1/2}u\|^\rho)\|A^{1/2}u\|\|A^{1/4}\theta\| \\
&\leq \varepsilon^2 K_{2,B}\|A^{1/2}u(t)\|^2 + \frac{1}{12}\|A^{1/4}\theta(t)\|^2.
\end{aligned}$$

Finally, by Hölder inequality, Young inequality and immersion $W_0 \hookrightarrow W_{-2\alpha}$, we have

$$\mathcal{I}_9 \leq \frac{2\varepsilon\kappa}{\lambda_1^{\alpha/2}}\|h\|\|\theta(t)\| \leq \frac{\varepsilon}{\varpi\lambda_1}\|h\|^2 + \frac{\varepsilon\kappa\varpi}{\lambda_1^{\alpha-1}}\|\theta(t)\|^2.$$

Thus substituting $\mathcal{I}_1, \ldots, \mathcal{I}_9$ in (3.40) and taking $\varepsilon > 0$ small enough such that

$$\begin{aligned}
&\frac{1}{2} - \varepsilon\left[12\lambda_1^{-1/2+\alpha}(1+\lambda_1^{-\beta/2})\right] \geq 0 \quad \text{and} \\
&\frac{1}{4} - \varepsilon\left[\varepsilon^{q-2}K_{0,\mathcal{B}} + K_{1,B} + K_{2,B} + \frac{3}{\lambda_1^{-\alpha+3/2}} + \frac{12\kappa^2}{\lambda_1^{\alpha-1/2}}\right] \geq 0
\end{aligned}$$

we obtain

$$\begin{aligned}
\frac{d}{dt}E^\varepsilon_{\beta,\kappa}(t) \leq & -\frac{1}{2}\|A^{1/4}\theta(t)\|^2 \\
& - \frac{\varepsilon}{2}\|u_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} - \frac{\varepsilon}{2}\|A^{1/2}u(t)\|^2 - \varepsilon\zeta_0\|A^{1/4}u(t)\|^2 \\
& - \varepsilon\zeta_1\|A^{1/4}u(t)\|^4 - \varepsilon(\widehat{f}(u), 1) + \varepsilon(h, u) \\
& + \frac{\varepsilon}{\varpi\lambda_1}\|h\|^2 + \varepsilon\kappa\left(2 + \frac{\varpi}{\lambda_1^{\alpha-1}}\right)\|\theta(t)\|^2
\end{aligned} \tag{3.41}$$

Still, using that

$$-\frac{1}{2}\|A^{1/4}\theta(t)\|^2 \leq -\frac{\lambda_1^{1/2}}{2}\|\theta(t)\|^2$$

and also taking ε such that $\frac{\lambda_1^{1/2}}{2} - \varepsilon\kappa(2 + \frac{\varpi}{\lambda_1^{\alpha-1}}) \geq \frac{\varepsilon\kappa}{2}$, returning to (3.41), we obtain

$$\frac{d}{dt}E^{\varepsilon}_{\beta,\kappa}(t) \leq -\,\varepsilon\widetilde{E}_{\beta,\kappa}(t) + \varepsilon L_1, \quad \text{where} \quad L_1 = \frac{2}{\varpi\lambda_1}\|h\|^2 + \frac{\zeta_0^2}{3\zeta_1} + K_f|\Omega|. \tag{3.42}$$

From (3.36), immersions $W_2 \hookrightarrow W_\beta \hookrightarrow W_0 \hookrightarrow W_{2\beta-2\alpha} \hookrightarrow W_{-1} \hookrightarrow W_{-2\alpha} \hookrightarrow W_{-2}$ and (2.34), we obtain

$$\begin{aligned}
\left|E^{\varepsilon}_{\beta,\kappa}(t) - \widetilde{E}_{\beta,\kappa}(t)\right| &\leq \varepsilon\left|(\mathcal{M}_\beta u_t, u)\right| + 2\varepsilon\kappa\left|(\mathcal{M}_\beta u_t, A^{-\alpha/2}\theta)\right| \\
&\leq \frac{\varepsilon}{\lambda_1^{1/2}}\|u_t(t)\|\|A^{1/2}u(t)\| + \frac{\varepsilon}{\lambda_1^{-\beta/4+1/2}}\|A^{\beta/4}u_t(t)\|\|A^{1/2}u(t)\| \\
&\quad + \frac{2\varepsilon\kappa}{\lambda_1^{\alpha/2}}\|u_t(t)\|\|\theta(t)\| + \frac{2\varepsilon\kappa}{\lambda_1^{-\beta/4+\alpha/2}}\|A^{\beta/4}u_t(t)\|\|\theta(t)\| \\
&\leq \varepsilon K_{\lambda_1}\widetilde{E}_{\beta,\kappa}(t),
\end{aligned}$$

where

$$K_{\lambda_1} = \left[\frac{4}{\lambda_1^{1/2}} + \frac{4}{\lambda_1^{-\beta/4+1/2}} + \frac{8\varepsilon\kappa^{1/2}}{\lambda_1^{\alpha/2}} + \frac{8\kappa^{1/2}}{\lambda_1^{-\beta/4+\alpha/2}}\right].$$

Taking $\varepsilon \leq \frac{1}{2K_{\lambda_1}}$, we have

$$\frac{1}{2}\widetilde{E}_{\beta,\kappa}(t) \leq E^{\varepsilon}_{\beta,\kappa}(t) \leq \frac{3}{2}\widetilde{E}_{\beta,\kappa}(t). \tag{3.43}$$

Substituting in (3.42), we have

$$\frac{d}{dt}E^{\varepsilon}_{\beta,\kappa}(t) \leq -\frac{2\varepsilon}{3}E^{\varepsilon}_{\beta,\kappa}(t) + \varepsilon L_1.$$

Integrating from 0 to t, we get

$$E^{\varepsilon}_{\beta,\kappa}(t) \leq E^{\varepsilon}_{\beta,\kappa}(0)e^{-\frac{2\varepsilon}{3}t} + \frac{3L_1}{2}$$

Again using (2.34) and (3.43), we obtain

$$\|U(t)\|^2_{\mathcal{H}_{\beta,\kappa}} \leq \widetilde{E}_{\beta,\kappa}(t) \leq 3\widetilde{E}_{\beta,\kappa}(0)e^{-\frac{2\varepsilon}{3}t} + 3L_1 \tag{3.44}$$

Taking $R = 12L_1$ and using, we obtain

$$||U(t)||^2_{\mathcal{H}_{\beta,\kappa}} \leq 12\widetilde{E}_{\beta,\kappa}(0)e^{-\frac{2\varepsilon}{3}t} + R. \tag{3.45}$$

□

Corollary 3.2 (Energy decay) *Under assumptions of Proposition 3.1 with $h \equiv 0$, $\zeta_0 = 0$ and $K_f = 0$ in (2.16), then the energy $E_{\beta,\kappa}(t)$ defined in (2.11) satisfies the following decay rates*

$$E_{\beta,\kappa}(U(t)) \leq Ce^{-\gamma t}, \quad \forall\, t > 0.$$

where the constants $C, \gamma > 0$ depending on $||U_0||_{\mathcal{H}\beta,\kappa}$.

Proof The proof is an immediate consequence of Proposition 3.1 by noting that (3.44) holds with $L_1 = 0$ and $\widetilde{E}_{\beta,\kappa}(t) = E_{\beta,\kappa}(t)$ when one takes $h = 0$, $\zeta_0 = 0$ and $K_f = 0$. □

Corollary 3.3 (Absorbing set) *Under assumptions of Proposition 3.1, let us consider any bounded set $B \subset \mathcal{H}_{\beta,\kappa}$. If $U_0 \in B$, then there exists $t_B > 0$ such that*

$$\|U(t)\|_{\mathcal{H}_{\beta,\kappa}} \leq \mathcal{R}, \quad \forall\, t > t_B, \tag{3.46}$$

where $U(t) = S_{\alpha,\beta,\kappa}(t)U_0$ is the weak solution of problem (1.1)–(1.3) and $\mathcal{R} > 0$ is a constant independent of U_0. In particular, the set

$$\mathcal{B} = \{U \in \mathcal{H}_{\beta,\kappa};\ \|U\|_{\mathcal{H}_{\beta,\kappa}} \leq \mathcal{R}\}$$

is a bounded absorbing set for $S_{\alpha,\beta,\kappa}(t)$ defined in (2.35). In other words, the dynamical system $(\mathcal{H}_{\beta,\kappa}, S_{\alpha,\beta,\kappa}(t))$ is dissipative.

Proof For initial data $U_0 \in B$ we obtain from estimates (3.45) that there exists $t_B > 0$ dependent of $B \subset \mathcal{H}_{\beta,\kappa}$ such that

$$\widetilde{E}_{\beta,\kappa}(t) \leq 2R, \quad \forall\, t > t_B.$$

From (2.34) one sees that (3.46) follows by taking $\mathcal{R} = 2\sqrt{2R} > 0$. □

3.2 Lipschitz Continuity and Stabilization Inequality

Proposition 3.4 *Let us take the assumptions of Proposition 3.1 with $|\zeta_0| \leq \frac{\lambda_1^{1/2}}{2}$. Given a bounded set $B \subset \mathcal{H}_{\beta,\kappa}$, let $U^i = (u^i, u^i_t, \theta^i)$, $i = 1, 2$, be two mild (or strong) solutions of problem (2.12) such that $U^i(0) = (u^i_0, u^i_1, \theta^i_0) \in B$. Then, there exist a uniform constant $\epsilon > 0$ and constants $\varrho_{0,B}, \varrho_{1,B}, \varrho_{2,B} > 0$ such that*

(i) *Lipschitz continuity:*

$$||U^1(t) - U^2(t)||^2_{\mathcal{H}_{\beta,\kappa}} \leq e^{\varrho_{0,B}t}||U^1(0) - U^2(0)||_{\mathcal{H}_{\beta,\kappa}}, \tag{3.47}$$

(ii) *Stabilization inequality: Assume that* $0 < \rho < \frac{n+2\beta}{n-4}$ *when* $\beta > 0$ *and* $n \geq 5$*,*

$$\begin{aligned} ||U^1(t) - U^2(t)||^2_{\mathcal{H}_{\beta,\kappa}} \leq \varrho_{1,B}||U^1(0) - U^2(0)||^2_{\mathcal{H}_{\beta,\kappa}} e^{-\epsilon t/4} \\ + \varrho_{2,B} \int_0^t e^{-\epsilon(t-s)/4} \|A^{(2-\beta)/4} w(s)\|^2 ds. \end{aligned} \tag{3.48}$$

for all $t > 0$*, where* $w = u^1 - u^2$*,* $z = \theta^1 - \theta^2$*.*

Proof (i) From (2.21) we have

$$U^1(t) - U^2(t) = e^{\mathcal{L}_{\alpha,\beta,\kappa}t}[U^1(0) - U^2(0)] + \int_0^t e^{\mathcal{L}_{\alpha,\beta,\kappa}(t-s)} \left[\mathcal{F}_\kappa(U^1(s)) - \mathcal{F}_\kappa(U^2(s))\right] ds.$$

Using (2.28) in Theorem 2.1 there exist a constant $\varrho_{0,B} > 0$ such that

$$||U^1(t) - U^2(t)||_{\mathcal{H}_{\beta,\kappa}} \leq ||U^1(0) - U^2(0)||_{\mathcal{H}_{\beta,\kappa}} + \varrho_{0,B} \int_0^t ||U^1(s) - U^2(s)||_{\mathcal{H}_{\beta,\kappa}} ds.$$

From Gronwall's lemma, we obtain

$$||U^1(t) - U^2(t)||_{\mathcal{H}_{\beta,\kappa}} \leq e^{\varrho_{0,B}t}||U^1(0) - U^2(0)||_{\mathcal{H}_{\beta,\kappa}}, \quad t \geq 0.$$

This proves inequality (3.47).

Proof (ii) In what follows we will prove inequality (3.48). We start by noting that function $(w, w_t, z) = U^1 - U^2$ is the mild (or strong) solution of problem

$$\begin{cases} \mathcal{M}_\beta w_{tt} + Aw + [\,\mathrm{S}_\Phi + \mathrm{S}_\Psi\,] A^{1/2}w + [\,\mathrm{D}_\Phi + \mathrm{D}_\Psi\,] A^{1/2}(u^1 + u^2) + \mathrm{D}_f - A^{\alpha/2}z = 0, \\ \kappa z_t + A^{1/2}z + A^{\alpha/2}w_t = 0, \end{cases} \tag{3.49}$$

with initial condition

$$U^1(0) - U^2(0) = (w(0), w_t(0), z(0)),$$

where

$$\begin{aligned} \mathrm{S}_\Phi &= \zeta_0 + \frac{\zeta_1}{2}\left[\,\|A^{1/4}u^1\|^2 + \|A^{1/4}u^2\|^2\,\right], \\ \mathrm{S}_\Psi &= \frac{1}{2}\left[\,\Psi(A^{1/2}u^1, u^1_t) + \Psi(A^{1/2}u^2, u^2_t)\,\right], \end{aligned}$$

$$
\begin{aligned}
\mathrm{D}_\Phi &= \frac{\zeta_1}{2}\Big[\,\|A^{1/4}u^1\|^2 - \|A^{1/4}u^2\|^2\,\Big],\\
\mathrm{D}_\Psi &= \frac{1}{2}\Big[\,\Psi(A^{1/2}u^1, u_t^1) - \Psi(A^{1/2}u^2, u_t^2)\Big],\\
\mathrm{D}_f &= f(u^1) - f(u^2).
\end{aligned}
$$

The energy functional to (3.49) is given by

$$
\mathcal{E}_{\beta,\kappa}(t) = \frac{1}{2}\Big[\,\|w_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} + \|A^{1/2}w(t)\|^2 + \kappa\|z(t)\|^2 + \mathrm{S}_\Phi\|A^{1/4}w(t)\|^2\,\Big].
$$

For simplicity, the same constant $K_B > 0$ will be used to denote several different constants depending on B in the next estimates. Initially, inequality (3.38) in Proposition 3.1 implies that

$$
\begin{aligned}
&||U^i(t)||^2_{\mathcal{H}_{\beta,\kappa}} + \zeta_2\int_0^t \Big|(A^{1/2}u^i, u_t^i)\Big|^q ds + \int_0^t \|A^{1/4}\theta^i(s)\|^2 ds\\
&\quad\le K_B, \quad \forall\, t \ge 0,\ i = 1, 2.
\end{aligned}
$$

From definition of $\mathcal{E}_{\beta,\kappa}$, immersion $W_2 \hookrightarrow W_1$ and using that $|\zeta_0| \le \frac{1}{2\lambda_1^{1/2}}$, we have

$$
\frac{1}{4}||U^1(t) - U^2(t)||^2_{\mathcal{H}_{\beta,\kappa}} \le \mathcal{E}_{\beta,\kappa}(t) \le \left[\,1 + \frac{\zeta_0}{\lambda_1^{1/2}} + \frac{\zeta_1}{\lambda_1}K_B\,\right]||U^1(t) - U^2(t)||^2_{\mathcal{H}_{\beta,\kappa}}. \tag{3.50}
$$

The terms S_Ψ, D_Φ and D_Ψ can be estimated as follows. Firstly, it is direct that

$$
|\mathrm{S}_\Psi| \le \frac{\zeta_2}{2}\Big[\,|(A^{1/2}u^1, u_t^1)|^{q-1} + |(A^{1/2}u^2, u_t^2)|^{q-1}\,\Big]. \tag{3.51}
$$

Using that $a^2 - b^2 = (a-b)(a+b)$, it is also direct that

$$
|\mathrm{D}_\Phi| \le \frac{\zeta_1}{2}\Big[\,\|A^{1/4}u^1(t)\| + \|A^{1/4}u^2(t)\|\,\Big]\|A^{1/4}w(t)\| \le K_B\|A^{1/4}w(t)\|. \tag{3.52}
$$

Using Mean Value Theorem in the function $\Psi(s) = |s|^{q-2}s$ and identity

$$
(A^{1/2}u^1, u_t^1) - (A^{1/2}u^2, u_t^2) = (A^{1/2}w,\ u_t) + (A^{1/2}u^2,\ w_t),
$$

we have

$$
|\mathrm{D}_\Psi| \le K_B\left[\,\Big|(A^{1/2}u^1, u_t^1)\Big|^{q-2} + \Big|(A^{1/2}u^2, u_t^2)\Big|^{q-2}\,\right]\Big[\,\|w_t(t)\| + \|A^{1/2}w(t)\|\,\Big]. \tag{3.53}
$$

Now, let us consider the perturbed functional

$$\mathcal{E}^{\epsilon}_{\beta,\kappa}(t) = \mathcal{E}_{\beta,\kappa}(t) + \epsilon\left(\mathcal{M}_\beta w_t, w\right) + 4\kappa\epsilon\left(\mathcal{M}_\beta w_t, A^{-\alpha/2}z\right). \tag{3.54}$$

From (3.54), we have

$$\begin{aligned}\mathcal{E}^{\epsilon}_{\beta,\kappa}(t) - \mathcal{E}_{\beta,\kappa}(t) &= \epsilon\,(w_t, w) + \epsilon\left(A^{\beta/4}w_t, A^{\beta/4}w\right)\\ &\quad + 4\epsilon\kappa\left(w_t, A^{-\alpha/2}z\right) + 4\epsilon\kappa\left(A^{\beta/4}w_t, A^{\beta/4-\alpha/2}z\right)\end{aligned}$$

Then, using that $W_2 \hookrightarrow W_\beta \hookrightarrow W_0 \hookrightarrow W_{-1} \hookrightarrow W_{\beta-2\alpha} \hookrightarrow W_{-2\alpha} \hookrightarrow W_{-2}$ we have

$$\begin{aligned}\left|\mathcal{E}^{\epsilon}_{\beta,\kappa}(t) - \mathcal{E}_{\beta,\kappa}(t)\right| &\leq \epsilon C_{0,\lambda_1}\|w_t(t)\|_{\mathcal{D}(\mathcal{M}_\beta^{1/2})}\|A^{1/2}w(t)\|\\ &+\epsilon C_{1,\lambda_1}\|w_t(t)\|_{\mathcal{D}(\mathcal{M}_\beta^{1/2})}\kappa^{1/2}\|z(t)\| \leq \epsilon C_{\lambda_1}\mathcal{E}_{\beta,\kappa}(t).\end{aligned}$$

Thus, taking $\epsilon \leq \frac{1}{2C_{\lambda_1}}$, we obtain

$$\frac{1}{2}\mathcal{E}_{\beta,\kappa}(t) \leq \mathcal{E}^{\epsilon}_{\beta,\kappa}(t) \leq \frac{3}{2}\mathcal{E}_{\beta,\kappa}(t). \tag{3.55}$$

Taking the derivative with respect to t in (3.54), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}^{\epsilon}_{\beta,\kappa}(t) &= \underbrace{\frac{d}{dt}\mathcal{E}_{\beta,\kappa}(t)}_{\mathcal{J}_1} + \underbrace{\epsilon\left(\mathcal{M}_\beta w_{tt}, w\right)}_{\mathcal{J}_2} + \epsilon\|w_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})}\\ &+ \underbrace{4\kappa\epsilon\left(\mathcal{M}_\beta w_t, A^{-\alpha/2}z_t\right)}_{\mathcal{J}_3} + \underbrace{4\kappa\epsilon\left(\mathcal{M}_\beta w_{tt}, A^{-\alpha/2}z\right)}_{\mathcal{J}_4}.\end{aligned} \tag{3.56}$$

In what follows we will estimate the terms $\mathcal{J}_1, \ldots, \mathcal{J}_4$ of equality (3.56). Firstly, taking the scalar product in W_0 of $(3.49)_1$ with w_t and $(3.49)_2$ with z, we get

$$\mathcal{J}_1 = -\|A^{1/4}z(t)\|^2 + \sum_{i=1}^{5}\mathcal{J}_{1,i},$$

where

$$\begin{aligned}\mathcal{J}_{1,1} &= -\frac{\zeta_1}{2}\sum_{i=1}^{2}(A^{1/2}u^i, u^i_t)\,\|A^{1/4}w(t)\|^2,\\ \mathcal{J}_{1,2} &= -\,\mathrm{D}_\Phi\left(A^{1/2}(u^1+u^2), w_t\right),\end{aligned}$$

$$\begin{aligned}\mathcal{J}_{1,3} &= -\,\mathrm{S}_\Psi\left(A^{1/2}w,\, w_t\right),\\ \mathcal{J}_{1,4} &= -\,\mathrm{D}_\Psi\left(A^{1/2}(u^1+u^2),\, w_t\right),\\ \mathcal{J}_{1,5} &= -\,(\mathrm{D}_f,\, w_t).\end{aligned}$$

Note that, it is direct that

$$\mathcal{J}_{1,1} \;\le\; K_B\|A^{1/4}w(t)\|^2.$$

Using (3.52) and Young inequality, also it is direct that

$$\mathcal{J}_{1,2} \le \frac{\epsilon}{2}\|w_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} + K_{B,\epsilon}\|A^{1/4}w(t)\|^2.$$

We define

$$\psi(t) = \left|(A^{1/2}u^1, u_t^1)\right|^q + \left|(A^{1/2}u^2, u_t^2)\right|^q.$$

From (3.51), definition of $\mathcal{E}_{\beta,\kappa}(t)$, (3.50) and Young inequality with $\frac{q-1}{q}+\frac{1}{q}=1$, we have

$$\mathcal{J}_{1,3} \le \left[\frac{\epsilon}{24} + K_\epsilon\psi(t)\right]\mathcal{E}_{\beta,\kappa}(t).$$

From (3.53), definition of $\mathcal{E}_{\beta,\kappa}(t)$, (3.50) and Young inequality with $\frac{q-2}{q}+\frac{2}{q}=1$, we obtain

$$\mathcal{J}_{1,4} \le \left[\frac{\epsilon}{24} + K_{B,\epsilon}\psi(t)\right]\mathcal{E}_{\beta,\kappa}(t).$$

When $n\le 4$ or $\beta=0$, using MVT, Assumption (2.15) and the Hölder inequality with $\frac{4}{2n}+\frac{n-2(2-\beta)}{2n}+\frac{n-2\beta}{2n}=1$, embedding $W_2\hookrightarrow L^{\frac{n\rho}{2}}(\Omega)$, $W_{2-\beta}\hookrightarrow L^{\frac{2n}{n-2(2-\beta)}}(\Omega)$ and $W_\beta\hookrightarrow L^{\frac{2n}{n-2\beta}}(\Omega)$, we obtain

$$\begin{aligned}\mathcal{J}_{1,5} &\le K_0\int_\Omega\left(1+|u^1(t)|^\rho+|u^2(t)|^\rho\right)|w(t)||w_t(t)|dx\\ &\le K_0\left[1+\|u^1(t)\|^\rho_{\frac{n\rho}{2}}+\|u^2(t)\|^\rho_{\frac{n\rho}{2}}\right]\|w(t)\|_{\frac{2n}{n-2(2-\beta)}}\|w_t(t)\|_{\frac{2n}{n-2\beta}(\Omega)}\\ &\le K_1\left[1+\|A^{1/2}u^1(t)\|^\rho+\|A^{1/2}u^2(t)\|^\rho\right]\|A^{(2-\beta)/4}w(t)\|\|A^{\beta/4}w_t(t)\|\\ &\le K_B\|A^{(2-\beta)/4}w(t)\|\|A^{\beta/4}w_t(t)\|\\ &\le \frac{\epsilon}{2}\|w_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})}+K_{B,\epsilon}\|A^{(2-\beta)/4}w(t)\|^2.\end{aligned}$$

When $n \geq 5$ and $\beta > 0$, it follows from $\rho < \frac{4+2\beta}{n-4}$ that there exists a $\delta : 0 < \delta \ll \beta$ such that $\rho \leq \frac{4+2\beta-2\delta}{n-4}$, which implies

$$\frac{2n\rho}{4+2\beta-2\delta} \leq \frac{2n}{n-4} \quad \text{and} \quad W_2 \hookrightarrow L^{\frac{2n\rho}{4+2\beta-2\delta}}(\Omega).$$

Thus, using MVT, Assumption (2.15) and the Hölder inequality with $\frac{4+2\beta-2\delta}{2n} + \frac{n-2(2-\delta)}{2n} + \frac{n-2\beta}{2n} = 1$, embedding $W_{2-\delta} \hookrightarrow L^{\frac{2n}{n-2(2-\delta)}}(\Omega)$ and $W_\beta \hookrightarrow L^{\frac{2n}{n-2\beta}}(\Omega)$, we obtain

$$\begin{aligned}
\mathcal{J}_{1,5} &\leq K_0 \left[1 + \|u^1(t)\|^{\rho}_{\frac{2n\rho}{4+2\beta-2\delta}} + \|u^2(t)\|^{\rho}_{\frac{2n\rho}{4+2\beta-2\delta}} \right] \|w(t)\|_{\frac{2n}{n-2(2-\delta)}} \|w_t(t)\|_{\frac{2n}{n-2\beta}(\Omega)} \\
&\leq K_1 \left[1 + \|A^{1/2}u^1(t)\|^{\rho} + \|A^{1/2}u^2(t)\|^{\rho} \right] \|A^{(2-\delta)/4}w(t)\| \|A^{\beta/4}w_t(t)\| \\
&\leq K_B \|A^{1/2}w(t)\|^{(\beta-\delta)/\beta} \|A^{(2-\beta)/4}w(t)\|^{\delta/\beta} \|A^{\beta/4}w_t(t)\| \\
&\leq \frac{\epsilon}{2} \|w_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} + \frac{\epsilon}{8} \|A^{1/2}w(t)\|^2 + K_{B,\epsilon} \|A^{(2-\beta)/4}w(t)\|^2.
\end{aligned}$$

Thus, substituting $\mathcal{J}_{1,1}, \ldots, \mathcal{J}_{1,5}$ in $\mathcal{J}_1$, we get

$$\begin{aligned}
\mathcal{J}_1 \leq &- \|A^{1/4}z(t)\|^2 + \epsilon \|w_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} + \frac{\epsilon}{8} \|A^{1/2}w(t)\|^2 \\
&+ K_{B,\epsilon} \left[\|A^{1/4}w(t)\|^2 + \|A^{(2-\beta)/4}w(t)\|^2 \right] + \left[\frac{\epsilon}{12} + K_{B,\epsilon}\psi(t) \right] \mathcal{E}_{\beta,\kappa}(t).
\end{aligned} \tag{3.57}$$

Now, taking Eq. $(3.49)_1$ into account, we can write the term $\mathcal{J}_2$ as follows

$$\mathcal{J}_2 = -\epsilon \|A^{1/2}w(t)\|^2 - \epsilon \mathrm{S}_\Phi \|A^{1/4}w(t)\|^2 + \sum_{i=1}^{5} \mathcal{J}_{2,i},$$

where

$$\begin{aligned}
\mathcal{J}_{2,1} &= -\epsilon \mathrm{D}_\Phi \left(A^{1/2}(u^1 + u^2), w \right), \\
\mathcal{J}_{2,2} &= -\epsilon \mathrm{S}_\Psi \|A^{1/4}w(t)\|^2, \\
\mathcal{J}_{2,3} &= -\epsilon \mathrm{D}_\Psi \left(A^{1/2}(u^1 + u^2), w \right), \\
\mathcal{J}_{2,4} &= \epsilon \left(A^{\alpha/2}z, w \right), \\
\mathcal{J}_{2,5} &= -\epsilon \left(\mathrm{D}_f, w \right).
\end{aligned}$$

From (3.52) and immersion $W_1 \hookrightarrow W_0$, we have

$$\mathcal{J}_{2,1} \leq K_B \|A^{1/4}w(t)\|^2.$$

From (3.51), it is direct that

$$\mathcal{J}_{2,2} \leq K_B \|A^{1/4}w(t)\|^2.$$

From (3.53), definition of $\mathcal{E}_{\beta,\kappa}(t)$, (3.50), immersion $W_2 \hookrightarrow W_1$ and Young inequality with $\frac{q-2}{q} + \frac{2}{q} = 1$, we obtain

$$\mathcal{J}_{2,3} \leq \left[\frac{\epsilon}{12} + K_{B,\epsilon}\psi(t) \right] \mathcal{E}_{\beta,\kappa}(t).$$

Using that $W_1 \hookrightarrow W_\alpha$, we have

$$\mathcal{J}_{2,4} = \epsilon \left(A^{\alpha/4}z, A^{\alpha/4}w \right) \leq \frac{\epsilon}{2\lambda_1^{-\alpha/2+1/2}} \left[\|A^{1/4}w(t)\|^2 + \|A^{1/4}z(t)\|^2 \right].$$

When $n \leq 4$ or $\beta = 0$, from MVT, Assumption (2.15) and Hölder inequality with $\frac{\rho}{2(\rho+1)} + \frac{\rho+2}{2(\rho+1)} = 1$, embedding $W_2 \hookrightarrow L^{2(\rho+1)}(\Omega)$ and $W_1 \hookrightarrow L^{\frac{4(\rho+1)}{\rho+2}}(\Omega)$, we have

$$\mathcal{J}_{2,5} \leq K[\, 1 + \|u^1(t)\|^\rho_{2(\rho+1)} + \|u^2(t)\|^\rho_{2(\rho+1)}\,]\|w(t)\|^2_{\frac{4(\rho+1)}{\rho+2}} \leq K_B \|A^{1/4}w(t)\|^2.$$

When $n \geq 5$ and $\beta > 0$, using MVT, Assumption (2.15) and Hölder inequality with $\frac{3}{n} + \frac{n-3}{n} = 1$ and embedding $W_{3/2} \hookrightarrow L^{\frac{n\rho}{3}}(\Omega)$ for

$$\frac{n\rho}{3} \leq \frac{n}{3} \cdot \frac{4+2\beta}{n-4} \leq \frac{n}{3} \cdot \frac{4\alpha+2}{n-4} \leq \frac{2n}{n-4},$$

we obtain

$$\begin{aligned} \mathcal{J}_{2,5} &\leq K[\, 1 + \|u^1(t)\|^\rho_{\frac{n\rho}{3}} + \|u^2(t)\|^\rho_{\frac{n\rho}{3}}\,]\|w(t)\|^2_{\frac{2n}{n-3}} \\ &\leq K_B \|A^{3/8}w(t)\|^2 \leq K_B \|A^{1/2}w(t)\| \|A^{1/4}w(t)\| \\ &\leq \frac{\epsilon}{8} \|A^{1/2}w(t)\|^2 + K_{B,\epsilon} \|A^{1/4}w(t)\|^2. \end{aligned}$$

Substituting $\mathcal{J}_{2,1}, \ldots, \mathcal{J}_{2,5}$ in $\mathcal{J}_2$, we obtain

$$\begin{aligned} \mathcal{J}_2 \;\leq & -\frac{7\epsilon}{8} \|A^{1/2}w(t)\|^2 - \epsilon \mathrm{S}_\Phi \|A^{1/4}w(t)\|^2 + K_{B,\epsilon} \|A^{1/4}w(t)\|^2 \\ & + \frac{\epsilon}{2\lambda_1^{-\alpha/2+1/2}} \|A^{1/4}z(t)\|^2 + \left[\frac{\epsilon}{12} + K_{B,\epsilon}\psi(t) \right] \mathcal{E}_{\beta,\kappa}(t). \end{aligned}$$

Using Eq. $(3.49)_2$ we get

$$\mathcal{J}_3 = -4\epsilon \|w_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} + \sum_{i=1}^{2} \mathcal{J}_{3,i},$$

where

$$\mathcal{J}_{3,1} = -4\epsilon\left(w_t, A^{1/2-\alpha/2}z\right),$$
$$\mathcal{J}_{3,2} = -4\epsilon\left(A^{\beta/4}w_t, A^{\beta/4-\alpha/2+1/2}z\right).$$

Using $W_1 \hookrightarrow W_{2-2\alpha}$, we have

$$\begin{aligned}\mathcal{J}_{3,1} &\le 4\epsilon\|w_t(t)\|\|A^{1/2-\alpha/2}z(t)\| \le \frac{4\epsilon}{\lambda_1^{-1/4+\alpha/2}}\|w_t(t)\|\|A^{1/4}z(t)\| \\ &\le \epsilon\|w_t(t)\|^2 + \frac{4\epsilon}{\lambda_1^{-1/2+\alpha}}\|A^{1/4}z(t)\|^2\end{aligned}$$

and, using $W_1 \hookrightarrow W_{\beta-2\alpha+2}$, we obtain

$$\begin{aligned}\mathcal{J}_{3,2} &\le 4\epsilon\|A^{\beta/4}w_t(t)\|\|A^{\beta/4-\alpha/2+1/2}z(t)\| \\ &\le \frac{4\epsilon}{\lambda_1^{-\beta/4+\alpha/2-1/4}}\|A^{\beta/4}w_t(t)\|\|A^{1/4}z(t)\| \\ &\le \epsilon\|A^{\beta/4}w_t(t)\|^2 + \frac{4\epsilon}{\lambda_1^{-\beta/2+\alpha-1/2}}\|A^{1/4}z(t)\|^2.\end{aligned}$$

Substituting $\mathcal{J}_{3,1}$, $\mathcal{J}_{3,2}$ in $\mathcal{J}_3$, we obtain

$$\mathcal{J}_3 \le -3\epsilon\|w_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} + \epsilon C_{\lambda_1}\|A^{1/4}z(t)\|^2.$$

Using Eq. (3.49)$_1$ we get

$$\mathcal{J}_4 = 4\kappa\epsilon\|z(t)\|^2 + \sum_{i=1}^{6}\mathcal{J}_{4,i},$$

where

$$\begin{aligned}\mathcal{J}_{4,1} &= -4\kappa\epsilon\left(Aw, A^{-\alpha/2}z\right),\\ \mathcal{J}_{4,2} &= -4\kappa\epsilon\mathrm{S}_\Phi\left(A^{1/2}w, A^{-\alpha/2}z\right),\\ \mathcal{J}_{4,3} &= -4\kappa\epsilon\mathrm{S}_\Psi\left(A^{1/2}w, A^{-\alpha/2}z\right),\\ \mathcal{J}_{4,4} &= -4\kappa\epsilon\mathrm{D}_\Phi\left(A^{1/2}(u^1+u^2), A^{-\alpha/2}z\right),\\ \mathcal{J}_{4,5} &= -4\kappa\epsilon\mathrm{D}_\Psi\left(A^{1/2}(u^1+u^2), A^{-\alpha/2}z\right),\\ \mathcal{J}_{4,6} &= -4\kappa\epsilon\left(\mathrm{D}_f, A^{-\alpha/2}z\right).\end{aligned}$$

Using that $W_1 \hookrightarrow W_{2-2\alpha} \hookrightarrow W_0$, we have

$$\begin{aligned}\mathcal{J}_{4,1} &= -4\kappa\epsilon\left(A^{1/2}w,\, A^{1/2-\alpha/2}z\right) \leq \frac{4\kappa\epsilon}{\lambda_1^{-1/4+\alpha/2}}\|A^{1/2}w(t)\|\|A^{1/4}z(t)\| \\ &\leq \frac{\epsilon}{4}\|A^{1/2}w(t)\|^2 + \frac{8\kappa^2\epsilon}{\lambda_1^{-1/2+\alpha}}\|A^{1/4}z(t)\|^2.\end{aligned}$$

From $W_0 \hookrightarrow W_{1-2\alpha} \hookrightarrow W_{-1}$ and using $|\mathrm{S}_\Phi|, |\mathrm{S}_\Psi| \leq K_B$, we obtain

$$\begin{aligned}\mathcal{J}_{4,2} + \mathcal{J}_{4,3} &= -4\kappa\epsilon(\mathrm{S}_\Phi + \mathrm{S}_\Psi)\left(A^{1/4}w,\, A^{1/4-\alpha/2}z\right) \\ &\leq \epsilon K_B\|A^{1/4}w(t)\|^2 + \kappa\epsilon\|z(t)\|^2.\end{aligned}$$

Using $W_0 \hookrightarrow W_{-1} \hookrightarrow W_{-2\alpha} \hookrightarrow W_{-2}$, we get

$$\mathcal{J}_{4,4} \leq K_B\|A^{1/4}w(t)\|^2 + \kappa\epsilon\|z(t)\|^2.$$

Finally, using $W_0 \hookrightarrow W_{-1} \hookrightarrow W_{-2\alpha} \hookrightarrow W_{-2}$, Young inequality with $\frac{q-2}{q} + \frac{2}{q} = 1$, we obtain

$$\mathcal{J}_{4,5} \leq \left[\frac{\epsilon}{12} + K_{B,\epsilon}\psi(t)\right]\mathcal{E}_{\beta,\kappa}(t).$$

Again from MVT, Assumption (2.15) and Hölder inequality with $\frac{2+4\alpha}{2n} + \frac{n-2}{2n} + \frac{n-4\alpha}{2n} = 1$, immersions $W_{2\alpha} \hookrightarrow L^{\frac{2n}{n-4\alpha}}(\Omega)$, $W_1 \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ and $W_2 \hookrightarrow L^{\frac{2n\rho}{2+4\alpha}}(\Omega)$ for

$$\frac{2n\rho}{2+4\alpha} \leq \frac{2n}{2+4\alpha}\cdot\frac{4+2\beta}{n-4} \leq \frac{2n}{n-4}\cdot\frac{4+2(2\alpha-1)}{2+4\alpha} = \frac{2n}{n-4} \quad \text{when} \quad n \geq 5,$$

we have

$$\begin{aligned}\mathcal{J}_{4,6} &\leq K\epsilon\kappa\left[1 + \|u^1(t)\|^\rho_{\frac{2n\rho}{2+4\alpha}} + \|u^2(t)\|^\rho_{\frac{2n\rho}{2+4\alpha}}\right]\|w(t)\|_{\frac{2n}{n-2}}\|A^{-\alpha/2}z(t)\|_{\frac{2n}{n-4\alpha}} \\ &\leq K\epsilon\kappa\left[1 + \|A^{1/2}u^1(t)\|^\rho + \|A^{1/2}u^2(t)\|^\rho\right]\|A^{1/4}w(t)\|\|z(t)\| \\ &\leq K_B\|A^{1/4}w(t)\|^2 + \kappa\epsilon\|z(t)\|^2.\end{aligned}$$

Returning to $\mathcal{J}_4$, we have

$$\begin{aligned}\mathcal{J}_4 \;&\leq\; \frac{\epsilon}{4}\|A^{1/2}w(t)\|^2 + 7\kappa\epsilon\|z(t)\|^2 + \frac{8\kappa^2\epsilon}{\lambda_1^{-1/2+\alpha}}\|A^{1/4}z(t)\|^2 \\ &+ \left[\frac{\epsilon}{12} + K_{B,\epsilon}\right]\mathcal{E}_{\beta,\kappa}(t) + K_B\|A^{1/4}w(t)\|^2. \qquad (3.58)\end{aligned}$$

Thus, substituting (3.57), $\cdots$, (3.58) in (3.56) and using $W_{2-\beta} \hookrightarrow W_1 \hookrightarrow W_0$, we obtain

$$\frac{d}{dt}\mathcal{E}^{\epsilon}_{\beta,\kappa}(t) \leq -\left(1-\epsilon K_{\lambda_1}\right)\|A^{1/4}z(t)\|^2 - \epsilon\|w_t(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} - \frac{\epsilon}{2}\|A^{1/2}w(t)\|^2$$
$$- \epsilon S_\Phi\|A^{1/4}w(t)\|^2 + \left[\frac{\epsilon}{4} + K_{B,\epsilon}\psi(t)\right]\mathcal{E}_{\beta,\kappa}(t) + K_B\|A^{(2-\beta)/4}w(t)\|^2.$$

Taking ϵ such that $1-\epsilon K_{\lambda_1} > \frac{\kappa\epsilon}{\lambda_1^{1/2}}$ and using that $-\|A^{1/4}z\|^2 \leq -\lambda_1^{1/2}\|z\|^2$, we obtain

$$\frac{d}{dt}\mathcal{E}^{\epsilon}_{\beta,\kappa}(t) + \left[\frac{\epsilon}{4} - K_{B,\epsilon}\psi(t)\right]\mathcal{E}_{\beta,\kappa}(t) \leq K_B\|A^{(2-\beta)/4}w(t)\|^2.$$

From (3.55), we obtain

$$\frac{d}{dt}\mathcal{E}^{\epsilon}_{\beta,\kappa}(t) + \Theta_\epsilon(t)\mathcal{E}^{\epsilon}_{\beta,\kappa}(t) \leq K_B\|A^{(2-\beta)/4}w(t)\|^2.$$

where $\Theta_\epsilon(t) = \frac{\epsilon}{6} - 2K_{B,\epsilon}\psi(t)$. By integrating from 0 to t, we obtain

$$\mathcal{E}^{\epsilon}_{\beta,\kappa}(t) \leq \mathcal{E}^{\epsilon}_{\beta,\kappa}(0)e^{-\int_0^t\Theta_\epsilon(s)ds} + K_B\int_0^t e^{-\int_s^t\Theta_\epsilon(\tau)d\tau}\|A^{(2-\beta)/4}w(s)\|^2 ds.$$

Using that $\psi \in L^1(0,t)$, we have

$$-\int_s^t \Theta_\epsilon(s)ds = -\frac{\epsilon}{4}(t-s) + K_{B,\epsilon}\int_s^t\psi(\tau)d\tau = -\frac{\epsilon}{4}(t-s) + K_B.$$

Thus,

$$\mathcal{E}^{\epsilon}_{\beta,\kappa}(t) \leq K_B\mathcal{E}^{\epsilon}_{\beta,\kappa}(0)e^{-\epsilon t/4} + K_B\int_0^t e^{-\epsilon(t-s)/4}\|A^{(2-\beta)/4}w(s)\|^2 ds.$$

Again from (3.55), we have

$$\mathcal{E}_{\beta,\kappa}(t) \leq 3K_B\mathcal{E}_{\beta,\kappa}(0)e^{-\epsilon t/4} + K_B\int_0^t e^{-\epsilon(t-s)/4}\|A^{(2-\beta)/4}w(s)\|^2 ds.$$

Finally, using (3.50) in the last inequality, we obtain (3.48) and the proof of Proposition 3.4 is now complete. □

3.3 Attractors and Their Properties

Our main results on global attractors for dynamical systems $(\mathcal{H}_{\beta,\kappa}, S_{\alpha,\beta,\kappa}(t))$ generated by (2.35) with $(\alpha,\beta,\kappa) \in \Upsilon = \left[\frac{1}{2},1\right] \times (0, 2\alpha-1] \times [0,1]$ are formulated below. To do so, we shall combine Proposition 3.4 with the abstracts concepts within the theory of infinite-dimensional dynamical systems, see e.g. [1, 11–13, 17, 23, 29, 34]. More specifically, we use the following notion of *quasi-stable* dynamical systems, accordingly to [13, Definition 7.9.2] which started with the prior work [11], restricted to our particular dynamical system.

Definition 3.1 The dynamical system $(\mathcal{H}_{\beta,\kappa}, S_{\alpha,\beta,\kappa}(t))$ generated by (2.35) is called to be *quasi-stable* on a set $B \subset \mathcal{H}_{\beta,\kappa}$ if there exist a compact seminorm $n_X(\cdot)$ on $X := W_2$ and nonnegative scalar functions $a(t)$ and $c(t)$ locally bounded in $[0, \infty)$, and $b(t) \in L^1(\mathbb{R}^+)$ with $\lim\limits_{t\to\infty} b(t) = 0$, such that

$$\|S_{\alpha,\beta,\kappa}(t)U_1 - S_{\alpha,\beta,\kappa}(t)U_2\|^2_{\mathcal{H}_{\beta,\kappa}} \le a(t)\|U_1 - U_2\|^2_{\mathcal{H}_{\beta,\kappa}}, \tag{3.59}$$

and

$$\begin{aligned} \|S_{\alpha,\beta,\kappa}(t)U_1 - S_{\alpha,\beta,\kappa}(t)U_2\|^2_{\mathcal{H}_{\beta,\kappa}} &\le b(t)\|U_1 - U_2\|^2_{\mathcal{H}_{\beta,\kappa}} \\ &+c(t) \sup_{s\in[0,t]} \Big[n_X(u^1(s) - u^2(s))\Big]^2, \end{aligned} \tag{3.60}$$

for any $U_1, U_2 \in B$, where we denote $S_{\alpha,\beta,\kappa}(t)U_i = (u^i(t), u^i_t(t), \theta^i(t)),\ i = 1, 2$.

The result of quasi-stability to the dynamical system $(\mathcal{H}_{\beta,\kappa}, S_{\alpha,\beta,\kappa}(t))$ defined in (2.35) is an immediate consequence of the Proposition 3.4.

Theorem 3.5 (Quasi-stability) *Under the assumptions of Proposition 3.4 with*

$$(\alpha, \beta, \kappa) \in \left[\frac{1}{2}, 1\right] \times (0, 2\alpha - 1] \times [0, 1],$$

the dynamical system $(\mathcal{H}_{\beta,\kappa}, S_{\alpha,\beta,\kappa}(t))$ generated by (2.35) is quasi-stable on any bounded positively invariant set $B \subset \mathcal{H}_{\beta,\kappa}$.

Proof Let $B \subset \mathcal{H}_{\beta,\kappa}$ be a bounded positively invariant set of $S_{\beta,\kappa}(t)$, $U_1, U_2 \in B$, and

$$S_{\beta,\kappa}(t)U_i = (u^i(t), u^i_t(t), \theta^i(t)),\ i = 1, 2, \quad u = u^1 - u^2.$$

Firstly, under the above notations, one sees promptly from (3.47) that (3.59) holds true with $a(t) = e^{\varrho_{0,B} t}$, being locally bounded in $[0, \infty)$. Then, setting

$$[n_X(u)]^2 := \|A^{(2-\beta)/4}u\|^2_2, \quad X = W_2,$$

and noting that embeddings $W_2 \hookrightarrow W_{2-\beta}$ are compact for $\beta \in (0, 2\alpha - 1]$ with $\alpha \in [1/2, 1]$, it follows that $n_X(\cdot)$ is a compact seminorm on X. Additionaly, from (3.48) one has

$$\|S_{\alpha,\beta,\kappa}(t)U_1 - S_{\alpha,\beta,\kappa}(t)U_2\|^2_{\mathcal{H}_{\beta,\kappa}} \le b(t)\|U_1 - U_2\|^2_{\mathcal{H}_{\beta,\kappa}} + c(t) \sup_{s\in[0,t]} [n_X(u(s))]^2,$$

where

$$b(t) = \varrho_{1,B} e^{-\epsilon t/4} \quad \text{and} \quad c(t) = \varrho_{2,B} \int_0^t e^{-\epsilon(t-s)/4} ds, \quad t \geq 0.$$

Thus, $b \in L^1(\mathbb{R}^+)$ with $\lim_{t\to\infty} b(t) = 0$, and $c(t)$ is globally bounded

$$c_\infty = \sup_{t\in\mathbb{R}^+} c(t) < \infty. \tag{3.61}$$

Hence, condition (3.60) also holds true, which completes the proof that $(\mathcal{H}_{\beta,\kappa}, S_{\alpha,\beta,\kappa}(t))$ is quasi-stable on any bounded positively invariant set in $\mathcal{H}_{\beta,\kappa}$. □

3.3.1 Global Attractor, Finite Dimensionality and Regularity

Theorem 3.6 *Let assumptions of Theorem 3.5 be valid. Then,*

(i) *Global attractor: the dynamical system $(\mathcal{H}_{\beta,\kappa}, S_{\alpha,\beta,\kappa}(t))$ generated by (2.35) possesses a compact global attractor $\{\mathfrak{A}_{\alpha,\beta,\kappa}\}_{(\alpha,\beta,\kappa)\in\left[\frac{1}{2},1\right]\times(0,2\alpha-1]\times[0,1]} = M^u_{\alpha,\beta,\kappa}(\mathcal{N}) \subset \mathcal{H}_{\beta,\kappa}$, where $M^u_{\alpha,\beta,\kappa}(\mathcal{N})$ is unstable manifold emanating from set $\mathcal{N}$ of stationary points.*

(ii) *Finite-dimensionality: the compact global attractor $\mathfrak{A}_{\alpha,\beta,\kappa}$ has finite fractal dimension $dim_f^{\mathcal{H}_{\beta,\kappa}} \mathfrak{A}_{\alpha,\beta,\kappa}$.*

(iii) *Regularity: any full trajectory $\chi = \{U(t) = (u(t), u_t(t), \theta(t));\ t \in \mathbb{R}\}$ from attractor $\mathfrak{A}_{\alpha,\beta,\kappa}$ enjoys the following regularity properties,*

$$\begin{aligned} &u_t \in L^\infty(\mathbb{R}; W_2) \cap C(\mathbb{R}, W_0), \quad u_{tt} \in L^\infty(\mathbb{R}; W_\beta), \quad \theta \in L^\infty(\mathbb{R}; W_0) \quad \text{if} \quad \kappa > 0, \\ &u_t \in L^\infty(\mathbb{R}; W_2) \cap C(\mathbb{R}, W_0), \quad u_{tt} \in L^\infty(\mathbb{R}; W_\beta), \quad \text{if} \quad \kappa = 0. \end{aligned} \tag{3.62}$$

Moreover, there exists $\mathfrak{R}_1 > 0$ such that

$$\|u_{tt}(t)\|^2_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} + \|A^{1/2}u_t(t)\|^2 + \kappa\|\theta_t(t)\|^2 \leq \mathfrak{R}_1^2, \quad t \in \mathbb{R}, \quad \forall \kappa \in [0, 1]. \tag{3.63}$$

where $\mathfrak{R}_1$ depends on the constant c_∞ on the seminorm η_X in Theorem 3.5.

Proof (i) It follows from Proposition 2.2 and Corollary 3.3 that the system $(\mathcal{H}_{\beta,\kappa}, S_{\alpha,\beta,\kappa}(t))$ is gradient and dissipative, respectively, and, from Proposition 3.4 that the system is quasi-stable. Therefore using [13, Proposition 7.9.4, Corollary 7.9.5 and Theorem 7.2.3] it follows that the dynamical system $(\mathcal{H}_{\beta,\kappa}, S_{\alpha,\beta,\kappa}(t))$ possesses a compact global attractor $\mathfrak{A}_{\alpha,\beta,\kappa}$ such that $\mathfrak{A}_{\alpha,\beta,\kappa} = M^u_{\beta,\kappa}(\mathcal{N}) \subset \mathcal{H}_{\beta,\kappa}$.

Proof (ii) It follows from [13, Theorem 7.9.6], because $(\mathcal{H}_{\beta,\kappa}, S_{\alpha,\beta,\kappa}(t))$ possesses a compact global attractor $\mathfrak{A}_{\alpha,\beta,\kappa}$ and is quasi-stable on $\mathfrak{A}_{\alpha,\beta,\kappa}$ by virtue of Theorem 3.5.

Proof (iii) From (i) the dynamical system $(\mathcal{H}_{\beta,\kappa}, S_{\alpha,\beta,\kappa}(t))$ possesses a compact global attractor $\mathfrak{A}_{\alpha,\beta,\kappa}$ and is quasi-stable on the attractor $\mathfrak{A}_{\alpha,\beta,\kappa}$. Since (3.60) holds with the function $c(t)$ possessing the property (3.61), it follows immediately from [13, Theorem 7.9.8] that any full trajectory $\chi(t) = \{U(t) = (u(t), u_t(t), \theta(t)); t \in \mathbb{R}\} \subset \mathcal{H}_{\beta,\kappa}$ from attractor $\mathfrak{A}_{\alpha,\beta,\kappa}$ satisfies (3.62)–(3.63). □

3.3.2 Upper Semicontinuity of Attractors

Now we will prove that the family of global attractors $\{\mathfrak{A}_{\alpha,\beta,\kappa}\}_{(\alpha,\beta,\kappa)\in\left[\frac{1}{2},1\right]\times(0,2\alpha-1]\times[0,1]}$ to problem (2.12)–(2.13) is upper semicontinuous at $(\alpha_0, \beta_0, \kappa_0) \in \left[\frac{1}{2}, 1\right] \times (0, 2\alpha - 1] \times [0, 1]$.

Theorem 3.7 (Upper semicontinuity) *Let assumptions of Theorem 3.5 be valid. Then, the family of the attractors* $\{\mathfrak{A}_{\alpha,\beta,\kappa}\}_{(\alpha,\beta,\kappa)\in\left(\frac{1}{2},1\right]\times[0,2\alpha-1]\times[0,1]}$ *to problem* (2.12)–(2.13) *in the space* $\mathcal{H}_{\beta,\kappa}$ *is upper semi-continuous on* $\left[\frac{1}{2}, 1\right] \times (0, 2\alpha - 1] \times [0, 1]$ *in the sense that*

(i) *for any* $\mu_0 = (\alpha_0, \beta_0, \kappa_0) \in \left[\frac{1}{2}, 1\right] \times (0, 2\alpha - 1] \times [0, 1]$ *we have that*

$$\lim_{(\alpha,\beta,\kappa)\to\mu_0} \sup_{y\in\mathfrak{A}_{\alpha,\beta,\kappa}} dist_{\mathcal{H}_{\beta_0,\kappa_0}}(y, \mathfrak{A}_{\mu_0}) = 0 \quad and \tag{3.64}$$

(ii) *for any* $\mu_0 = (\alpha_0, \beta_0, 0) \in \left[\frac{1}{2}, 1\right] \times (0, 2\alpha - 1] \times [0, 1]$ *we have*

$$\lim_{(\alpha,\beta,\kappa)\to\mu_0} \sup_{y\in\mathfrak{A}_{\alpha,\beta,\kappa}} dist_{\mathcal{H}_{\beta_0,1}}(y, \widehat{\mathfrak{A}}_{\mu_0}) = 0, \tag{3.65}$$

where $\widehat{\mathfrak{A}}_{\mu_0} = \{(u_0, u_1, -u_1) \in \mathcal{H}_{\alpha_0,\beta_0,1};\ (u_0, u_1) \in \mathfrak{A}_{\alpha_0,\beta_0,0}\}$.

Proof Firstly, using Eq. $(2.12)_2$, immersion $W_2 \hookrightarrow W_{2\alpha}$ and regularity (3.63), we get

$$\begin{aligned} ||A^{1/2}\theta(t)\| =\ & \|\kappa\,\theta_t(t) + A^{\alpha/2}u_t(t)\| \le \kappa\|\theta_t(t)\| \\ & + \frac{1}{\lambda_1^{-\alpha/2+1/2}}\|A^{1/2}u_t(t)\| \le C_{1,\mathfrak{R}}, \quad t \in \mathbb{R} \end{aligned} \tag{3.66}$$

where the constant $C_{1,\mathfrak{R}}$ is independent of α, β, and κ. Now, using Eq. $(2.12)_1$, Assumption (2.15), (3.63), immersion $W_2 \hookrightarrow W_{2\alpha}$ and (3.66), we obtain

$$\begin{aligned} ||Au(t)\| \le \|u_{tt}(t)\|_{\mathcal{D}(\mathcal{M}_\beta^{1/2})} + |\mathrm{B}(u, u_t)|\,\|A^{1/2}u(t)\| + \|f(u)\| \\ + \|A^{\alpha/2}\theta(t)\| + \|h\| \le C_{2,\mathfrak{R}}, \end{aligned} \tag{3.67}$$

$t \in \mathbb{R}$, where the constant $C_{2,\mathfrak{R}}$ is also independent of α, β, and κ. Finally, from (3.63), (3.66) and (3.67) we deduce

$$\sup_{t\in\mathbb{R}} \left\{\|Au(t)\|^2 + \|A^{1/2}u_t(t)\|^2 + \kappa\|A^{1/2}\theta(t)\|^2\right\} \le C_{\mathfrak{R}}. \tag{3.68}$$

This showed that the attractor $\mathfrak{A}_{\alpha,\beta,\kappa}$ is a bounded set in the space $W_4 \times W_2 \times W_2$.

We argue by contradiction in order to prove (3.64) (or (3.65) in the case $\kappa_* = 0$). if (3.64) (or (3.65)) does not hold. Then there exist a positive constant δ, a sequence $\mu_n \equiv (\alpha_n, \beta_n, \kappa_n)$ such that

$$\mu_n \longrightarrow \mu_* = (\alpha_*, \beta_*, \kappa_*)$$

and a sequence $U^n = (u^n, u_t^n, \theta^n) \in \mathfrak{A}_{\alpha_n,\beta_n,\kappa_n}$ such that

$$\mathrm{dist}_{\mathcal{H}_{\beta_*,\kappa_*}}\left(U^n, \mathfrak{A}_{\alpha_*,\beta_*,\kappa_*}\right) \geq \delta > 0, \quad n = 1, 2, \ldots. \tag{3.69}$$

Let $\chi^n = \{(u^n(t), u_t^n(t), \theta^n(t)); t \in \mathbb{R}\}$ be a full trajectory from the attractor $\mathfrak{A}_{\alpha_n,\beta_n,\kappa_n}$ with $U^n(0) = U^n$. Noting that embeddings $W_4 \hookrightarrow W_2$, $W_2 \hookrightarrow \mathcal{D}(\mathcal{M}_\beta^{1/2})$ and $W_2 \hookrightarrow W_0$ are compact, it follows from (3.68) and Aubin's compactness theorem that there exists a sequence $\{n_k\}$ and a function $U(t) \in C_{bnd}(\mathbb{R}; \mathcal{H}_{\beta_*,\kappa_*})$ such that

$$\lim_{k\to\infty} \max_{t\in[-T,T]} ||U^{n_k}(t) - U(t)||_{\mathcal{H}_{\beta_*,\kappa_*}} = 0.$$

Then, taking in (2.12) in the case $\kappa_* > 0$ (or (2.13) in the case $\kappa_* = 0$) the limit as

$$\mu_{n_k} \longrightarrow \mu_*$$

one as see that $U(t) = (u(t), u_t(t), \theta(t))$ solves the Eq. (2.12) in the case $\kappa_* > 0$ (or (2.13) in the case $\kappa_* = 0$). Since $\|U(t)\|_{\mathcal{H}_{\beta_*,\kappa_*}} \leq R$ for all $t \in \mathbb{R}$ and some $R > 0$ the trajectory $\chi = \{(u(t), u_t(t), \theta(t)); t \in \mathbb{R}\}$ belongs to the attractor $\mathfrak{A}_{\alpha_*,\beta_*,\kappa_*}$. Therefore, $U^{n_k} \to U(0) \in \mathfrak{A}_{\alpha_*,\beta_*,\kappa_*}$ which is contradict (3.69). ◻

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