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Research Article

Xin-Guang Yang*, Shubin Wang, and Marcio A. Jorge Silva

Lamé system with weak damping and nonlinear time-varying delay

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Abstract: This article is concerned with the stability and dynamics for the weak damped Lamé system with nonlinear time-varying delay in a bounded domain. Under some appropriate assumptions, the global well-posedness and asymptotic stability are shown in the case where the delay coefficient is upper dominated by the damping one. Moreover, the finite dimensional global and exponential attractors have also been presented by relying on quasi-stability arguments. The results in this article is an extension of Ma, Mesquita, and Seminario-Huertas's recent work [*Smooth dynamics of weakly damped Lamé systems with delay*, SIAM J. Math. Anal. **53** (2021), no. 4, 3759–3771].

Keywords: Lamé system, nonlinear time-varying delay, quasi-stability

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1 Introduction

The Lamé system is a classical model to describe the conservation law for isotropic elasticity, which can be represented as follows:

$$\partial_{tt}u(x,t) - \mu\Delta u(x,t) - (\lambda+\mu)\nabla \operatorname{div} u(x,t) = 0 \quad \text{in } \Omega \times \mathbb{R}^+,$$

where $u = (u_1, u_2, u_3)$ stands for the displacements of the elastic body in a bounded domain $\Omega \subset \mathbb{R}^3$, the positive coefficients μ and λ are the Lamé constants, and the physical background can be seen in, for instance, [9,10,14,28,36,38] and literatures therein. For mathematical analysis of damped Lamé system, especially the stabilization and control, we refer the readers to [1,6,7,23,29].

Physical reasons, non-instant transmission phenomena, memory processes, and specific biological motivations make retarded differential equations as an important area in applied mathematics. The effect of delay was first investigated from mathematical analysis in differential equations, and applications to control theory and engineering, see [11,12,18,20,21,24,35,40] and references therein. The investigation of stabilization and control of wave equations featuring delay effects acting on the boundary or interior can be seen in that studies of Nicaise and Pignotti [31,32], Nicaise et al. [33,34], which also give the damping-delay interaction. Moreover, the delay on hyperbolic systems has been considered, for instance, viscoelastic equations with delay in [15,17,18,22,26,39], Breese-Timoshenko system with delay in [19], and wave equation with memory in [3,8].

Although fruitful results on the hyperbolic systems with linear or distributed delays have been obtained in related literatures, as our best acknowledge, there are still less results on the nonlinear or unbounded delays

*** Corresponding author: Xin-Guang Yang**, Department of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, P. R. China, e-mail: yangxinguangyxg@163.com
Shubin Wang: School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, P. R. China, e-mail: wangshubin@zzu.edu.cn
Marcio A. Jorge Silva: Department of Mathematics, State University of Londrina, Londrina 86057-970, PR, Brazil, e-mail: marcioajs@uel.br

acting on hyperbolic equations, which is our motivation for this research. This article is presented to investigate the stability and dynamics for the following Lamé system with weak damping and nonlinear time-varying delay in a bounded domain $x \in \Omega \subset \mathbb{R}^3$ and $t \in \mathbb{R}^+$:

$$\begin{cases} \partial_{tt}u - \Delta_e u + \alpha_1\partial_t u + \alpha_2\partial_t u(x, t - \rho(t)) + f(u(x, t)) = h(x) \quad \text{in } \Omega \times \mathbb{R}^+, \\ u(x, t) = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x) \quad \text{in } \Omega, \\ \partial_t u(x, t - \rho(t)) = g_0(x, t - \rho(t)) \quad \text{in } \Omega \times (0, \rho(t)), \end{cases} \tag{1.1}$$

where Δ_e denotes the Lamé operator defined as

$$-\Delta_e u = -[\mu\Delta u + (\lambda + \mu)\nabla \operatorname{div} u]; \tag{1.2}$$

α_1 and α_2 denote positive parameters; h be the source term; u_0, u_1, and g_0 represent the general initial and delay conditions, respectively; and the function $\rho(t) : \mathbb{R}^+ \to \mathbb{R}^+$ denotes the nonlinear time-varying delay. If $\rho(t)$ reduces to constant, then what we consider is the same as in the study by Ma et al. [29]. Moreover, if a memory term is added to equation (1.1), $\lambda + \mu = 0$ and nonlinear term $f(u(t, x)) = 0$, in (1.1), then equation (1.1) is the same as in [39].

In order to deal with the nonlinear time-varying delay term in equation (1.1), originated from the techniques proposed by Conti and Pata [15], Dafermos [16], Datko [18], Nicaise and Pignotti [31,32], Ma et al. [29], and Yang et al. [39], we introduce the following new unknown variable to represent the delay term:

$$z(x, \eta, t) = \partial_t u(x, t - \eta\rho(t)). \tag{1.3}$$

Then, $z(x, \eta, t)$ satisfies

$$\begin{cases} \rho\partial_t z(x, \eta, t) + (1 - \eta\rho')\partial_\eta z(x, \eta, t) = 0 \quad \text{in } \Omega \times (0, 1) \times \mathbb{R}^+, \\ z(x, 1, t) = \partial_t u(x, t - \rho(t)) \quad \text{in } \Omega \times \mathbb{R}^+, \\ z(x, 0, t) = \partial_t u(x, t) \quad \text{in } \Omega \times \mathbb{R}^+, \\ z(x, \eta, 0) = g_0(x, -\eta\rho(0)) \quad \text{in } \Omega. \end{cases} \tag{1.4}$$

Using equations (1.3) and (1.4), the system (1.1) can be written as equivalent form as follows:

$$\begin{cases} \partial_{tt}u(x, t) - \Delta_e u(t, x) + \alpha_1\partial_t u(x, t) + \alpha_2 z(x, 1, t) + f(u(x, t)) = h(x) \quad \text{in } \Omega \times \mathbb{R}^+, \\ \rho\partial_t z(x, \eta, t) + (1 - \eta\rho')\partial_\eta z(x, \eta, t) = 0 \quad \text{in } \Omega \times (0, 1) \times \mathbb{R}^+, \\ u(x, t) = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x) \quad \text{in } \Omega, \\ z(x, 0, t) = \partial_t u(x, t) \quad \text{in } \Omega \times \mathbb{R}^+, \\ z(x, \eta, 0) = g_0(x, -\eta\rho(0)) \quad \text{in } \Omega \times (0, 1). \end{cases} \tag{1.5}$$

Our analysis in this article is to study the well-posedness, stability, and dynamics of the Lamé system (1.5), since the delay term is a source that may destabilize the asymptotic stability of an evolutionary system. The main results and features of this article are summarized as follows:

(I) Since the delay in equation (1.5) is nonlinear, the semigroup approach is not easy to be used as in lot of literatures to obtain the well-posedness. The new Dafermos transformation and phase space with delay are used to deal with the nonlinear delay firstly. Then, the global well-posedness and stability are obtained by Galerkin's approximation method and compact argument similar as in [25], see Theorem 3.1. Using the multiplier method, the estimates of energy functional has been shown under assumption on generic nonlinear term $f(u)$, see Theorem 4.1. Moreover, the finite dimensional global and exponential attractors have been derived by verifying the quasi-stability of gradient system for equation (1.5), and the structure of global attractor has been described by unstable manifold of equilibrium, see Theorem 5.4. In comparison to [39], because of the nonlinear increasing term $f(u)$, we cannot derive the exponential decay of total energy as in [39] and only present the uniform boundedness and monotone of energy functional $\mathcal{E}(t)$.

(II) The interface between nonlinear delay and weak damping is a key condition for establishing the global well-posedness, asymptotic stability of energy functional, and quasi-stability of gradient system, i.e., $\alpha_1 > \alpha_2$

and $\frac{\rho\alpha_2}{1-\rho'} < \xi < \rho(2\alpha_1 - \alpha_2)$. Conversely, as in [26,33,34], the total energy functional is instable for equation (1.1) with nonlinear time-varying delay $\rho(t)$ for $\alpha_1 \le \alpha_2$.

(III) The main difficulty here is the balance between time-varying delay and nonlinear $f(u)$ in energy estimates, uniform boundedness of energy functional, and quasi-stability of gradient system. In comparison to [39], our system contains a nonlinear term $f(u)$, which complicates the application of multiplier technique, as well as the balance between constants of weak damping and time-varying delay. Moreover, we also modified some unreasonable hypotheses on the delay term proposed by Yang et al. [39].

(IV) From mathematical analysis, our results in this article is an extension of Ma et al.'s recent work [29], which also cover the work by Yang et al. [39]. Since the Lamé system models seismic waves composed of P-waves and S-waves, the nonlinear delay in equation (1.1) is more prevalent than the linear case from physical interpretation, which means the motion of wave is not only dependent on the current state, but also the past, such as the delay and memory effect see [4,9,10,14,28,36,38].

The structure of this article is organized as follows: in Section 2, the preliminaries have been listed for preparation; the well-posedness has been presented in Section 3; in Section 4, the asymptotic stability and energy estimates are also shown; and the dynamics for gradient systems is proved in Section 5.

2 Preliminaries

2.1 Assumptions

To proceed with the analysis of our problem, the following assumptions is forced on time-varying delay term:

(H1) The nonlinear time-varying delay $\rho(\cdot)$ is a C^1-function satisfying

$$0 < \rho(t) \le \rho_1, \quad 0 < \rho'(t) \le \tilde{\rho}_0 < 1 \quad \text{for } t \ge 0, \tag{2.1}$$

such as the choice of $\rho(s) = \frac{e^{-s}}{2}$.

We also make the following assumptions on the external force h and generic nonlinear term $f = (f_1, f_2, f_3)$:

(H2) The external force $h \in (L^2(\Omega))^3$, e.g., $h(x) = B\sin x$ with positive constant $B > 0$.

(H3) For each vector field $u = (u_1, u_2, u_3)$, denote $f_i(u) = f_i(u_i)$ for $i = 1, 2, 3$, we assume that $f_i \in C^2(\mathbb{R})$, which has at most critical growth in Sobolev embedding with three dimension, i.e., $|f_i(s)|$ has at most cubic growth as in that study by Arrieta et al. [2] and Cavalcanti et al. [8]. Moreover, there exists $c_f > 0$ such that

$$|f_i''(u)| \le c_f(1 + |u|), \quad u \in \mathbb{R}, \ i = 1, 2, 3. \tag{2.2}$$

In addition, there exist positive constants $m, m_f > 0$ such that

$$F(u) = \sum_{i=1}^{3}\int_0^{u_i} f_i(s)\mathrm{d}s \ge -\frac{\tilde{m}}{2}|u|^2 - m_f \tag{2.3}$$

and

$$f(u)\cdot u \ge -\tilde{m}|u|^2 - m_f \tag{2.4}$$

hold for the vector field $u \in \mathbb{R}^3$ and $0 < \tilde{m} < \mu\lambda_1$, where λ_1 is the first eigenvalue of the Laplacian operator $-\Delta$ in $H_0^1(\Omega)$.

Examples for the nonlinear term $f(u)$: It is easy to verify that $f(u) = u^3$ and $f(u) = u\sin u$ satisfy the assumption (H3).

(H4) Moreover, the viscosity coefficients satisfy

$$\mu + \lambda \ge 0 \quad \text{and} \quad \mu > 0.$$

(H5) In addition, we assume that

$$\alpha_1 > \alpha_2. \tag{2.5}$$

2.2 Functional settings

Some properties of the Lamé operator $-\Delta_e$ and the energy space for the global solution are presented in this section for preparation. The spectrum analysis of $-\Delta_e$ and its corresponding domain have been considered in the study by Belishev and Lasiecka [6], which has been used for the construction of energy space similar as in [7,13,19,29,39] and literatures therein.

• **Functional spaces**:

Throughout this article, the norm for $L^p(\Omega)$ $(p \geq 1)$ is defined as follows:

$$\|\cdot\|_p^p = \int_\Omega |\cdot|^p \mathrm{d}x \quad \text{for } \cdot \in L^p(\Omega) \quad \text{and} \quad \|u\|_p^p = \sum_{i=1}^{3} \|u_i\|_p^p \quad \text{for } u = (u_1, u_2, u_3) \in (L^p(\Omega))^3.$$

The generic inner product for $u, v \in (L^2(\Omega))^3$ is defined as $(u, v) = \sum_{i=1}^{3}(u_i, v_i)$, where $(u_i, v_i) = \int_\Omega u_i v_i \mathrm{d}x$. Analogously, the inner product of $(H_0^1(\Omega))^3$ is

$$(\nabla u, \nabla v) = \sum_{i=1}^{3} (\nabla u_i, \nabla v_i),$$

with $(\nabla u_i, \nabla v_i) = \int_\Omega \nabla u_i \cdot \nabla v_i \mathrm{d}x$ for $u, v \in (H_0^1(\Omega))^3$, see, e.g., [13,37] and related literatures.

• **The Lamé operator**:

To deal with the Lamé operator $-\Delta_e$ defined in equation (1.2), one defines the inner product and norm of corresponding operator $-\Delta_e$ in $(H_0^1(\Omega))^3$ as follows:

$$(u, v)_e = \mu(\nabla u, \nabla v) + (\lambda + \mu)(\operatorname{div} u, \operatorname{div} v) \tag{2.6}$$

and

$$\|u\|_e^2 = \mu\|\nabla u\|_2^2 + (\lambda + \mu)\|\operatorname{div} u\|_2^2, \tag{2.7}$$

respectively. The norm (2.7) is equivalent to the generic one of $(H_0^1(\Omega))^3$ and satisfies

$$\mu\|\nabla u\|_2^2 \leq \|u\|_e^2 \leq a_0\|\nabla u\|_2^2$$

for $u \in (H_0^1(\Omega))^3$, where $a_0 = \max\{\mu, 3(\lambda + \mu)\}$, see the literature [6].

Definition 2.1. (See [6]) The operator $-\Delta_e$ is defined by

$$(-\Delta_e u, v) = (u, v)_e \tag{2.8}$$

for $u \in D(-\Delta_e)$ and $v \in (H_0^1(\Omega))^3$, where $D(-\Delta_e) = (H^2(\Omega) \cap H_0^1(\Omega))^3$.

From equations (2.6) and (2.7), the Lamé operator is positive and self-adjoint in $(L^2(\Omega))^3$. Hence, there exist eigenvalues $\tilde{\lambda}_i$ and its corresponding eigenfunctions $\{e_i\}_{i=1}^\infty \in (H_0^1(\Omega))^3$ such that

$$\begin{cases} -\Delta_e e_i = \tilde{\lambda}_i e_i & \text{in } \Omega, \\ e_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

where $0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$ are the corresponding eigenvalues. Moreover, $\{e_i\}_{i=1}^\infty$ is the orthonormal basis for $(H_0^1(\Omega))^3$ and $(L^2(\Omega))^3$, see [6,14,29] for more details. The problem (2.9) implies

$$(-\Delta_e)u = \sum_{i=1}^\infty \tilde{\lambda}_i (u, e_i) e_i$$

for $u \in D(-\Delta_e)$. The fractional power of the Lamé operator is defined as

$$(-\Delta_e)^{\alpha}u = \sum_{i=1}^{\infty}\tilde{\lambda}_i^{\alpha}(u, e_i)e_i$$

for $u \in D((-\Delta_e)^{\alpha})$ with $\alpha \in (0, 1)$, especially, $D((-\Delta_e)^{1/2}) = ((H_0^1(\Omega))^3, \|\cdot\|_e)$.

- **Energy spaces**:

Assume that $\xi > 0$ satisfies

$$\frac{\rho\alpha_2}{1-\rho'} < \xi < \rho(2\alpha_1 - \alpha_2) \tag{2.10}$$

be fixed such that for the delay term $y = y(x, \eta, t), z = z(x, \eta, t) \in (L^2(\Omega \times (0, 1)))^3$, and we can define the inner product and norm as

$$(z, y)_{\xi} = \xi\int_{\Omega}\int_{0}^{1} z(x, \eta, t)\cdot y(x, \eta, t)\mathrm{d}\eta\mathrm{d}x$$

and

$$\|z\|_{\xi}^2 = \xi\int_{\Omega}\int_{0}^{1}|z(x, \eta, t)|^2\mathrm{d}\eta\mathrm{d}x$$

respectively, see the similar notations in the study by Yang et al. [39]. Moreover, the assumption (2.1) assume that equation (2.10) and the above definition is reasonable.

The energy space of global solution to system (1.5) can be defined as the following Hilbert space:

$$\mathcal{H} = D((-\Delta_e)^{1/2}) \times (L^2(\Omega))^3 \times (L^2(\Omega \times (0, 1)))^3 \tag{2.11}$$

endowed with inner product

$$((u_1, v_1, z_1), (u_2, v_2, z_2))_{\mathcal{H}} = (u_1, u_2)_e + (v_1, v_2) + (z_1, z_2)_{\xi}$$

and norm

$$\|(u, v, z)\|_{\mathcal{H}}^2 = \|u\|_e^2 + \|v\|_2^2 + \|z\|_{\xi}^2.$$

3 Global well-posedness: existence and uniqueness

Next, we will use Faedo-Galerkin approximation method and compact argument to prove the well-posedness, which needs a lemma to verify the initial data.

Lemma 3.1. (See Ball [5]) *Let X and X_0 be two Banach spaces, $X = (X_0)'$. If*

$$\begin{aligned} &u_m \overset{*}{\rightharpoonup} u \quad \text{in } L^p(0, T; X),\\ &\partial_t u_m \overset{*}{\rightharpoonup} \partial_t u \quad \text{in } L^p(0, T; X) \end{aligned}$$

for $p \in (1, \infty]$, then $u_n(0) \overset{*}{\rightharpoonup} u(0)$ in X.

Theorem 3.1. *Under the hypotheses (H1)–(H5) and the given initial data $(u_0, u_1, g_0) \in \mathcal{H}$, the problem (1.5) possesses a unique global weak solution $(u, \partial_t u, z)$, which satisfies*

$$(u, \partial_t u, z) \in C([0, T]; \mathcal{H}). \tag{3.1}$$

for arbitrary $T > 0$.

Moreover, for a bounded set $B \subset \mathcal{H}$, there exists a positive constant $C = C(T, B)$ such that it yields the continuous dependence on initial data as follows:

$$\|Z^1(t) - Z^2(t)\|_{\mathcal{H}}^2 \le C(T, B)\|Z_0^1 - Z_0^2\|_{\mathcal{H}}^2 \tag{3.2}$$

for $t \in [0, T]$ *and any initial data* $Z_0^i \in B$ *with* $i = 1, 2$, *where* $Z^i(t) = (u^i, \partial_t u^i, z^i)$ *is the weak solution.*

Proof. The proof is divided into the following steps.

Step 1: Local solution of Galerkin's approximated equations

Let $W_m = \text{span}\{e_1, ..., e_m\}$ is the finite dimensional subspace of $(H_0^1(\Omega))^3 = D((-\Delta_e)^{1/2})$, since $(H_0^1(\Omega))^3$ is dense in $(L^2(\Omega))^3$, there exists two sequences $\{u_{0m}\}$ and $\{u_{1m}\}$ in W_m such that

$$u_{0m} = \sum_{i=1}^m a_{im} e_i \to u_0 \quad \text{in } (H_0^1(\Omega))^3, \tag{3.3}$$

$$u_{1m} = \sum_{i=1}^m \beta_{im} e_i \to u_1 \quad \text{in } (L^2(\Omega))^3 \tag{3.4}$$

as $m \to \infty$. Let $\{\omega_i(x, \eta)\}_{i=1}^\infty$ be the orthonormal basis for the Hilbert space $(L^2(\Omega \times [0, 1]))^3$ satisfying $\omega_i(x, 1) = e_i(x)$, and let $V_m = \text{span}\{\omega_1, \omega_2, ..., \omega_m\}$ is the m-dimensional subspace of $(L^2(\Omega \times [0, 1]))^3$. Similarly, we can choose a sequence $\{z_{0m} = \sum_{i=1}^m \gamma_{im}\omega_i\}$ such that

$$z_{0m} \to g_0(x, -\eta\rho(0)) \tag{3.5}$$

in $(L^2(\Omega \times [0, 1]))^3$ with $z_{0m} \in V_m$ as $m \to \infty$.

By the definition

$$u_m(t, x) = \sum_{j=1}^n \alpha_j^m(t) e_j(x), \quad z_m(t, x, \eta) = \sum_{j=1}^n \beta_j^m(t)\omega_j(x, \eta),$$

we can construct the approximated solution $(u_m(x, t), z_m(t, x, \eta))$ with

$$\alpha_j^m(t) = (u_m(t, x), e_j(x)), \quad \beta_j^m(t) = z_m(t, x, \eta), \omega_j(x, \eta).$$

Thus, $\alpha_j^m(t)$ and $\beta_j^m(t)$ satisfy the Cauchy problem of functional differential equations

$$\begin{cases} \ddot{\alpha}_j^m(t)\int_\Omega e_j \cdot e_j \mathrm{d}x + \alpha_j^m(t)\left[\mu\int_\Omega \nabla e_j \cdot \nabla e_j \mathrm{d}x + (\lambda + \mu)\int_\Omega \operatorname{div} e_j \cdot \operatorname{div} e_j \mathrm{d}x\right] \\ \quad + \alpha_1\dot{\alpha}_j^m(t)\int_\Omega e_j \cdot e_j \mathrm{d}x + \alpha_2\beta_j^m(t)\int_\Omega \omega_j(x, 1)e_j \mathrm{d}x + \left(f\left(\sum_{j=1}^m \alpha_j^m(t)e_j\right), e_j\right) = \int_\Omega h(x)e_j \mathrm{d}x, \\ \dot{\alpha}_j^m(t) = \sum_{j=1}^m \beta_j^m(t)\int_\Omega \omega_j(x, 0)e_j \mathrm{d}x, \\ \alpha_j^m(0) = \int_\Omega u_{0m} \cdot e_j \mathrm{d}x, \quad \dot{\alpha}_j^m(0) = \int_\Omega u_{1m} \cdot e_j \mathrm{d}x \end{cases} \tag{3.6}$$

and

$$\begin{cases} \rho(t)\dot{\beta}_j^m(t) + \sum_{i=1}^m \beta_i^m(t)\int_0^1\int_\Omega (1 - \eta\rho')\partial_\eta(\omega_i(x, \eta))\omega_j(x, \eta)\mathrm{d}x\mathrm{d}\eta = 0, \\ \beta_j^m(0) = \int_0^1\int_\Omega g_{0m} \cdot \omega_j(x, \eta)\mathrm{d}x\mathrm{d}\eta. \end{cases} \tag{3.7}$$

The systems (3.6) and (3.7) are the first- and second-order Cauchy problem for functional differential equations. Since $f_i \in C^2(\mathbb{R})$, by the local existence of unique solutions for functional differential equations, equations (3.6) and (3.7) possess local unique solutions $\alpha_j^m(t)$ and $\beta_j^m(t)$ on time interval $[0, T_m)$ for some $0 < T_m < T$, respectively, i.e., the local existence of unique solutions $u_m(t, x)$ and $z_m(t, x, \eta)$.

Next, the local approximated solutions can be extended to global by the following uniformly *a priori* estimates.

Step 2: Uniformly a priori estimate

By Step 1, the approximated systems can be written as follows:

$$\begin{cases} (\partial_{tt}u_m(t), e_j) + [\mu(\nabla u_m, \nabla e_j) + (\lambda+\mu)(\operatorname{div} u_m, \operatorname{div} e_j)] \\ \quad + (\alpha_1\partial_t u_m(t,x) + \alpha_2 z_m(x,1,t), e_j) + (f(u_m), e_j) = \int\limits_\Omega h(x)e_j\mathrm{d}x, \\ z_m(x,0,t) = \partial_t u_m(x,t), \\ (u_m(0), \partial_t u_m(0)) = (u_{0m}, u_{1m}) \end{cases} \tag{3.8}$$

and

$$\begin{cases} \int\limits_0^1 (\rho\partial_t z_m(x,\eta,t), \omega_j) + ((1-\eta\rho')\partial_\eta z_m(x,\eta,t), \omega_j)\mathrm{d}\eta = 0, \\ z_m(x,\eta,0) = z_{0m}. \end{cases} \tag{3.9}$$

Multiply equation (3.8) by $\dot{\alpha}_j^m$ and take the sum of j from 1 to m, which yields

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\limits_\Omega(\partial_t u_m)^2\mathrm{d}x + \mu\int\limits_\Omega(\nabla u_m)^2\mathrm{d}x + (\lambda+\mu)\int\limits_\Omega(\operatorname{div} u_m)^2\mathrm{d}x\right) \\ &\quad + \alpha_1\int\limits_\Omega(\partial_t u_m)^2\mathrm{d}x + \alpha_2\int\limits_\Omega z_m(x,1,t)\cdot\partial_t u_m\mathrm{d}x + \int\limits_\Omega f(u_m)\cdot\partial_t u_m\mathrm{d}x \\ &= \int\limits_\Omega h(x)\cdot\partial_t u_m\mathrm{d}x. \end{aligned} \tag{3.10}$$

Integrating with respect to time variable from 0 to t, we derive

$$\begin{aligned} &\frac{1}{2}\left(\int\limits_\Omega(\partial_t u_m)^2\mathrm{d}x + \mu\int\limits_\Omega(\nabla u_m)^2\mathrm{d}x + (\lambda+\mu)\int\limits_\Omega(\operatorname{div} u_m)^2\mathrm{d}x\right) \\ &\quad + \alpha_1\int\limits_0^t\|\partial_s u_m(s)\|_2^2\mathrm{d}s + \alpha_2\int\limits_0^t\int\limits_\Omega z_m(x,1,s)\cdot\partial_s u_m(s)\mathrm{d}x\mathrm{d}s + \int\limits_0^t\int\limits_\Omega f(u_m)\cdot\partial_s u_m\mathrm{d}x\mathrm{d}s \\ &= \int\limits_0^t\int\limits_\Omega h(x)\cdot\partial_s u_m(s)\mathrm{d}x\mathrm{d}s + \frac{1}{2}(\|u_{1m}\|_2^2 + \mu\|\nabla u_{0m}\|_2^2 + (\lambda+\mu)\|\operatorname{div} u_{0m}\|_2^2). \end{aligned} \tag{3.11}$$

Multiply equation (3.9) by $\beta_j^m(t)/\rho(t)$ and take the sum of j from 1 to m, which lead to

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_0^1\int\limits_\Omega z_m^2(x,\eta,t)\mathrm{d}x\mathrm{d}\eta + \frac{1}{2\rho}\int\limits_\Omega z_m^2(x,1,t) - z_m^2(x,0,t)\mathrm{d}x \\ &\quad - \frac{\rho'}{2\rho}\int\limits_\Omega z_m^2(x,1,t)\mathrm{d}x + \frac{\rho'}{2\rho}\int\limits_0^1\int\limits_\Omega z_m^2(x,\eta,t)\mathrm{d}x\mathrm{d}\eta = 0. \end{aligned} \tag{3.12}$$

By integration of equation (3.12) with time variable, we obtain that

$$\begin{aligned} &\frac{1}{2}\int\limits_0^1\int\limits_\Omega z_m^2(x,\eta,t)\mathrm{d}x\mathrm{d}\eta + \int\limits_0^t\int\limits_\Omega\frac{1}{2\rho}[z_m^2(x,1,s) - z_m^2(x,0,s)]\mathrm{d}x\mathrm{d}s \\ &\quad - \int\limits_0^t\int\limits_\Omega\frac{\rho'}{2\rho}z_m^2(x,1,s)\mathrm{d}x\mathrm{d}s + \int\limits_0^t\int\limits_0^1\int\limits_\Omega\frac{\rho'}{2\rho}z_m^2(x,\eta,s)\mathrm{d}x\mathrm{d}\eta\mathrm{d}s \\ &= \frac{1}{2}\int\limits_0^1\int\limits_\Omega z_{0m}^2(x,\eta,t)\mathrm{d}x\mathrm{d}\eta. \end{aligned} \tag{3.13}$$

Multiplying equation (3.13) by ξ, then summing up with equation (3.11), and setting

$$E_m(t) = \frac{1}{2}[\|\partial_t u_m\|_2^2 + \mu\|\nabla u_m\|_2^2 + (\lambda+\mu)\|\operatorname{div} u_m\|_2^2 + \|z_m(t)\|_\xi^2] = \frac{1}{2}[\|\partial_t u_m\|_2^2 + \|u_m\|_e^2 + \|z_m(t)\|_\xi^2], \quad (3.14)$$

we have

$$\begin{aligned} &E_m(t) + \left(\alpha_1 - \frac{\xi}{2\rho}\right)\int_0^t \|\partial_s u_m(s)\|_2^2 \mathrm{d}s + \frac{\xi(1-\rho')}{2\rho}\int_0^t \|z_m(x,1,s)\|_2^2 \mathrm{d}s \\ &+ \int_0^t\int_0^1\int_\Omega \frac{\xi\rho'}{2\rho} z_m^2(x,\eta,s)\mathrm{d}x\mathrm{d}\eta\mathrm{d}s + \alpha_2\int_0^t\int_\Omega z_m(x,1,s)\cdot\partial_s u_m(s)\mathrm{d}x\mathrm{d}s + \int_\Omega F(u_m)\mathrm{d}x \\ &= \int_0^t\int_\Omega h(x)\cdot\partial_s u_m(s)\mathrm{d}x\mathrm{d}s + E_m(0) + \int_\Omega F(u_{0m})\mathrm{d}x. \end{aligned} \quad (3.15)$$

Next, we shall estimate equation (3.15) under the hypothesis $\alpha_1 > \alpha_2$ to achieve the uniformly bounded for approximated solution in energy space.

From the energy identity (3.15), using the Young inequality, we obtain

$$\begin{aligned} &E_m(t) + \left(\alpha_1 - \frac{\xi}{2\rho} - \frac{\alpha_2}{2}\right)\int_0^t \|\partial_s u_m(s)\|_2^2 \mathrm{d}s + \left(\frac{\xi(1-\rho')}{2\rho} - \frac{\alpha_2}{2}\right)\int_0^t \|z_m(x,1,s)\|_2^2 \mathrm{d}s \\ &+ \int_0^t\int_0^1\int_\Omega \frac{\xi\rho'}{2\rho} z_m^2(x,\eta,s)\mathrm{d}x\mathrm{d}\eta\mathrm{d}s + \int_\Omega F(u_m)\mathrm{d}x \\ &\le \int_0^t\int_\Omega h(x)\cdot\partial_s u_m(s)\mathrm{d}x\mathrm{d}s + E_m(0) + \int_\Omega F(u_{0m})\mathrm{d}x. \end{aligned} \quad (3.16)$$

From the assumption (2.3) and the equivalent of norm between $D((-\Delta_e)^{1/2})$ and $(H_0^1(\Omega))^3$, we can see that

$$\int_\Omega F(u_m)\mathrm{d}x \ge -\frac{\tilde m}{2\mu\lambda_1}\|u_m\|_e^2 - m_f|\Omega|. \quad (3.17)$$

Since (u_{0m}, u_{1m}, z_{0m}) are the projection of $(u_0, u_1, g_0(x, -\eta\rho(0)))$ in $W_m \times W_m \times V_m$, we can see that

$$E_m(0) \le \frac{1}{2}(\|\nabla u_0\|_2^2 + \|u_1\|_2^2 + \|g_0(x, -\eta\tau(0))\|_\xi^2) = E(0). \quad (3.18)$$

The hypothesis (2.10) implies $\alpha_1 - \frac{\xi}{2\rho} - \frac{\alpha_2}{2} > 0$ and $\frac{\xi(1-\rho')}{2\rho} - \frac{\alpha_2}{2} > 0$, and the condition (H3) deduces $\frac{1}{2} - \frac{\tilde m}{2\mu\lambda_1} > 0$. By hypotheses (H1) and (H2), from equation (3.18), we conclude that there exist positive constants c_1, c_2, and c_3 and a small enough parameter $\varepsilon_0 > 0$ such that

$$\begin{aligned} &\frac{1}{2}[\|\partial_t u_m\|_2^2 + \|z_m(t)\|_\xi^2] + \left(\frac{1}{2} - \frac{\tilde m}{2\mu\lambda_1}\right)\|u_m\|_e^2 \\ &+ (c_1 - \varepsilon_0)\int_0^t \|\partial_s u_m(s)\|_2^2 \mathrm{d}s + c_2\int_0^t \|z_m(x,1,s)\|_2^2 \mathrm{d}s + c_3\int_0^t \|z_m(x,\eta,s)\|_\xi^2 \mathrm{d}s \\ &\le \frac{C}{\varepsilon_0}\|h\|_2^2 + \left(1 + \frac{\tilde m}{2}\right)E(0) + 2m_f|\Omega|. \end{aligned} \quad (3.19)$$

Choosing $\tilde c_0 = \min\left\{\frac{1}{2}, \frac{1}{2} - \frac{\tilde m}{2\mu\lambda_1}\right\} > 0$, the energy inequality (3.19) leads to the uniform boundedness as follows:

$$E_m(t) \le C \quad (3.20)$$

for all $t > 0$, here C is a positive constant independent on m. Moreover, the nonlinear term is also bounded as

$$\left|\int_{\Omega} F(u_m)\mathrm{d}x\right| \leq C. \tag{3.21}$$

Thus, the inequalities (3.19) and (3.20) imply the uniform boundedness of $(u_m, \partial_t u_m, z_m)$ in appropriate functional space as

$$\{u_m\}_{m=1}^{\infty} \text{ is bounded in } L^{\infty}(0, T;\ D((-\Delta_e)^{1/2})), \tag{3.22}$$

$$\{\partial_t u_m\}_{m=1}^{\infty} \text{ is bounded in } L^{\infty}(0, T;\ (L^2(\Omega))^3) \cap L^2(0, T;\ (L^2(\Omega))^3), \tag{3.23}$$

$$\{z_m(x, 1, t) = \partial_t u_m(x, t - \rho(t))\}_{m=1}^{\infty} \text{ is bounded in } L^2(0, T;\ (L^2(\Omega))^3), \tag{3.24}$$

$$\{z_m\}_{m=1}^{\infty} \text{ is bounded in } L^{\infty}(0, T;\ (L^2(\Omega \times [0, 1]))^3) \cap L^2(0, T;\ (L^2(\Omega \times [0, 1]))^3). \tag{3.25}$$

By the Dunford-Pettis theorem, there exists a subsequence (still denote $(u_m, \partial_t u_m, z_m)$ without confusion), such that the convergences

$$u_m \overset{*}{\rightharpoonup} u \text{ weakly * in } L^{\infty}(0, T;\ D((-\Delta_e)^{1/2})), \tag{3.26}$$

$$\partial_t u_m \overset{*}{\rightharpoonup} \partial_t u \text{ weakly * in } L^{\infty}(0, T;\ (L^2(\Omega))^3), \tag{3.27}$$

$$\partial_t u_m \rightharpoonup \partial_t u \text{ weakly in } L^2(0, T;\ (L^2(\Omega))^3), \tag{3.28}$$

$$z_m \overset{*}{\rightharpoonup} z \text{ weakly * in } L^{\infty}(0, T;\ (L^2(\Omega \times [0, 1]))^3), \tag{3.29}$$

$$z_m \rightharpoonup z \text{ weakly in } L^2(0, T;\ (L^2(\Omega \times [0, 1]))^3) \tag{3.30}$$

hold.

Step 3: Compact argument and passing to limit

Since the embedding $(H_0^1(\Omega))^3 \hookrightarrow (L^2(\Omega))^3$ is compact, we conclude that

$$u_m \to u \text{ strongly in } L^2(0, T;\ (L^2(\Omega))^3), \tag{3.31}$$

$$u_m \to u \text{ a.e. in } (L^2(\Omega))^3 \tag{3.32}$$

for some $u \in L^{\infty}(0, T;\ D((-\Delta_e)^{1/2}))$ and $z \in L^2(0, T;\ (L^2(\Omega \times [0, 1]))^3)$.

By the uniform boundedness of $\{u_m\}$ in $L^{\infty}(0, T;\ D((-\Delta_e)^{1/2}))$, we can obtain

$$\partial_t u_m(x, t - \rho(t)) \rightharpoonup \partial_t u(x, t - \rho(t)) \text{ weakly in } L^2(0, T;\ (L^2(\Omega))^3). \tag{3.33}$$

Since $f(\cdot) \in C^2$, by using equation (2.3) and the uniform boundedness by equation (3.21), from the compact embedding $D((-\Delta_e)^{1/2}) \hookrightarrow (L^4(\Omega))^3$, we have the weak convergence

$$F(u_m) \rightharpoonup F(u) \text{ weakly in } L^2(0, T;\ (L^2(\Omega))^3). \tag{3.34}$$

Thus, using the convergences (3.26)–(3.34), passing to the limit, we can conclude that $(u, \partial_t u, z)$ is a weak solution to system (1.5).

Step 4: The verification of initial data

From equations (3.26) and (3.27), we derive

$$u_m \in C([0, T];\ (L^2(\Omega))^3), \tag{3.35}$$

which implies $u_m(0) \overset{*}{\rightharpoonup} u(0)$. By the assumption, it is easy to check

$$u_m(0) = u_{0m} \to u_0 \quad \text{in } (H_0^1(\Omega))^3. \tag{3.36}$$

Since $(H_0^1(\Omega))^3$ is dense in $(L^2(\Omega))^3$, it yields $u(0) = u_0$.

From equation (3.27), it is easy to check

$$(\partial_t u_m(t), e_j) \overset{*}{\rightharpoonup} (\partial_t u(t), e_j) \tag{3.37}$$

in $L^{\infty}(0, T)$. Equation (3.8) and convergence (3.26) lead to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle u_m(t), e_j\rangle = \langle \partial_{tt}u_m, e_j\rangle \overset{*}{\rightharpoonup} \langle \partial_{tt}u, e_j\rangle = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle u(t), e_j\rangle, \tag{3.38}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual product between $(H_0^1(\Omega))^3$ and $(H^{-1}(\Omega))^3$, also $(L^2(\Omega))^3$ to itself.

If $\partial_t u(t) \in (L^2(\Omega))^3$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\partial_t u_m(t), e_j) \overset{*}{\rightharpoonup} \frac{\mathrm{d}}{\mathrm{d}t}(\partial_t u(t), e_j). \tag{3.39}$$

Using Lemma 3.1, we deduce

$$(\partial_t u_m(0), e_j) \to (\partial_t u(0), e_j). \tag{3.40}$$

By the assumption, we conclude that $(\partial_t u(0), e_j) = (u_1, e_j)$, which implies $\partial_t u(x, 0) = u_1(x)$ in $(L^2(\Omega))^3$.

Passing to limit of equation (3.13), combining equations (3.5) and (3.28)–(3.30), and comparing with resulting equation and equation (3.13), we can conclude that $z_m(x, \eta, 0) = g(x, -\eta\rho(0))$.

Step 5: Continuous dependence on initial data and uniqueness

Let $Z^1 = (u^1(t), \partial_t u^1(t), z^1(x, \eta, t))$ and $Z^2 = (u^2(t), \partial_t u^2(t), z^2(x, \eta, t))$ be two global weak solutions for equation (1.5) and denote $Z = (u, \partial_t u, z) = (u^1 - u^2, \partial_t u^1 - \partial_t u^2, z^1 - z^2)$, then for any $\phi \in C^1([0, T]; (L^1(\Omega))^3) \cap C([0, T]; (H_0^1(\Omega))^3)$ with $\phi(x, T) = 0$ and $\psi(x, \eta, t) \in C^1([0, T]; (L^2(\Omega \times [0, 1]))^3)$ with $\psi(x, \eta, T) = 0$, the difference $Z(t, x)$ satisfies

$$\begin{cases} \displaystyle\int_0^T [-(\partial_t w(t), \partial_t\phi(t)) + (w(t), \phi(t))_e + \alpha_1(\partial_t u(t), \phi(t)) + \alpha_2(z(x, 1, t), \phi(t)) \\ \qquad + (f(u^1(t)) - f(u^2(t)), \phi(t))]\mathrm{d}t = 0, \\ \displaystyle\int_0^T [(\rho\partial_t z(x, \eta, t), \psi(x, \eta, t)) + ((1 - \eta\rho')\partial_\eta z(x, \eta, t), \psi(x, \eta, t))]\mathrm{d}t = 0, \\ w(x, t)|_{\partial\Omega} = 0, \\ w(x, 0) = u_0^1(x) - u_0^2(x), \quad \partial_t w(x, 0) = u_1^1(x) - u_1^2(x), \\ z^1(x, 0, t) - z^2(x, 0, t) = \partial_t w(x, t), \\ z^1(x, \eta, 0) - z^2(x, \eta, 0) = g_0^1(x, -\eta\rho(0)) - g_0^2(x, -\eta\rho(0)). \end{cases} \tag{3.41}$$

For $s \in [0, T]$, setting

$$\phi(t) = \begin{cases} -\displaystyle\int_s^t \bar\phi(\tau)\mathrm{d}\tau, & t \le s, \\ 0, & t > s \end{cases} \tag{3.42}$$

for $\psi(t, \eta, t) = \frac{1}{\rho}z(x, \eta, t)$ and $\bar\psi = \int_0^t z(x, \eta, \tau)\mathrm{d}\tau$, noting that $f(u) \in C^2$, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\partial_t u(t)\|_2^2 + \|u(t)\|_e^2 + \int_0^1\int_\Omega z^2(x, \eta, t)\mathrm{d}x\mathrm{d}\eta\right] + \left(\alpha_1 - \frac{1}{2\rho}\right)\|\partial_t u\|_2^2 \\ &\quad + \frac{1 - \rho'}{2\rho}\int_\Omega z^2(x, 1, t)\mathrm{d}x + \frac{\rho'}{2\rho}\int_0^1\int_\Omega z^2(x, \eta, t)\mathrm{d}x\mathrm{d}\eta \\ &\le |\alpha_1(z(x, 1, t), \partial_t u)| + |(f(u^1) - f(u^2), \partial_t u)|. \end{aligned} \tag{3.43}$$

Hence, there exists a $L_f > 0$ such that

$$
\begin{aligned}
&\frac{1}{2}\left[\|\partial_t u(t)\|_2^2 + \|u(t)\|_e^2 + \int_0^1\int_\Omega z^2(x,\eta,t)\mathrm{d}x\mathrm{d}\eta\right] + \left(\alpha_1 - \frac{1}{2\rho}\right)\int_0^t \|\partial_t u(s)\|_2^2\mathrm{d}s \\
&\quad + \frac{1-\rho'}{4\rho}\int_0^t\int_\Omega z^2(x,1,s)\mathrm{d}x\mathrm{d}s + \frac{\rho'}{2\rho}\int_0^t\int_0^1\int_\Omega z^2(x,\eta,s)\mathrm{d}x\mathrm{d}\eta\mathrm{d}s \qquad (3.44)\\
&\le \left[\frac{\rho\alpha_2^2}{1-\rho'} + \frac{1}{2\lambda_1}\right]\int_0^t \|\partial_s u(s)\|_2^2\mathrm{d}s + \frac{L_f^2}{2}\int_0^t \|u(s)\|_e^2\mathrm{d}s.
\end{aligned}
$$

Choosing appropriate parameter $C_5 = \min\{\frac{\rho\alpha_2^2}{1-\rho'} + \frac{1}{2\lambda_1}, \frac{L_f^2}{2}\}$ for $\alpha_1 - \frac{1}{2\rho} > 0$, and $C_6 = \min\{\frac{\rho\alpha_2^2}{1-\rho'} + \frac{1}{2\lambda_1} + \frac{1}{2\rho} - \alpha_1, \frac{L_f^2}{2}\}$ for $\alpha_1 - \frac{1}{2\rho} < 0$, then using Gronwall's lemma, we have the continuous dependence on initial data (3.2) for $C(T,B) = \max\{C_5, C_6\}$, which implies the uniqueness of global weak solution. □

Remark 3.2. Similar as in [39], the global well-posedness cannot hold if $\alpha_1 = \alpha_2$, since the energy estimate (3.19) not holds, which leads to the invalidity of compact argument and passing to the limit for the case $\alpha_1 = \alpha_2$.

Corollary 3.3. *Based on Theorem* 3.1, *the weak solution in equation* (3.1) *generates a nonlinear C_0-operator $\{S(t)\}_{t\ge 0}$ in $\mathcal{H}$ defined by*

$$
\begin{cases} S(t): \mathcal{H} \to \mathcal{H}, \\ S(t)(u_0, u_1, g_0) = (u(x,t), \partial_t u(x,t), z(x,\eta,t)), \quad t > 0, \end{cases}
$$

where $(u(x,t), \partial_t u(x,t), z(x,\eta,t))$ is the weak solution corresponding to the initial data (u_0, u_1, g_0). The corresponding dynamic system can be denoted by $(\mathcal{H}, S(t))$.

4 Energy estimates and asymptotic stability

The energy functional for problem (1.5) is defined as

$$
\mathcal{E}(t) = \frac{1}{2}\|(u, \partial_t u, z)\|_{\mathcal{H}}^2 + \int_\Omega F(u(t))\mathrm{d}x - \int_\Omega h\cdot u\mathrm{d}x, \qquad (4.1)
$$

then we have following asymptotic stable result for the decay rate of the energy functional (4.1), which plays an important role in control theory and engineering.

By using the multiplier technique, such as in Liu and Zheng [27], Ma et al. [29], Yang et al. [39], we will show the energy estimates and its exponential decay in this section.

Lemma 4.1. *The energy defined in equation* (4.1) *for system* (1.5) *is decreasing, and there exists a positive constant C such that*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \le -C(\|\partial_t u(t)\|_2^2 + \|z(x,1,t)\|_2^2 + \|z(x,\eta,t)\|_\xi^2). \qquad (4.2)
$$

Proof. By equations (1.4) and (1.5) and computation, we have

$$
\begin{aligned}
\int_\Omega \partial_t u(t)\cdot\partial_{tt}u(t)\mathrm{d}x &= \int_\Omega \Big[\mu\nabla(\partial_t u(t))\cdot\nabla u + (\lambda+\mu)\,\mathrm{div}(\partial_t u(t))\,\mathrm{div}\,u\Big]\mathrm{d}x - \alpha_1\int_\Omega(\partial_t u)^2(t)\mathrm{d}x \\
&\quad - \alpha_2\int_\Omega \partial_t u(t)\cdot z(x,1,t)\mathrm{d}x - \int_\Omega f(u)\cdot\partial_t u(t)\mathrm{d}x + \int_\Omega h\cdot\partial_t u(t)\mathrm{d}x
\end{aligned} \qquad (4.3)
$$

and

$$\begin{aligned}
&\int_0^1\int_\Omega z(x,\eta,t)\cdot\partial_t z(x,\eta,t)\mathrm{d}x\mathrm{d}\eta \\
&\quad= -\int_0^1\int_\Omega \frac{1-\eta\rho'}{\rho} z(x,\eta,t)\cdot\partial_\eta z(x,\eta,t)\mathrm{d}x\mathrm{d}\eta \\
&\quad= -\frac{1-\rho'}{2\rho}\int_\Omega z^2(x,1,t)\mathrm{d}x + \frac{1}{2\rho}\int_\Omega(\partial_t u(t))^2\mathrm{d}x - \frac{\rho'}{2\rho}\int_0^1\int_\Omega z^2(x,\eta,t)\mathrm{d}x\mathrm{d}\eta.
\end{aligned} \tag{4.4}$$

Hence,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) &= -\alpha_1\int_\Omega(\partial_t u)^2(t)\mathrm{d}x - \alpha_2\int_\Omega \partial_t u(t)\cdot z(x,1,t)\mathrm{d}x - \frac{\xi(1-\rho')}{2\rho}\int_\Omega z^2(x,1,t)\mathrm{d}x \\
&\quad+ \frac{\xi}{2\rho}\int_\Omega(\partial_t u(t))^2\mathrm{d}x - \frac{\xi\rho'}{2\rho}\int_0^1\int_\Omega z^2(x,\eta,t)\mathrm{d}x\mathrm{d}\eta \\
&\le -\left(\alpha_1 - \frac{\xi}{2\rho} - \frac{\alpha_2}{2}\right)\int_\Omega(\partial_t u)^2(t)\mathrm{d}x - \left(\frac{\xi(1-\rho')}{2\rho} - \frac{\alpha_2}{2}\right)\int_\Omega z^2(x,1,t)\mathrm{d}x - \frac{\xi\rho'}{2\rho}\int_0^1\int_\Omega z^2(x,\eta,t)\mathrm{d}x\mathrm{d}\eta,
\end{aligned} \tag{4.5}$$

and the assumption (H1) and equation (2.10) assure that $\alpha_1 - \frac{\xi}{2\rho} - \frac{\alpha_2}{2} > 0$, $\frac{\xi(1-\rho')}{2\rho} - \frac{\alpha_2}{2}$, and $\frac{\xi\rho'}{2\rho} > 0$, which implies the lemma. □

Lemma 4.2. *Define*

$$\mathcal{G}(t) = \int_\Omega \partial_t u(t)\cdot u(t)\mathrm{d}x. \tag{4.6}$$

Then, we have

$$\frac{\mathrm{d}\mathcal{G}(t)}{\mathrm{d}t} \le C_1\|\partial_t u(t)\|_2^2 + C_2\int_\Omega z^2(x,1,t)\mathrm{d}x - \int_\Omega f(u)\cdot u\mathrm{d}x + \int_\Omega h\cdot u\mathrm{d}x - \frac{1}{2}\|u(t)\|_e^2 \tag{4.7}$$

for some positive constants C_1 and C_2.

Proof. Noting that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \partial_t u\cdot u\mathrm{d}x &= -\|u(t)\|_e^2 + \|\partial_t u(t)\|_2^2 - \alpha_1\int_\Omega\partial_t u\cdot u\mathrm{d}x - \alpha_2\int_\Omega z(x,1,t)\cdot u\mathrm{d}x \\
&\quad- \int_\Omega f(u)\cdot u\mathrm{d}x + \int_\Omega h\cdot u\mathrm{d}x
\end{aligned} \tag{4.8}$$

and

$$\left|-\alpha_1\int_\Omega\partial_t u\cdot u\mathrm{d}x - \alpha_2\int_\Omega z(x,1,t)\cdot u\mathrm{d}x\right| \le \left(\frac{\alpha_1^2}{\mu\lambda_1}\int_\Omega(\partial_t u)^2\mathrm{d}x + \frac{\alpha_2^2}{\mu\lambda_1}\int_\Omega z^2(x,1,t)\mathrm{d}x\right) + \frac{\mu}{2}\int_\Omega|\nabla u(t)|^2\mathrm{d}x \tag{4.9}$$

hold, we can derive the result

$$\frac{\mathrm{d}\mathcal{G}(t)}{\mathrm{d}t} \le \left(1 + \frac{\alpha_1^2}{\mu\lambda_1}\right)\|\partial_t u(t)\|_2^2 + \frac{\alpha_2^2}{\mu\lambda_1}\int_\Omega z^2(x,1,t)\mathrm{d}x - \int_\Omega f(u)\cdot u\mathrm{d}x + \int_\Omega h\cdot u\mathrm{d}x - \frac{1}{2}\|u(t)\|_e^2, \tag{4.10}$$

which implies the desired result. □

Theorem 4.1. *Assume the hypotheses* (*H*1)–(*H*5) *and the given initial data* $(u_0, u_1, g_0) \in \mathcal{H}$ *for problem* (1.5) *hold, then there exist constants* $C_1, C_2, C_3, C_4 > 0$ *such that the energy for solution trajectory satisfies the estimates*

$$C_1\|(u(t), \partial_t u(t), z(t))\|_{\mathcal{H}}^2 - C_2 \le \mathcal{E}(t) \le C_3(1 + \|(u(t), \partial_t u(t), z(t))\|_{\mathcal{H}}^2) \tag{4.11}$$

and

$$\|(u(t), \partial_t u(t), z(t))\|_{\mathcal{H}}^2 \le C_4(1 + \|(u_0, u_1, g_0)\|_{\mathcal{H}}^2). \tag{4.12}$$

Proof. By the perturbed energy functional technique, the asymptotic stability can be proved. The key idea for the proof is by constructing a perturbed functional $\mathcal{F}(t)$, which is equivalent to $\mathcal{E}(t)$, such that $\mathcal{F}(t)$ satisfies the exponential decay $\frac{\mathrm{d}\mathcal{F}(t)}{\mathrm{d}t} \le -c\mathcal{F}(t)$ for $c > 0$ and uniform boundedness of energy functional for inhomogeneous case.

For $\varepsilon > 0$, the perturbed Lyapunov functional $\mathcal{F}(t)$ is defined as follows:

$$\mathcal{F}(t) = \mathcal{E}(t) + \varepsilon\mathcal{G}(t). \tag{4.13}$$

Since

$$|\mathcal{G}(t)| \le \max\left\{2, \frac{2}{\lambda_1}\right\}\mathcal{E}(t), \tag{4.14}$$

and choosing appropriate ε, there exists $0 < \nu_1 < \nu_2$ such that the equivalence

$$\nu_1\mathcal{E}(t) \le \mathcal{F}(t) \le \nu_2\mathcal{E}(t). \tag{4.15}$$

holds.

Combining the estimates (4.2) and (4.7), and noting that

$$\int_\Omega f(u(t))\cdot u(t)\mathrm{d}x \ge -\frac{\tilde{m}}{\lambda_1}\|\nabla u\|_2^2 - m_f|\Omega| \ge -\frac{\tilde{m}}{\mu\lambda_1}\|u\|_e^2 - m_f|\Omega|, \tag{4.16}$$

we have

$$\begin{aligned}\frac{\mathrm{d}\mathcal{F}(t)}{\mathrm{d}t} &\le -\left(\alpha_1 - \frac{\xi}{2\rho} - \frac{\alpha_2}{2} - \varepsilon\left(1 + \frac{\alpha_1^2}{\mu\lambda_1}\right)\right)\|\partial_t u(t)\|_2^2 \\ &\quad - \left(\frac{\xi(1-\rho')}{2\rho} - \frac{\alpha_2}{2} - \frac{\varepsilon\alpha_2^2}{\mu\lambda_1}\right)\int_\Omega z^2(x,1,t)\mathrm{d}x \\ &\quad - \frac{\rho'}{2\rho}\|z(x,\eta,t)\|_\xi^2 - \varepsilon\int_\Omega f(u)\cdot u\mathrm{d}x + \varepsilon\int_\Omega h\cdot u\mathrm{d}x - \frac{\varepsilon}{2}\|u(t)\|_e^2 \\ &\le -\left(\alpha_1 - \frac{\xi}{2\rho} - \frac{\alpha_2}{2} - \varepsilon\left(1 + \frac{\alpha_1^2}{\mu\lambda_1}\right)\right)\|\partial_t u(t)\|_2^2 \\ &\quad - \left(\frac{\xi(1-\rho')}{2\rho} - \frac{\alpha_2}{2} - \frac{\varepsilon\alpha_2^2}{\mu\lambda_1}\right)\int_\Omega z^2(x,1,t)\mathrm{d}x - \frac{\rho'}{2\rho}\|z(x,\eta,t)\|_\xi^2 \\ &\quad + \varepsilon\int_\Omega h\cdot u\mathrm{d}x - \frac{\varepsilon}{2}\|u(t)\|_e^2 + \frac{\varepsilon\tilde{m}}{\mu\lambda_1}\|u\|_e^2 + \varepsilon m_f|\Omega|.\end{aligned} \tag{4.17}$$

Choosing suitable ε small enough such that

$$\alpha_1 - \frac{\xi}{2\rho} - \frac{\alpha_2}{2} - \varepsilon\left(1 + \frac{\alpha_1^2}{\mu\lambda_1}\right) > 0, \quad \frac{\xi(1-\rho')}{2\rho} - \frac{\alpha_2}{2} - \frac{\varepsilon\alpha_2^2}{\mu\lambda_1} > 0 \quad \text{and} \quad \frac{\rho'}{2\rho} > 0,$$

we can find some positive constant $\beta_1 = \min\left\{\alpha_1 - \frac{\xi}{2\rho} - \frac{\alpha_2}{2} - \varepsilon\left(1 + \frac{\alpha_1^2}{\mu\lambda_1}\right), \frac{\rho'}{2\rho}, \frac{\varepsilon}{2}\right\}$ such that

$$\mathcal{F}'(t) \le -\beta_1\mathcal{E}(t) + \frac{\varepsilon\tilde{m}}{\mu\lambda_1}\|u\|_e^2 + \varepsilon m_f|\Omega|. \tag{4.18}$$

Since $\mathcal{F}(t)$ is equivalent to $\mathcal{E}(t)$, the asymptotic stability of $\mathcal{F}(t)$ and $\mathcal{E}(t)$ as $t \to \infty$ can be estimated as

$$\mathcal{F}(t) \le \mathcal{F}(0)e^{-\beta_1 t} + \int_0^t e^{-\beta_1(t-s)}\left(\frac{\varepsilon\tilde{m}}{\mu\lambda_1}\|u(s)\|_e^2 + \varepsilon m_f|\Omega|\right)\mathrm{d}s$$

and

$$\mathcal{E}(t) \le C\mathcal{E}(0)e^{-\beta_1 t} + \int_0^t e^{-\beta_1(t-s)}\left(\frac{\varepsilon\tilde{m}}{\mu\lambda_1}\|u(s)\|_e^2 + \varepsilon m_f|\Omega|\right)\mathrm{d}s \le C_4(1 + \mathcal{E}(0)).$$

Using equation (2.3) and Lemma 4.1, it yields $\int_\Omega F(u(t))\mathrm{d}x \ge -\frac{\tilde{m}}{2\mu\lambda_1}\|u\|_e^2 - m_f|\Omega|$, and we can finish the proof of Theorem 4.1. □

Remark 4.3. An interesting issue is that Nicaise and Pignotti [31] considered the linear wave system with constant delay

$$u_{tt} - \Delta u + \alpha_1 u_t(x, t) + \alpha_2 u_t(x, t - \tau) = 0, \tag{4.19}$$

which obtained exponential stability of solution only if $\alpha_1 > \alpha_2$, instability occurs for $\alpha_1 \le \alpha_2$ with counter-examples. In the study by Kirane and Said-Houari [22], the addition of memory term helps to save the case $\alpha_1 = \alpha_2$ for exponential stability, while for $\alpha_1 < \alpha_2$, the system is still unstable.

In our research, the nonlinear time-varying delay offsets the stabilizing effects of weak damping, Thus, to obtain suitable energy estimates for stability and dynamic systems, we still need the assumption $\alpha_1 > \alpha_2$.

5 Dynamics: global and exponential attractors

5.1 Theory of dynamic systems

In this section, we will review the theory of global attractors, which can be seen in the studies of Miranville and Zelik [30] and Temam [37], the quasi-stability theory as shown in the studies of Chapman [10], Lasiecka and Chueshov [13], and literatures therein.

- **Some definitions**

Definition 5.1.
(a) (Dissipation) A set $B_0 \subset X$ is called an absorbing set for the semigroup $S(t)$ ($t \ge 0$) if, for any bounded set $B \subset X$, there exists a time $t_1 = t_1(B) > 0$ such that for all $t > t_1$, $S(t)B \subseteq B_0$.
(b) (Asymptotic smoothness) The semigroup $S(t)$ ($t \ge 0$) is said to be asymptotically smooth in X if, for any closed bounded subset $B \subset X$ satisfying $S(t)B \subset B$, there exists a nonempty compact set $K = K(B) \subset X$ such that $\mathrm{dist}(S(t)B, K(B)) \to 0$ as $t \to \infty$.
(c) (Asymptotic compactness) A dynamical system $(X, S(t))$ is asymptotically compact if, for any bounded set $B \subset X$, and sequence $\{x_k\} \subset B$, the sequence $\{S(t_k)x_k\}$ has convergent subsequence as $t_k \to \infty$.

Definition 5.2. A compact set $\mathcal{A} \subset X$ is called a global attractor of the semigroup $S(t)$ if
(i) $\mathcal{A}$ is strictly invariant with respect to $S(t)$, i.e., for all $t \ge 0$, $S(t)\mathcal{A} = \mathcal{A}$.
(ii) $\mathcal{A}$ attracts any bounded set $B \subset X$: for any $\varepsilon > 0$, then there exists a time $t_1 = t_1(\varepsilon, B) > 0$ such that for all $t \ge t_1(\varepsilon, B)$, $S(t)B \subseteq O_\varepsilon(\mathcal{A})$, where $O_\varepsilon(\mathcal{A})$ is an ε-neighborhood of $\mathcal{A}$ in X.

Definition 5.3. Given a compact set M in a metric space X, the fractal dimension of M is defined as follows:

$$\dim_f^X M = \limsup_{\varepsilon\to 0}\frac{\ln N(M,\varepsilon)}{\ln(1/\varepsilon)},$$

where $N(M,\varepsilon)$ is the minimal number of closed balls with radius $\varepsilon > 0$, which covers M.

Remark 5.4. The Lorentz system, 2D incompressible Navier-Stokes equations, and reactive-diffusion equation with sub-critical case are dissipative systems, which possess global attractors with fractional dimension, see [37] for more details.

- **Quasi-stability:**

Definition 5.5. The unstable manifold $M_+(\mathcal{N})$ is defined as the family of $y \in X$ such that there exists a full trajectory $u(t)$ satisfying

$$u(0) = y, \quad \text{and} \quad \lim_{t\to-\infty} \operatorname{dist}_X(u(t),\mathcal{N}) = 0, \tag{5.1}$$

where $\mathcal{N}$ is the set of equilibrium for $S(t)$.

Theorem 5.1. (See Chapman [10], Chueshov and Lasiecka [13]) *Assume that the gradient system $(S(t),X)$ with corresponding Lyapunov functional Φ is asymptotically compact. Moreover, assume that*
(*I*) $\Phi(S(t)z) \to \infty$ *if and only if* $||z||_X \to \infty$,
(*II*) *the set of equilibrium $\mathcal{N}$ is bounded.*

Then, the gradient system $(S(t),X)$ possesses a compact global attractor $\mathcal{A} \subset X$, which has the structure $\mathcal{A} = M_+(\mathcal{N})$.

Definition 5.6. (See Chapman [10], Chueshov and Lasiecka [13]) The dynamic system $(S(t),X)$ is quasi-stable on a set $B \subset X$ if there exists a compact semi-norm n_Y on Y, the subspace of X, and nonnegative scalar functions $a(t)$ and $c(t)$, locally bounded on $[0,\infty)$ and $b(t) \in L^1(\mathbb{R}^+)$ with $\lim_{t\to\infty} b(t) = 0$, such that for $U_1, U_2, \in B$

$$||S(t)U_1 - S(t)U_2||_X^2 \le a(t)||U_1 - U_2||_X^2, \tag{5.2}$$

$$||S(t)U_1 - S(t)U_2||_X^2 \le b(t)||U_1 - U_2||_X^2 + c(t)\sup_{0<s<t}[n_Y(y_1(s) - y_2(s))]^2. \tag{5.3}$$

Inequality (5.3) is usually called stabilizability inequality.

Theorem 5.2. (See Chapman [10], Chueshov and Lasiecka [13]) *Let $(X,S(t))$ be a dynamical system and suppose that the system is quasi-stable on every bounded positively invariant set $B \subset X$. Then, $(X,S(t))$ is asymptotically compact.*

- **Fractal dimensional global and exponential attractors:**

The key idea for the investigation of smooth dynamics is to verify the following proposition for the gradient system. A dynamical system $(X,S(t))$ is called a gradient system if it admits a strict Lyapunov function or functional, i.e., a functional $\Phi : X \to \mathbb{R}$ is a strict Lyapunov function or functional for the system $(X,S(t))$ if
(i) the map $t \to \Phi(S(t)z)$ is non-increasing for any $z \in X$;
(ii) if $\Phi(S(t)z) = \Phi(z)$ for all t, then z is a stationary point of $S(t)$.

Theorem 5.3. (Chapman [10], Chueshov and Lasiecka [13]) *Let $(X,S(t))$ be a gradient system and suppose that the system is quasi-stable on every bounded positively invariant set $B \subset X$. Then, $(X,S(t))$ has a global attractor $\mathcal{A} = M_+(\mathcal{N})$ with finite fractal dimension, where $\mathcal{N}$ is the set of equilibrium for $S(t)$ and $M_+(\mathcal{N})$ is the unstable manifold for $\mathcal{N}$. Moreover, the generalized finite fractal dimensional exponential attractor also exists under suitable condition for $S(t)$.*

5.2 Finite dimensional dynamics of problem (1.1)

Based on the theory of finite fractal dimensional global and exponential attractor for the quasi-stability of gradient system proposed by Chapman [13] and Chueshov and Lasiecka [10], we will present the smooth dynamic for problem (1.5). From the global well-posedness for the system (1.5) in Theorem 3.1, the global weak solution generates a gradient system $(\mathcal{H}, S(t))$, which possesses finite dimensional global and exponential attractors as Theorem 5.4.

- **Gradient structure:**

Lemma 5.7. *Assume the hypotheses* (*H1*)–(*H5*) *and the given initial data* $(u_0, u_1, g_0) \in \mathcal{H}$ *for problem* (1.5) *hold, then the dynamical system* $(\mathcal{H}, S(t))$ *is gradient.*

Proof. For the given initial data $Z_0 = (u_0, u_1, g_0) \in \mathcal{H}$, it yields that

$$Z(t) = S(t)Z_0 = (u(t), \partial_t u(t), z(x, \eta, t))$$

is a solution trajectory for the dynamical system $(\mathcal{H}, S(t))$ of problem (1.5). Define the total energy $\Psi(S(t)Z_0)$ as Lyapunov functional by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(S(t)Z_0) = \frac{1}{2}\|(u, \partial_t u, z)\|^2_{\mathcal{H}} + \int_\Omega F(u(t))\mathrm{d}x - \int_\Omega h(x)\cdot u\mathrm{d}x. \tag{5.4}$$

By Lemma 4.1, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(S(t)Z_0) \le -C(\|\partial_t u(t)\|_2^2 + \|z(x, 1, t)\|_2^2 + \|z(x, \eta, t)\|_\xi^2) \tag{5.5}$$

for $t \ge 0$, which implies that the mapping $t \mapsto \Psi(S(t)Z_0)(t)$ is non-increasing.

By the definition of gradient system, suppose that $\Psi(S(t)Z_0) = \Psi(Z_0)$ for some $Z_0 \in \mathcal{H}$ and $t \ge 0$. Then, $\frac{\mathrm{d}}{\mathrm{d}t}\Psi(S(t)Z_0) = 0$. Using the similar argument as in the study by Ma et al. [29] and energy inequality (5.5), we can conclude that $Z_0 = (u_0, 0, 0)$ is a stationary point of dynamical system $(\mathcal{H}, S(t))$, which implies that $(\mathcal{H}, S(t))$ is a gradient system. □

- **Dissipation property of semigroup:**

Lemma 5.8. *Assume the hypotheses* (*H*1)–(*H*5) *and the given initial data* $(u_0, u_1, g_0) \in \mathcal{H}$, *then the gradient system* $(\mathcal{H}, S(t))$ *corresponding to system* (1.5) *has a bounded absorbing set* $\mathcal{B}$ *in* $\mathcal{H}$.

Proof. Analogous to the proof of Theorem 4.1, we consider the functional $\tilde{\mathcal{E}}(t)$,

$$\tilde{\mathcal{E}}(t) = \frac{1}{2}[\|u(t)\|_e^2 + \|\partial_t u(t)\|_2^2 + \|z(t)\|_\xi^2]. \tag{5.6}$$

Then, simple computation gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}(t) = &\int_\Omega [\mu\nabla u(t)\cdot\nabla\partial_t u(t) + (\lambda + \mu)(\operatorname{div} u(t))\operatorname{div}(\partial_t u(t))]\mathrm{d}x \\ &+ \int_\Omega \partial_t u(t)\cdot\partial_{tt}u(t)\mathrm{d}x + \int_0^1\int_\Omega z(x, \eta, t)\cdot\partial_t z(x, \eta, t)\mathrm{d}x\mathrm{d}\eta.\end{aligned} \tag{5.7}$$

Hence, the inequality $|F(u)| \le \frac{m}{2\mu\lambda_1}|u|^2 + m_f|\Omega|$ implies

$$\tilde{\mathcal{E}}(t) - \frac{m}{2\mu\lambda_1}\|u\|_e^2 - m_f|\Omega| - \int_\Omega h(x)\cdot u(x)\mathrm{d}x \le \mathcal{E}(t) \le \mathcal{E}(0)e^{-\beta t}. \tag{5.8}$$

Using the Young inequality, we can derive that

$$\begin{aligned}&\frac{1}{2}\left(\frac{1}{2}-\frac{m}{2\mu\lambda_1}\right)\|u(t)\|_e^2+\frac{1}{2}\|\partial_t u(t)\|_2^2+\frac{1}{2}\|z(x,\eta,t)\|_\xi^2\\&\le\left[\frac{1}{2}\tilde{\mathcal{E}}(0)+\frac{1}{\mu}\left(\frac{m}{2\mu\lambda_1^2}+\frac{1}{2\lambda_1}\right)\|u_0\|_e^2+m_f|\Omega|+\frac{1}{2}\|h(x)\|_2^2\right]e^{-\beta t}+\frac{1}{\mu\lambda_1-m}\|h(x)\|_2^2+m_f|\Omega|.\end{aligned}\tag{5.9}$$

Then, choosing $\gamma=\min\left\{\frac{1}{2}(\frac{1}{2}-\frac{m}{2\mu\lambda_1}),\frac{1}{2}\right\}$ and $\gamma_1=\frac{1}{2}+\frac{1}{\mu}\left(\frac{m}{2\mu\lambda_1^2}+\frac{1}{2\lambda_1}\right)$, it yields that

$$\gamma\tilde{\mathcal{E}}(t)\le\left[\gamma\tilde{\mathcal{E}}(0)+m_f|\Omega|+\frac{1}{2}\|h(x)\|_2^2\right]e^{-\beta t}+\frac{1}{\mu\lambda_1-m}\|h(x)\|_2^2+m_f|\Omega|.\tag{5.10}$$

Therefore, by the definition of phase space, we deduce that

$$\|\nabla u(t)\|_2^2+\|\partial_t u(t)\|_2^2+\|z(t)\|_\xi^2\le\frac{1}{\gamma}\left[\gamma\tilde{\mathcal{E}}(0)+m_f|\Omega|+\frac{1}{2}\|h(x)\|_2^2\right]e^{-\beta t}+\frac{1}{\gamma(\mu\lambda_1-m)}\|h(x)\|_2^2+\frac{m_f|\Omega|}{\gamma}.\tag{5.11}$$

This implies that there exists a closed ball $\mathcal{B}(0,R)$ with radius

$$R=\frac{2}{\gamma(\mu\lambda_1-m)}\|h(x)\|_2^2+\frac{2m_f|\Omega|}{\gamma}\tag{5.12}$$

for $t\ge 1+\left[\frac{1}{\beta}\ln\left(\frac{\frac{1}{\gamma}[\gamma\tilde{\mathcal{E}}(0)+m_f\mid\Omega\mid+\frac{1}{2}\|h(x)\|_2^2]}{\frac{1}{\gamma(\mu\lambda_1-m)}\|h(x)\|_2^2+\frac{m_f\mid\Omega\mid}{\gamma}}\right)\right]$ such that $\mathcal{B}(0,R)$ can be chosen as a bounded absorbing set for gradient system $(\mathcal{H},S(t))$. Then we end the proof of Lemma 5.8. □

- **Quasi-stability:**

For the purpose of verifying the quasi-stability of gradient system $(\mathcal{H},S(t))$ on the difference of two trajectories, let us take the initial data

$$Z_0^i=(u_0^i(x,t),u_1^i(x,t),g_0^i(x,-\eta\rho(0)))\in\mathcal{H}\tag{5.13}$$

inside a forward invariant bounded set $\mathcal{B}\subset\mathcal{H}$ for the system (1.5). Thus, the corresponding solution trajectories of equation (1.5) can be denoted by

$$Z^i=S(t)Z_0^i=(u^i(x,t),\partial_t u^i(x,t),z^i(x,\eta,t)).\tag{5.14}$$

The difference of solution trajectories $W=(\Phi,\partial_t\Phi,Z)$ is denoted by

$$\Phi(t)=u^1(t)-u^2(t),\quad \partial_t\Phi=\partial_t u^1(t)-\partial_t u^2(t),\quad \mathcal{Z}=z^1(x,\eta,t)-z^2(x,\eta,t),\tag{5.15}$$

then we will present the quasi-stability on $\mathcal{B}$ for difference of solution trajectories as

$$\|(\Phi,\partial_t\Phi,Z)\|_{\mathcal{H}}^2\le b_{\mathcal{B}}\|Z_0^1-Z_0^2\|_{\mathcal{H}}^2+c_{\mathcal{B}}\sup_{0\le s\le t}\|u^1(t)-u^2(t)\|_e^2.\tag{5.16}$$

Lemma 5.9. *Assume the hypotheses* ($H1$)–($H5$) *and the given initial data* $(u_0,u_1,g_0)\in\mathcal{H}$, *then the gradient system* $(\mathcal{H},S(t))$ *corresponding to system* (1.5) *satisfies the quasi-stability property* (5.16) *for any initial conditions* Z_0^i $(i=1,2)$ *in* $\mathcal{B}(0,R)\subset\mathcal{H}$.

Proof. Step 1: For any initial condition $(u_0^i,u_1^i,g_0^i)\in\mathcal{B}(0,R)$, let $Z^i=(u^i,\partial_t u^i,z^i)$ be the corresponding solution for (u_0^i,u_1^i,g_0^i) with $i=1,2$, then the difference of trajectories $W(t)$ satisfies

$$\begin{cases}\partial_{tt}\Phi(x,t) - \Delta_e\Phi(x,t) + \alpha_1\partial_t\Phi + \alpha_2\mathcal{Z}(x,1,t) + f(u^1) - f(u^2) = 0, & x\in\Omega, \quad t\geq 0,\\ \rho\mathcal{Z}_t(x,\eta,t) + (1-\eta\rho')\mathcal{Z}_\eta(x,\eta,t) = 0, \quad x\in\Omega, \quad 0<\eta\leq 1, \quad t\geq 0,\\ \mathcal{Z}(x,0,t) = \partial_t\Phi(x,t), \quad x\in\Omega, \quad t\geq 0,\\ \Phi(x,t) = 0, \quad x\in\partial\Omega, \quad t\geq 0,\\ \Phi(x,0) = u_0^1(x) - u_0^2(x), \quad \partial_t\Phi(x,0) = u_1^1(x) - u_1^2(x), \quad x\in\Omega,\\ \mathcal{Z}(x,\eta,0) = g_0^1(x,-\eta\rho(0)) - g_0^2(x,-\eta\rho(0)), \quad x\in\Omega.\end{cases} \tag{5.17}$$

Define the norm of $W(t) = Z^1(t) - Z^2(t)$ in $\mathcal{H}$ as

$$\tilde{\mathcal{E}}_W(t) = \frac{1}{2}[\|\Phi(t)\|_e^2 + \|\partial_t\Phi(t)\|_2^2 + \|\mathcal{Z}(t)\|_\xi^2] \tag{5.18}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_W(t) = &\int_\Omega[\mu\nabla\Phi(t)\cdot\nabla(\partial_t\Phi(t)) + (\lambda+\mu)\,\mathrm{div}\,u(t)(\,\mathrm{div}\,\partial_t u(t))]\mathrm{d}x\\ &+\int_\Omega\partial_t\Phi(t)\cdot\partial_{tt}\Phi(t)\mathrm{d}x + \int_0^1\int_\Omega\mathcal{Z}(t)\cdot\partial_t\mathcal{Z}(t)\mathrm{d}x\mathrm{d}\eta.\end{aligned} \tag{5.19}$$

Using the similar technique as in the proof of equation (4.2), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_W(t) \leq &-\frac{\xi\tau'}{2\tau}\int_0^1\int_\Omega\mathcal{Z}^2(x,\eta,t)\mathrm{d}x\mathrm{d}\eta - \left(\alpha_1 - \frac{\xi}{2\tau} - \frac{\alpha_2}{2}\right)\int_\Omega(\partial_t\Phi(t))^2\mathrm{d}x\\ &-\left(\frac{\xi(1-\tau')}{2\tau} - \frac{\alpha_2}{2}\right)\int_\Omega\mathcal{Z}^2(x,1,t)\mathrm{d}x + \int_\Omega(f(u^1) - f(u^2))\cdot\partial_t\Phi\mathrm{d}x.\end{aligned} \tag{5.20}$$

Step 2: To deal with the difference of nonlinear terms $f(u^1) - f(u^2)$, we will present a known result from Ma et al. [29] [Lemma 3.1] directly as follows:

Proposition 5.10. *Under the assumptions of Lemma* 5.9, *for a given* $\varepsilon > 0$, *there exist two positive parameters* b_ε *and* b_1 *depending on* $\mathcal{B}$ *such that*

$$\begin{aligned}&\int_0^t\int_\Omega e^{\tilde{\gamma}s}(f(u^1(s)) - f(u^1(s)))\cdot\partial_t\Phi(s)\mathrm{d}x\mathrm{d}s\\ &\leq\; b_1e^{\tilde{\gamma}t}\sup_{0<s<t}\|\Phi(s)\|_4^2 + \varepsilon\int_0^t e^{\tilde{\gamma}s}\|\Phi\|_e^2\mathrm{d}s + b_\varepsilon\int_0^t e^{\tilde{\gamma}s}(\|\partial_t u^1(s)\|_2^2 + \|\partial_t u^2(s)\|_2^2)\|\Phi(s)\|_e^2\mathrm{d}s\end{aligned} \tag{5.21}$$

for any $\tilde{\gamma} > 0$ *and* $t \geq 0$. *Moreover, there exists a constant* $b_2 > 0$ *depending on* $\mathcal{B}$ *such that*

$$\int_\Omega(f(u^1) - f(u^2))\cdot\,\Phi\mathrm{d}x \leq b_2\|\Phi\|_4^2 \tag{5.22}$$

for any $t \geq 0$.

Step 3: Defining $\mathcal{G}_\Phi(t) = \int_\Omega\partial_t\Phi(t)\cdot\,\Phi(t)\mathrm{d}x$, Then, simple computation yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_\Phi(t) = &\int_\Omega(\partial_t\Phi(t))^2\mathrm{d}x + \int_\Omega\Phi(t)\cdot\,\Delta_e\Phi(t)\mathrm{d}x - \alpha_1\int_\Omega\partial_t\Phi(t)\cdot\,\Phi(t)\mathrm{d}x\\ &-\alpha_2\int_\Omega\Phi(t)\cdot\,\mathcal{Z}(x,1,t)\mathrm{d}x - \int_\Omega f(u^1) - f(u^2)\cdot\,\Phi\mathrm{d}x,\end{aligned} \tag{5.23}$$

and by the similar method in proving equation (4.7), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_\Phi(t) \le \left(1+\frac{\alpha_1^2}{\mu\lambda_1}\right)\|\partial_t\Phi(t)\|_2^2 - \frac{1}{2}[\mu\|\nabla\Phi(t)\|_2^2 + (\lambda+\mu)\|\operatorname{div}\Phi(t)\|_2^2] \\ + \frac{\alpha_2^2}{\mu\lambda_1}\int_\Omega \mathcal{Z}^2(x,1,t)\mathrm{d}x - \int_\Omega f(u^1)-f(u^2)\cdot\Phi\mathrm{d}x.\end{aligned} \tag{5.24}$$

Defining

$$\tilde{\mathcal{G}}_Z(t) = N\tilde{\mathcal{E}}_W(t) + \mathcal{G}_\Phi(t),$$

it is easy to see that $\tilde{\mathcal{G}}_Z(t)$ is equivalent to $\tilde{\mathcal{E}}_W(t)$.

Using Proposition 5.10 and combining the estimates (5.20) and (5.23), we conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{G}}_Z(t) \le -\frac{N\xi\rho'}{2\rho}\int_0^1\int_\Omega \mathcal{Z}^2(x,\eta,t)\mathrm{d}x\mathrm{d}\eta - \frac{1}{2}\|\Phi(t)\|_e^2 \\ - \left(N\left(\frac{\xi(1-\rho')}{2\rho} - \frac{\alpha_2}{2}\right) - \frac{\alpha_2^2}{\mu\lambda_1}\right)\int_\Omega \mathcal{Z}^2(x,1,t)\mathrm{d}x \\ - \left(N\left(\alpha_1 - \frac{\xi}{2\tau} - \frac{\alpha_2}{2}\right) - \left(1+\frac{\alpha_1^2}{\mu\lambda_1}\right)\right)\int_\Omega(\partial_t\Phi(t))^2\mathrm{d}x \\ + N\int_\Omega f(u^1)-f(u^2)\cdot\partial_t\Phi\mathrm{d}x + b_2\|\Phi\|_4^2.\end{aligned} \tag{5.25}$$

Step 4: First, for appropriate $N>0$ and fixed, there exist $\tilde{\beta}_1>0$ and $\tilde{\beta}_2>0$ such that

$$\tilde{\beta}_2\tilde{\mathcal{E}}_W(t) \le \tilde{\mathcal{G}}_Z(t) \le \tilde{\beta}_2\tilde{\mathcal{E}}_W(t). \tag{5.26}$$

Then, the choosing of parameter $N>0$ can deduce

$$\begin{aligned} N\left(\frac{\xi(1-\rho')}{2\rho} - \frac{\alpha_2}{2}\right) - \frac{\alpha_2^2}{\mu\lambda_1} > 0, \\ N\left(\alpha_1 - \frac{\xi}{2\tau} - \frac{\alpha_2}{2}\right) - \left(1+\frac{\alpha_1^2}{\mu\lambda_1}\right) > 0, \quad \text{and} \quad \frac{N\xi\rho'}{2\rho} > 0.\end{aligned} \tag{5.27}$$

Setting $\tilde{C} = \min\left\{N\left(\alpha_1 - \frac{\xi}{2\tau} - \frac{\alpha_2}{2}\right) - \left(1+\frac{\alpha_1^2}{\mu\lambda_1}\right), \frac{N\xi\rho'}{2\rho}, \frac{1}{4}\right\}$, the estimate (5.25) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{G}}_Z(t) \le -\frac{\tilde{C}}{\tilde{\beta}_2}\tilde{\mathcal{G}}_Z(t) + \mathcal{M}(t)$$

with $\mathcal{M}(t) = -\frac{1}{4}\|\Phi(t)\|_e^2 + N\int_\Omega f(u^1)-f(u^2)\cdot\partial_t\Phi\mathrm{d}x + b_2\|\Phi\|_4^2$, which leads to

$$\tilde{\mathcal{G}}_Z(t) \le \tilde{\mathcal{G}}_Z(0)e^{-\frac{\tilde{C}}{\tilde{\beta}_2}t} + \int_0^t e^{-\frac{\tilde{C}}{\tilde{\beta}_2}(t-s)}\mathcal{M}(s)\mathrm{d}x. \tag{5.28}$$

By the equivalence (5.26), and using Proposition 5.10, we have

$$\begin{aligned}\tilde{\mathcal{E}}_W(t) \le \frac{\tilde{\beta}_2}{\tilde{\beta}_1}e^{-\frac{\tilde{C}}{\tilde{\beta}_2}t}\tilde{\mathcal{E}}_W(0) - \frac{e^{-\frac{\tilde{C}}{\tilde{\beta}_2}t}}{4\tilde{\beta}_1}\int_0^t e^{\frac{\tilde{C}}{\tilde{\beta}_2}s}\|\Phi(s)\|_e^2\mathrm{d}s \\ + \left(\frac{Nb_1}{\tilde{\beta}_1} + \frac{b_2\tilde{\beta}_2}{\tilde{\beta}_1\tilde{C}}\right)\sup_{0<s<t}\|\Phi(s)\|_4^2 + \frac{\varepsilon Ne^{-\frac{\tilde{C}}{\tilde{\beta}_2}t}}{\tilde{\beta}_1}\int_0^t e^{\frac{\tilde{C}}{\tilde{\beta}_2}s}\|\Phi\|_e^2\mathrm{d}s \\ + \frac{2b_\varepsilon Ne^{-\frac{\tilde{C}}{\tilde{\beta}_2}t}}{\tilde{\beta}_1}\int_0^t e^{\frac{\tilde{C}}{\tilde{\beta}_2}s}(\|\partial_t u^1(s)\|_2^2 + \|\partial_t u^2(s)\|_2^2)\tilde{\mathcal{E}}_W(t)\mathrm{d}s.\end{aligned} \tag{5.29}$$

Choosing $\varepsilon < \frac{1}{4N}$ in equation (5.29), we have

$$\begin{aligned}\tilde{\mathcal{E}}_W(t) \le \frac{\tilde{\beta}_2}{\tilde{\beta}_1} e^{-\frac{\tilde{C}}{\tilde{\beta}_2}t}\tilde{\mathcal{E}}_W(0) + \left(\frac{Nb_1}{\tilde{\beta}_1} + \frac{b_2\tilde{\beta}_2}{\tilde{\beta}_1\tilde{C}}\right)\sup_{0<s<t}\|\Phi(s)\|_4^2 \\ + \frac{2b_\varepsilon N e^{-\frac{\tilde{C}}{\tilde{\beta}_2}t}}{\tilde{\beta}_1}\int_0^t e^{\frac{\tilde{C}}{\tilde{\beta}_2}s}(\|\partial_t u^1(s)\|_2^2 + \|\partial_t u^2(s)\|_2^2)\tilde{\mathcal{E}}_W(t)\mathrm{d}s.\end{aligned} \tag{5.30}$$

Multiplying equation (5.30) with $e^{\frac{\tilde{C}}{\tilde{\beta}_2}t}$, the Gronwall lemma gives the estimate

$$e^{\frac{\tilde{C}}{\tilde{\beta}_2}t}\tilde{\mathcal{E}}_W(t) \le \left[\frac{\tilde{\beta}_2}{\tilde{\beta}_1}\tilde{\mathcal{E}}_W(0) + \left(\frac{Nb_1}{\tilde{\beta}_1} + \frac{b_2\tilde{\beta}_2}{\tilde{\beta}_1\tilde{C}}\right)e^{\frac{\tilde{C}}{\tilde{\beta}_2}t}\sup_{0<s<t}\|\Phi(s)\|_4^2\right]e^{\frac{2b_\varepsilon N}{\tilde{\beta}_1}\int_0^t(\|\partial_t u^1(s)\|_2^2 + \|\partial_t u^2(s)\|_2^2)\mathrm{d}s}. \tag{5.31}$$

By the uniform bounded of Z in $\mathcal{H}$, we can denote $\int_0^t(\|\partial_t u^1(s)\|_2^2 + \|\partial_t u^2(s)\|_2^2)\mathrm{d}s = C_0$. Then, combining the embedding $D(-\Delta_e) \hookrightarrow (L^4(\Omega))^3$, the inequality (5.31) implies

$$\begin{aligned}\tilde{\mathcal{E}}_W(t) &\le \frac{\tilde{\beta}_2 e^{\frac{2b_\varepsilon NC_0}{\tilde{\beta}_1}}}{\tilde{\beta}_1}\tilde{\mathcal{E}}_W(0)e^{-\frac{\tilde{C}}{\tilde{\beta}_2}t} + \left(\frac{Nb_1}{\tilde{\beta}_1} + \frac{b_2\tilde{\beta}_2}{\tilde{\beta}_1\tilde{C}}\right)e^{\frac{2b_\varepsilon NC_0}{\tilde{\beta}_1}}\sup_{0<s<t}\|\Phi(s)\|_4^2 \\ &\le \frac{\tilde{\beta}_2 e^{\frac{2b_\varepsilon NC_0}{\tilde{\beta}_1}}}{\tilde{\beta}_1}\|Z_0^1 - Z_0^2\|_{\mathcal{H}}^2 e^{-\frac{\tilde{C}}{\tilde{\beta}_2}t} + \left(\frac{Nb_1}{\tilde{\beta}_1} + \frac{b_2\tilde{\beta}_2}{\tilde{\beta}_1\tilde{C}}\right)e^{\frac{2b_\varepsilon NC_0}{\tilde{\beta}_1}}\sup_{0<s<t}\|u^1(t) - u^2(t)\|_e^2,\end{aligned}$$

which implies the quasi-stability inequality in equation (5.16). This finishes the proof of Lemma 5.9. □

Theorem 5.4. *Assume the hypotheses (H1)–(H5) and the given initial data* $(u_0, u_1, g_0) \in \mathcal{H}$ *for problem* (1.5) *hold, then we have the following results*:

The gradient system $(\mathcal{H}, S(t))$ *for the problem* (1.5) *possesses a compact finite fractal dimensional global attractor* $\mathcal{A} \subset \mathcal{H}$, *which has the following structure*:

$$\mathcal{A} = M_+(\mathcal{N}) \tag{5.32}$$

where $\mathcal{N} = \{y \in \mathcal{H}, S(t)y = y\}$ *for all* $t > 0$ *is the set of stationary points and* $M_+(\mathcal{N})$ *be the unstable manifold containing all trajectories emanating from the set* $\mathcal{N}$.

Moreover, the gradient system has a generalized exponential attractor $\mathcal{A}^{exp} \subset \mathcal{H}$ *with finite fractal dimension.*

Proof. The quasi-stability in Lemma 5.9 gives the asymptotic compactness of gradient system $(\mathcal{H}, S(t))$, and Lemma 5.8 presents the dissipation. By using the existence theory of finite dimensional attractors in Theorem 5.3, we can derive that the gradient system $(\mathcal{H}, S(t))$ for equation (1.5) possesses finite dimensional global and exponential attractors $\mathcal{A}$ and $\mathcal{A}^{exp}$, respectively. Moreover, $\mathcal{A}$ has the structure $\mathcal{A} = M_+(\mathcal{N})$. Thus, Theorem 5.4 has been proved. □

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