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Arched beams of Bresse type: New thermal couplings and pattern of stability

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Abstract. This is the second paper of a trilogy intended by the authors in what concerns a unified approach to the stability of thermoelastic arched beams of Bresse type under Fourier's law. Differently of the first one, where the thermal couplings are regarded on the axial and bending displacements, here the thermal couplings are taken over the shear and bending forces. Such thermal effects still result in a new prototype of partially damped Bresse system whose stability results demand a proper approach. Combining a novel path of local estimates by means of the resolvent equation along with a control-observability analysis developed for elastic non-homogeneous systems of Bresse type proposed in trilogy's first paper, we are able to provide a unified methodology of the asymptotic stability results, by proving the pattern of them with respect to boundary conditions and the action of temperature couplings, which is in compliance with our previous and present goal.

Keywords: Bresse system, thermoelasticity, stability, optimality

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1. Introduction

Inspired by the thermoelastic constitutive laws approached singly in [12] and [25], we address in the present article (second one within the trilogy) the following Bresse system with thermal couplings located on the shear force and the bending moment under Furier's law, which can be mathematically described as

$$\begin{aligned}
&\rho A\varphi_{tt} = Q_x + lN,\\
&\rho I\psi_{tt} = M_x - Q,\\
&\rho A w_{tt} = N_x - lQ,\\
&\rho c_v\theta_t = \theta_{xx} - k_1T_0(\varphi_x + \psi + lw)_t,\\
&\rho c_v\vartheta_t = \vartheta_{xx} - k_2T_0\psi_{xt},
\end{aligned} \tag{1.1}$$

where $\varphi = \varphi(x,t)$, $\psi = \psi(x,t)$, and $w = w(x,t)$ stand for the vertical displacement, rotation angle and longitudinal displacement, respectively, for $x \in [0, L]$, $t \geqslant 0$, and L being the beam length. Also, the functions $\theta = \theta(x,t)$ and $\vartheta = \vartheta(x,t)$ are the temperature deviations from the reference temperature T_0 along the longitudinal and vertical directions. For the remaining constants, ρ is the mass density per

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unit of the reference area, A is the cross-sectional area, I is the second moment of area of the cross-section, $l = R^{-1}$ where R is the curvature ratio of the beam, k_1 and k_2 are coupling constants and c_v is the material heat capacity. In addition, the quantities Q, N, and M represent, respectively, the shear force, the axial force, and the bending moment, whose corresponding constitutive (thermo-)elastic laws are considered in the present work as

$$Q = k'GA(\varphi_x + \psi + lw) - k_1\theta, \qquad N = EA(w_x - l\varphi), \qquad M = EI\psi_x - k_2\vartheta, \tag{1.2}$$

where E is the modulus of elasticity, G is the shear modulus and k' is the coefficient of shear.

Replacing, as usual, (1.2) in (1.1) one gets the following thermoelastic Bresse system

$$\begin{aligned}
&\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + k_1\theta_x = 0 \quad \text{in } (0, L) \times \mathbb{R}^+,\\
&\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) - k_1\theta + k_2\vartheta_x = 0 \quad \text{in } (0, L) \times \mathbb{R}^+,\\
&\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) - k_1 l\theta = 0 \quad \text{in } (0, L) \times \mathbb{R}^+,\\
&\rho_3\theta_t - \gamma_1\theta_{xx} + k_1(\varphi_x + \psi + lw)_t = 0 \quad \text{in } (0, L) \times \mathbb{R}^+,\\
&\rho_4\vartheta_t - \gamma_2\vartheta_{xx} + k_2\psi_{xt} = 0 \quad \text{in } (0, L) \times \mathbb{R}^+,
\end{aligned} \tag{1.3}$$

where we have simplified the notation on the constants as follows

$$\begin{aligned}
&\rho_1 = \rho A, \qquad \rho_2 = \rho I, \qquad k = k'GA, \qquad k_0 = EA,\\
&b = EI, \qquad \rho_3 = \rho_4 = \frac{\rho c_v}{T_0}, \qquad \gamma_1 = \gamma_2 = \frac{1}{\rho c_v T_0}.
\end{aligned}$$

Even having the identities $\gamma_1 = \gamma_2$ and $\rho_3 = \rho_4$ from the physical point of view, we are going to see that it can be mathematically studied as general (and different) constants in computations. Also, we still stress that the governing model (1.3) stands for a linear planar, shearable, and flexible thermoelastic beam vibration, which is a special case of networks on flexible thermoelastic (possibly nonlinear) beams as structured by Lagnese, Leugering & Schmidt [17,18]. All physical aspects including the modeling of thermoelastic Bresse systems, specially for the constitutive laws (1.2), will be explored in the third trilogy's paper (coming work). Here, our goal is to proceed with the second article within the series started in [2] on the asymptotic stability for beams of Bresse type with, let us say, possible double thermal couplings. In this direction, we consider the thermoelastic Bresse system (1.3) subject to initial conditions

$$\begin{aligned}
&\varphi(x, 0) = \varphi_0(x), \qquad \varphi_t(x, 0) = \varphi_1(x), \qquad \psi(x, 0) = \psi_0(x),\\
&\psi_t(x, 0) = \psi_1(x), \qquad w(x, 0) = w_0(x), \qquad w_t(x, 0) = w_1(x),\\
&\theta(x, 0) = \theta_0(x), \qquad \vartheta(x, 0) = \vartheta_0(x), \quad x \in (0, L),
\end{aligned} \tag{1.4}$$

and either the full Dirichlet or mixed Dirichlet–Neumann boundary conditions

$$\varphi(x, t) = \psi(x, t) = w(x, t) = \theta(x, t) = \vartheta(x, t) = 0, \quad x \in \{0, L\}, t \geqslant 0, \tag{1.5}$$

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or

$$\varphi(x,t) = \psi_x(x,t) = w_x(x,t) = \theta_x(x,t) = \vartheta(x,t) = 0, \quad x \in \{0, L\}, t \geqslant 0. \tag{1.6}$$

As far as the authors know, there is no previous study on the stability properties for the IBVP (1.3)–(1.6). The closest, but different, thermoelastic Bresse systems approached in the literature can be found in [12] and [25], where a single (thermal) coupling is considered. Also, the first paper of the trilogy [2] brings some similar aspects, though important differences arise physically and mathematically, the last being explained below. We still quote [1,6–9,11,14,16,23] where some different couplings and laws for the heat flux of conduction are regarded.

1.1. State of the art, main goal and contributions

In [12] the authors study the thermoelastic Bresse system with thermal dissipation acting only on the bending moment. Therein it is proved that the exponential decay of the system is directly related to the equal speeds of wave propagation (EWS for short). More precisely, the authors show the lack of exponential decay for certain boundary conditions when $\frac{k}{\rho_1} \neq \frac{b}{\rho_2}$ or $k \neq k_0$. Furthermore, they reach exponential decay only if $\frac{k}{\rho_1} = \frac{b}{\rho_2}$ and $k = k_0$. On other hand, when the EWS assumption does not occur, the system has only a kind of *semi-uniformly*[1] polynomial decay rate obeying the cases:

$$\begin{cases} \frac{1}{t^{1/6-\epsilon}} & \text{in case } \frac{k}{\rho_1} \neq \frac{b}{\rho_2} \text{ and } k \neq k_0; \\ \frac{1}{t^{1/3-\epsilon}} & \text{in case } \frac{k}{\rho_1} \neq \frac{b}{\rho_2} \text{ and } k = k_0; \end{cases}$$

where $\epsilon > 0$ is small enough. By following a similar structure of stability results, but now with thermal coupling just on the shear force, the authors in [25] also obtain the lack of exponential decay for the system by considering boundary conditions of the Dirichlet–Neumann type. The exponential stability of the system is obtained through the EWS assumption and, when one of the identities does not happen, a semi-uniform polynomial decay rate is obtained depending again on the boundary conditions. Indeed, for boundary conditions of the Dirichlet type, the decay rate $\frac{1}{t^{1/4}}$ is obtained whereas for boundary conditions of the Dirichlet–Neumann one, the faster decay rate $\frac{1}{t^{1/2}}$ is in place, the latter being optimal. Such stability dichotomy also appears in thermoelastic problems of Bresse type under other laws, cf. [24].

As a consequence of the aforementioned papers, one can see that the stability results depend not only on the EWS assumption but also on the boundary condition in turn. However, such a dichotomy does not seem to have a physical (nor mathematical) explanation. Here, our objective is to study the asymptotic stability of the problem (1.3)–(1.6) by proving that all results on stability (exponential and polynomial) are achieved independently of the boundary conditions (1.5) or (1.6), except for the optimality one. This achievement will be possible via refined computations with cut-off multipliers in the resolvent equation and the application of the *observability inequality* developed in the first paper of the trilogy (cf. [2]). Then, the results will follow by applying the classical result in the semigroup theory, namely, Gearhart–Huang–Prüss and Borichev–Tomilov's Theorems (cf. [20, Theorem 1.3.2] and [3, Theorem 2.4]).

[1] Here, similar to the previous article of the trilogy, the notion of *semi-uniform stability* is always invoked when the stability of the semigroup solution does not occur for all weak initial data (say at the same energy level of solutions), but only for more regular initial data, e.g. data in the domain of the infinitesimal generator of the semigroup.

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More precisely, we show the IBVP (1.3)–(1.6) is exponentially stable if $k = k_0$, see Theorem 2.5. The lack of exponential stability when considering the mixed boundary conditions (1.6) is approached in Corollary 2.6. Also, when $k \neq k_0$, the semi-uniform polynomial stability with the solely decay rate $\frac{1}{t^{1/2}}$ is proved, regardless of the boundary condition taken into account, see Theorem 2.2. Additionally, the optimality of this decay rate in case of b.c. (1.6) is proved, according to Theorem 2.3.

Therefore, the main contributions in this second paper of the series are highlighted as follows:

I. New Couplings. The stability of Bresse systems has been studied for over a decade. We can find such systems in different ways and with different couplings, but it is the first time that the thermoelastic Bresse system with thermoelastic coupling in the shear force and in the bending moment, as given in (1.3), is regarded in the literature. The motivation of this study is directly related to the study carried out previously in [2] whose difference is in the exchange of thermal coupling from axial force to shear force.

II. Pattern of Stability. In addition to the first item, the studies carried out on the Bresse thermoelastic system with a single coupling (e.g. [12,25]) promote polynomial decay rates depending on the EWS assumption, which is usual, but also depend on the boundary conditions, which can be considered as a dichotomy. Though our system is different from [2] where (therein) the thermal couplings are placed on the axial (N) and bending (M) displacements, here we address thermal couplings on the shear (Q) and bending (M) forces (see (1.2)), we still prove here the same semi-uniform polynomial decay rate $\frac{1}{t^{1/2}}$, independently of the boundary conditions in turn, by providing a pattern of stability promised in this trilogy with respect to (double) thermal couplings and boundary conditions. Furthermore, in case of Dirichlet–Neumann boundary conditions, we prove the optimality of the decay rate.

III. Path of Proofs. In order to clarify the technical difference when compared to [2] (and the statements in the right above item II), we stress that there is a difference in the path of the proofs as explained in the next diagrams. Indeed, when compared to the first trilogy's paper [2] where the it is considered the thermal laws applied to the axial force and the bending moment (cf. [19])

$$Q = k'GA(\varphi_x + \psi + lw), \qquad N = EA(w_x - l\varphi) - k_2\eta \qquad M = EI\psi_x - k_1\vartheta, \tag{1.7}$$

rather than (1.2), the propagation of dissipativity to the whole terms of the solution obeys the following path

$$\boxed{\eta, \vartheta \xrightarrow{1^{\text{st}}} w_x - l\varphi, \psi_x \xrightarrow{2^{\text{nd}}} \varphi_x + \psi + lw} \tag{1.8}$$

whereas here it is necessary a new (and different) way for the estimates. In fact, such a way (1.8) is not useful in the present scenario where the thermal law is applied to the shear force and the bending moment as in (1.2). This means that the same technical computations therein can not be applied here in its full essence. Instead, we highlight that the estimates shall be provided in a different configuration as follows

$$\boxed{\theta, \vartheta \xrightarrow{1^{\text{st}}} \varphi_x + \psi + lw, \psi_x \xrightarrow{2^{\text{nd}}} w_x - l\varphi} \tag{1.9}$$

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being all computations clarified in Section 3.1. The path (1.9) reveals that the propagation of dissipativity through the whole energy terms requires technical arguments different from [2] once developed to (1.8).

The remaining paper is organized as follows: in Section 2 we set the problem in a semigroup framework and state our main results on stability. In Section 3 we provide all proofs. We end this work with Section 4 where brief remarks on the results are considered and Appendix A where we recall the observability inequality for systems of Bresse type.

2. Main results on asymptotic stability

Recalling, for $\mathbb{R}^+ = (0, \infty)$ and $L > 0$, we consider the following Bresse system with thermoelastic coupling on the shear force and the bending moment:

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + k_1\theta_x = 0 \quad \text{in } (0, L) \times \mathbb{R}^+, \tag{2.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) - k_1\theta + k_2\vartheta_x = 0 \quad \text{in } (0, L) \times \mathbb{R}^+, \tag{2.2}$$

$$\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) - k_1 l\theta = 0 \quad \text{in } (0, L) \times \mathbb{R}^+, \tag{2.3}$$

$$\rho_3\theta_t - \gamma_1\theta_{xx} + k_1(\varphi_x + \psi + lw)_t = 0 \quad \text{in } (0, L) \times \mathbb{R}^+, \tag{2.4}$$

$$\rho_4\vartheta_t - \gamma_2\vartheta_{xx} + k_2\psi_{xt} = 0 \quad \text{in } (0, L) \times \mathbb{R}^+, \tag{2.5}$$

subject to initial conditions

$$\begin{aligned} &\varphi(\cdot, 0) = \varphi_0(\cdot), \qquad \varphi_t(\cdot, 0) = \varphi_1(\cdot), \qquad \psi(\cdot, 0) = \psi_0(\cdot), \qquad \psi_t(\cdot, 0) = \psi_1(\cdot), \\ &w(\cdot, 0) = w_0(\cdot), \qquad w_t(\cdot, 0) = w_1(\cdot), \qquad \theta(\cdot, 0) = \theta_0(\cdot), \qquad \vartheta(\cdot, 0) = \vartheta_0(\cdot) \quad \text{in } (0, L), \end{aligned} \tag{2.6}$$

and Dirichlet boundary conditions

$$\varphi(x, t) = \psi(x, t) = w(x, t) = \theta(x, t) = \vartheta(x, t) = 0, \quad \text{for } t \geqslant 0, x \in \{0, L\}, \tag{2.7}$$

or the mixed Neumann–Dirichlet one

$$\varphi(x, t) = \psi_x(x, t) = w_x(x, t) = \theta_x(x, t) = \vartheta(x, t) = 0, \quad \text{for } t \geqslant 0, x \in \{0, L\}, \tag{2.8}$$

where ρ_1, ρ_2, ρ_3, ρ_4, k, b, k_0, γ_1, γ_2, k_1, k_2 are positive coefficients, whose physical meanings are very well understood and come from the material making up the beam with length $L > 0$, and the unknown functions φ, ψ, w, θ and ϑ are related to the transversal displacement, rotation angle, longitudinal displacement, and temperature deviations, respectively.

2.1. Notations and semigroup framework

Let us initially denote

$$L^2_*(0, L) = \left\{ u \in L^2(0, L) : \frac{1}{L}\int_0^L u(x)\,dx = 0 \right\}$$

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and

$$H^1_*(0, L) = H^1(0, L) \cap L^2_*(0, L).$$

Hereafter, in order to simplify the notations, we shall neglect the range $(0, L)$ of the spaces $H^2(0, L)$, $H^1(0, L)$, $L^2(0, L)$, $H^1_0(0, L)$, $H^1_*(0, L)$, and $L^2_*(0, L)$.

We start by considering the Hilbert phase spaces

$$\mathcal{H}_1 = H^1_0 \times L^2 \times H^1_0 \times L^2 \times H^1_0 \times L^2 \times L^2 \times L^2 \quad \text{for (2.7)},$$

and

$$\mathcal{H}_2 = H^1_0 \times L^2 \times H^1_* \times L^2_* \times H^1_* \times L^2_* \times L^2_* \times L^2 \quad \text{for (2.8)},$$

both with inner product

$$\begin{aligned}(U, U^*)_{\mathcal{H}_j} = \int_0^L \big[&\rho_1 \Phi \overline{\Phi^*} + \rho_2 \Psi \overline{\Psi^*} + \rho_1 W \overline{W^*} + b\psi_x \overline{\psi_x^*} + k(\varphi_x + \psi + lw)\overline{(\varphi_x^* + \psi^* + lw^*)} \\ &+ k_0(w_x - l\varphi)\overline{(w_x^* - l\varphi^*)} + \rho_3 \theta \overline{\theta^*} + \rho_4 \vartheta \overline{\vartheta^*}\big]\, dx,\end{aligned} \tag{2.9}$$

and induced norm

$$\begin{aligned}\|U\|^2_{\mathcal{H}_j} = \int_0^L \big[&\rho_1 |\Phi|^2 + \rho_2 |\Psi|^2 + \rho_1 |W|^2 + b|\psi_x|^2 + k|\varphi_x + \psi + lw|^2 + k_0 |w_x - l\varphi|^2 \\ &+ \rho_3 |\theta|^2 + \rho_4 |\vartheta|^2\big]\, dx,\end{aligned} \tag{2.10}$$

for all $U = (\varphi, \Phi, \psi, \Psi, w, W, \theta, \vartheta)$, $U^* = (\varphi^*, \Phi^*, \psi^*, \Psi^*, w^*, W^*, \theta^*, \vartheta^*) \in \mathcal{H}_j$, $j = 1, 2$.

Remark 2.1. As highlighted e.g. in [2, Remark 3.1], the bilinear map (2.9) does define an inner product in $\mathcal{H}_1$, whereas in $\mathcal{H}_2$ it is an inner product only if $Ll \neq n\pi$, $n \in \mathbb{Z}$. Therefore, from now on, when working in the space (2.8), such condition is implicitly assumed.

Denoting $\varphi_t = \Phi$, $\psi_t = \Psi$, $w_t = W$ and $U = (\varphi, \Phi, \psi, \Psi, w, W, \theta, \vartheta)$, we can convert the thermoelastic system of second-order (2.1)–(2.6) into the following Cauchy problem

$$\begin{cases} \frac{d}{dt} U = \mathcal{A}_j U, \quad t > 0, \\ U(0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \theta_0, \vartheta_0) := U_0, \end{cases} \tag{2.11}$$

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where $\mathcal{A}_j : D(\mathcal{A}_j) \subset \mathcal{H}_j \to \mathcal{H}_j$ is defined by

$$\mathcal{A}_j U = \begin{bmatrix} \Phi \\ \frac{k}{\rho_1}(\varphi_x + \psi + lw)_x + \frac{k_0 l}{\rho_1}(w_x - l\varphi) - \frac{k_1}{\rho_1}\theta_x \\ \Psi \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + lw) + \frac{k_1}{\rho_2}\theta - \frac{k_2}{\rho_2}\vartheta_x \\ W \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{kl}{\rho_1}(\varphi_x + \psi + lw) + \frac{k_1 l}{\rho_1}\theta \\ \frac{\gamma_1}{\rho_3}\theta_{xx} - \frac{k_1}{\rho_3}(\Phi_x + \Psi + lW) \\ \frac{\gamma_2}{\rho_4}\vartheta_{xx} - \frac{k_2}{\rho_4}\Psi_x \end{bmatrix}, \quad U \in D(\mathcal{A}_j),\ j = 1, 2, \tag{2.12}$$

with domain

$$D(\mathcal{A}_1) = \left\{U \in \mathcal{H}_1 : \varphi, \psi, w, \theta, \vartheta \in H^2 \cap H_0^1;\ \Phi, \Psi, W \in H_0^1\right\} \quad \text{for (2.7)},$$

and

$$D(\mathcal{A}_2) = \left\{U \in \mathcal{H}_2 : \varphi, \vartheta \in H^2;\ \varphi, \Phi, \psi_x, w_x, \theta_x, \vartheta \in H_0^1;\ \Psi, W \in H_*^1\right\} \quad \text{for (2.8)}.$$

Under the above notations, the existence and uniqueness of solution to (2.11) and, consequently, to (2.1)–(2.6), reads as follows:

Theorem 2.1. *Under the above notations, we have:*

(i) *If $U_0 \in \mathcal{H}_j$, then problem (2.11) has a unique mild solution*

$$U \in C^0\big([0, \infty), \mathcal{H}_j\big), \quad j = 1, 2.$$

(ii) *If $U_0 \in D(\mathcal{A}_j)$, then problem (2.11) has a unique regular solution*

$$U \in C^0\big([0, \infty), D(\mathcal{A}_j)\big) \cap C^1\big([0, \infty), \mathcal{H}_j\big), \quad j = 1, 2.$$

(iii) *If $U_0 \in D(\mathcal{A}_j^n)$, $n \geqslant 2$ integer, then the solution is more regular*

$$U \in \bigcap_{\nu=0}^{n} C^{n-\nu}\big([0, \infty), D\big(\mathcal{A}_j^\nu\big)\big), \quad j = 1, 2.$$

Sketch of the proof. The proof can be done similarly to a combination of arguments provided by [12, 25]. For the sake of future computations, here it follows the necessary clarifications.

First, it is not difficult to check that $0 \in \rho(\mathcal{A}_j)$, where $\rho(\mathcal{A}_j)$ stands for the resolvent set of $\mathcal{A}_j$, $j = 1, 2$. Then, a straightforward computation shows that $\mathcal{A}_j$ is dissipative with

$$\mathrm{Re}(\mathcal{A}_j U, U)_{\mathcal{H}_j} = -\int_0^L \gamma_1 |\theta_x|^2\, dx - \int_0^L \gamma_2 |\vartheta_x|^2\, dx \leqslant 0, \quad U \in D(\mathcal{A}_j),\ j = 1, 2. \tag{2.13}$$

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Therefore, employing the classical Lummer–Phillips Theorem (cf. [21, Theorem 4.6]) we have that $\mathcal{A}_j$ is the infinitesimal generator of a C_0-semigroup of contractions $S_j(t) := e^{\mathcal{A}_j t}$ on $\mathcal{H}_j$, $j = 1, 2$. Consequently, the solution of (2.11) satisfying (i)–(iii) is given by

$$U(t) = e^{\mathcal{A}_j t} U_0, \quad t \geqslant 0, \, j = 1, 2. \qquad \square$$

2.2. Main results

Our first main result asserts that problem (2.1)–(2.6) is, in general, only semi-uniformly stable with the polynomial rate depending on the regularity of initial data. However, it is independent of the boundary conditions. In any case, the asymptotic stability will depend on the following number

$$\chi_0 := k_0 - k. \tag{2.14}$$

Theorem 2.2 (Semi-uniform Polynomial Decay)**.** *Let us assume that $\chi_0 \neq 0$ in (2.14). Then, for every integer $n \geqslant 1$, there exists a constant $C_n > 0$ independent of $U_0 \in D(\mathcal{A}_j{}^n)$ such that the semigroup solution $U(t) = e^{\mathcal{A}_j t} U_0$ satisfies*

$$\|U(t)\|_{\mathcal{H}_j} \leqslant \frac{C_n}{t^{n/2}} \|U_0\|_{D(\mathcal{A}_j^n)}, \quad j = 1, 2, \text{ as } t \to +\infty. \tag{2.15}$$

In other words, the thermoelastic system (2.1)–(2.6) with either boundary conditions (2.7) or (2.8) is (semi-uniformly) polynomially stable with the decay rate depending on the regularity of initial data.

In addition to Theorem 2.2, one can show that the semi-uniform polynomial decay is optimal for the boundary condition (2.8). This is proved for $n = 1$, namely when initial data belong to the domain of the operator. More precisely, we have:

Theorem 2.3 (Optimality)**.** *Let us assume that $\chi_0 \neq 0$ and take $U_0 \in D(\mathcal{A}_2)$. Then, the semi-uniform polynomial rate $1/t^{1/2}$ obtained (2.15) is optimal in the following sense: there is no constant $\nu_0 > 0$ such that*

$$\|U(t)\|_{\mathcal{H}_2} \leqslant \frac{C}{t^{\frac{1}{2}+\nu_0}} \|U_0\|_{D(\mathcal{A}_2)}, \quad t \to +\infty. \tag{2.16}$$

In particular, the thermoelastic Bresse system (2.1)–(2.6) with boundary condition (2.8) is not exponentially stable if $\chi_0 \neq 0$.

As an immediate consequence of Theorem 2.3, we deduce the next result.

Corollary 2.4 (Non-uniform Stability)**.** *Under the conditions of Theorem 2.3, the system (2.1)–(2.6) with boundary condition (2.8) is never uniformly stable for initial data $U_0 \in \mathcal{H}_2$. More precisely, there is no positive function $\Upsilon(t)$ vanishing at infinity such that*

$$\|U(t)\|_{\mathcal{H}_2} \leqslant C_0 \Upsilon(t), \quad \forall U_0 \in \mathcal{H}_2, t \to +\infty, \tag{2.17}$$

where $C_0 = C_0(\|U_0\|_{\mathcal{H}_2}) > 0$ is a constant depending on U_0.

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Proof. It follows promptly from Theorem 2.3 and [5, Remark 3.1]. □

Our fourth main result in this section deals with the uniform (exponential) stability of system (2.1)–(2.8) when the assumption on equal wave speeds is taken into account.

Theorem 2.5 (Uniform Exponential Stability)**.** *Let us assume that $\chi_0 = 0$ in (2.14). Then, there exist constants $C, \omega > 0$ independent of $U_0 \in \mathcal{H}_j$ such that the semigroup solution $U(t) = e^{\mathcal{A}_j t} U_0$, $j = 1, 2$, satisfies*

$$\|U(t)\|_{\mathcal{H}_j} \leqslant C e^{-\omega t} \|U_0\|_{\mathcal{H}_j}, \quad t > 0. \tag{2.18}$$

In other words, the thermoelastic system (2.1)–(2.6) with either boundary conditions (2.7) or (2.8) is (uniformly) exponentially stable if $\chi_0 = 0$.

Corollary 2.6. *The thermoelastic Bresse system (2.1)–(2.6) with boundary condition (2.8) is exponentially stable if and only if $\chi_0 = 0$.*

Proof. Immediately from Theorems 2.3 and 2.5. □

The conclusion of the proofs of Theorems 2.2 to 2.5 shall be given in the next section. Indeed, we first explore the preliminary tools for this goal, namely, we provide some technical lemmas with localized estimates through the resolvent equation and then combine with the observability inequality previously obtained in the first trilogy paper [2] for systems of Bresse type. Hence, the proofs will follow from the general theory in linear semigroup, see e.g. [3,10,13,15,20,22].

3. Proofs

3.1. Technical results via resolvent equation

The resolvent equation associated with problem (2.11) is given by

$$i\beta U - \mathcal{A}_j U = F, \quad j = 1, 2, \tag{3.1}$$

with $U = (\varphi, \Phi, \psi, \Psi, w, W, \theta, \vartheta)$, $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)$ and $\mathcal{A}_j$ defined in (2.12), which in terms of its components takes the form

$$i\beta\varphi - \Phi = f_1, \tag{3.2}$$

$$i\beta\rho_1\Phi - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + k_1\theta_x = \rho_1 f_2, \tag{3.3}$$

$$i\beta\psi - \Psi = f_3, \tag{3.4}$$

$$i\beta\rho_2\Psi - b\psi_{xx} + k(\varphi_x + \psi + lw) - k_1\theta + k_2\vartheta_x = \rho_2 f_4, \tag{3.5}$$

$$i\beta w - W = f_5, \tag{3.6}$$

$$i\beta\rho_1 W - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) - k_1 l\theta = \rho_1 f_6, \tag{3.7}$$

$$i\beta\rho_3\theta - \gamma_1\theta_{xx} + k_1(\Phi_x + \Psi + lW) = \rho_3 f_7, \tag{3.8}$$

$$i\beta\rho_4\vartheta - \gamma_2\vartheta_{xx} + k_2\Psi_x = \rho_4 f_8. \tag{3.9}$$

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Lemma 3.1. *Under the above notations, we have $i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$, where $\rho(\mathcal{A}_j)$ stands for the resolvent set of $\mathcal{A}_j$, $j = 1, 2$, given in (2.12).*

Proof. Since the embedding $D(\mathcal{A}_j) \hookrightarrow \mathcal{H}_j$ is compact (whose computations can be done as a good exercise), it is enough to prove the following property for any $\beta \in \mathbb{R}$,

$$i\beta U - \mathcal{A}_j U = 0 \quad \Longrightarrow \quad U = 0,$$

and then employing the results in [10, Proposition 5.8 and Corollary 1.15] we conclude that the spectrum of $\mathcal{A}_j$ consists only of eigenvalues. As a prompt consequence, one concludes that $i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$.

This methodology has been hugely employed lately and its mathematical justifications relies on similar arguments as presented e.g. in [12,25] (see also [2, Lemma 3.7]). □

Hereafter, to simplify the notations, we will use a parameter $C > 0$ to denote several different positive constants in the computations below. As usual, $\|\cdot\|_2$ stands for the norm in L^2. Hölder and Poincaré's inequalities will be constantly regarded, sometimes implicitly in the estimates without mentioning them to avoid so many repetitions, and also $|\beta| > 1$ large enough can be taken w.l.o.g. in the estimates.

Lemma 3.2. *Under the above notations, there exists a constant $C > 0$ such that*

$$\|\theta_x\|_2^2 + \|\vartheta_x\|_2^2 \leqslant C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j}, \quad \text{for } j = 1, 2. \tag{3.10}$$

Proof. Estimate (3.10) is a direct consequence of (2.13) and (3.1). □

To the next results, we shall invoke some useful auxiliary cut-off functions in order to obtain localized estimates. This allows us to work with both boundary conditions at the same time without trouble with possible boundary point-wise terms coming from integration by parts.

Let us consider $l_0 \in (0, L)$ and $\delta > 0$ such that $(l_0 - \delta, l_0 + \delta) \subset (0, L)$. Then, we set $s_1 \in C^2(0, L)$ satisfying

$$\operatorname{supp} s_1 \subset (l_0 - \delta, l_0 + \delta), \quad 0 \leqslant s_1(x) \leqslant 1, x \in (0, L), \tag{3.11}$$

and

$$s_1(x) = 1 \quad \text{for } x \in [l_0 - \delta/2, l_0 + \delta/2]. \tag{3.12}$$

Remark 3.1. An explicit example of such a cut-off function is given in [2, Remark 3.2]. The geometric idea of s_1 can be seen e.g. in [4, Fig. 1].

Lemma 3.3. *Under the above notations, there exists a constant $C > 0$ such that*

$$\int_{l_0-\delta}^{l_0+\delta} s_1|\varphi_x + \psi + lw|^2\,dx \leqslant C\|\theta_x\|_2\left(\int_{l_0-\delta}^{l_0+\delta} s_1|\Phi|^2\,dx\right)^{1/2} + \frac{C}{|\beta|}\|\theta_x\|_2\|U\|_{\mathcal{H}_j} + \frac{C}{|\beta|}\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + \frac{C}{|\beta|}\|F\|_{\mathcal{H}_j}^2, \tag{3.13}$$

for $j = 1, 2$.

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Proof. Deriving the equation (3.2) and adding with (3.4) and (3.6), we have

$$\Phi_x + \Psi + lW = i\beta(\varphi_x + \psi + lw) - (f_{1,x} + f_3 + lf_5), \tag{3.14}$$

and replacing (3.14) in (3.8),

$$k_1 i\beta(\varphi_x + \psi + lw) = \gamma_1\theta_{xx} - i\beta\rho_3\theta + \rho_3 f_7 + k_1(f_{1,x} + f_3 + lf_5). \tag{3.15}$$

Taking the multiplier $s_1 k\overline{(\varphi_x + \psi + lw)}$ in (3.15), performing integration by parts and using the equations (3.2), (3.3) and (3.4), we have

$$\begin{aligned} i\beta k_1 k\int_0^L s_1|\varphi_x + \psi + lw|^2\,dx = {} & \underbrace{-\gamma_1 k\int_0^L s_1\theta_x\overline{(\varphi_x + \psi + lw)_x}\,dx}_{:=I_1} - \rho_3 k\int_0^L s_1\theta_x\overline{(\Phi + f_1)}\,dx \\ & - \gamma_1 k\int_0^L s_1'\theta_x\overline{(\varphi_x + \psi + lw)}\,dx - \rho_3 k\int_0^L s_1'\theta\overline{(\Phi + f_1)}\,dx \\ & + k\int_0^L s_1\big[k_1(f_{1,x} + f_3 + lf_5) + \rho_3 f_7\big]\overline{(\varphi_x + \psi + lw)}\,dx \\ & + \rho_3 k\int_0^L s_1\theta\overline{(\Psi + lW + f_3 + lf_5)}\,dx. \end{aligned} \tag{3.16}$$

Replacing the equation (3.3) in I_1, we get

$$i\beta k_1 k\int_0^L s_1|\varphi_x + \psi + lw|^2\,dx = i\beta\gamma_1\rho_1\int_0^L s_1\theta_x\overline{\Phi}\,dx + I_2, \tag{3.17}$$

where

$$\begin{aligned} I_2 = {} & \gamma_1\int_0^L s_1\theta_x\overline{\big[k_0 l(w_x - l\varphi) - k_1\theta_x + \rho_2 f_2\big]}\,dx - \gamma_1 k\int_0^L s_1'\theta_x\overline{(\varphi_x + \psi + lw)}\,dx \\ & + k\int_0^L s_1\big[k_1(f_{1,x} + f_3 + lf_5) + \rho_3 f_7\big]\overline{(\varphi_x + \psi + lw)}\,dx - \rho_3 k\int_0^L s_1\theta_x\overline{(\Phi + f_1)}\,dx \\ & - \rho_3 k\int_0^L s_1'\theta\overline{(\Phi + f_1)}\,dx + \rho_3 k\int_0^L s_1\theta\overline{(\Psi + lW + f_3 + lf_5)}\,dx. \end{aligned} \tag{3.18}$$

From condition (3.11) on s_1, Hölder's inequality and Lemma 3.2, we have that

$$|I_2| \leqslant C\|\theta_x\|_2\|U\|_{\mathcal{H}_j} + C\|\theta_x\|_2\|F\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j},$$

for some constant $C > 0$ and for $j = 1, 2$. Therefore, from (3.17) and using Hölder's inequality along with the condition (3.11), we obtain

$$\int_{l_0-\delta}^{l_0+\delta} s_1|\varphi_x + \psi + lw|^2\,dx \leqslant C\|\theta_x\|_2\left(\int_{l_0-\delta}^{l_0+\delta} s_1|\Phi|^2\,dx\right)^{1/2} + \frac{C}{|\beta|}|I_2|, \tag{3.19}$$

for some constant $C > 0$. Thus, from Young's inequality and (3.10), follow the desired in (3.13). □

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Lemma 3.4. *Under the above notations, there exists a constant $C > 0$ such that*

$$\int_{l_0-\delta}^{l_0+\delta} s_1|\Phi|^2\,dx \leqslant \frac{C}{|\beta|}\|\theta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j}, \tag{3.20}$$

for $j = 1, 2$.

Proof. Multiplying the equation (3.3) by $-s_1\overline{\varphi}$ and integrating by parts, we have

$$\begin{aligned}-i\beta\rho_1\int_0^L s_1\Phi\overline{\varphi}\,dx &= k\int_0^L s_1(\varphi_x+\psi+lw)\overline{\varphi_x}\,dx - k_0 l\int_0^L s_1(w_x-l\varphi)\overline{\varphi}\,dx\\ &\quad+\int_0^L s_1[k_1\theta_x-\rho_1 f_2]\overline{\varphi}\,dx + k\int_0^L s_1'(\varphi_x+\psi+lw)_x\overline{\varphi}\,dx.\end{aligned}$$

Integrating by parts and using the equation (3.2), it follows that

$$\rho_1\int_0^L s_1|\Phi|^2\,dx = k\int_0^L s_1|\varphi_x+\psi+lw|^2\,dx + \underbrace{k\int_0^L s_1'\varphi_x\overline{\varphi}\,dx}_{:=I_3} + I_4, \tag{3.21}$$

where

$$\begin{aligned}I_4 &= -k\int_0^L s_1(\varphi_x+\psi+lw)\overline{(\psi+lw)}\,dx + k\int_0^L s_1'(\psi+lw)\overline{\varphi}\,dx - k_0 l\int_0^L s_1 w_x\overline{\varphi}\,dx\\ &\quad+k_0 l^2\int_0^L s_1|\varphi|^2\,dx + \int_0^L s_1[k_1\theta_x-\rho_1 f_2]\overline{\varphi}\,dx - \rho_1\int_0^L s_1\Phi\overline{f_1}\,dx.\end{aligned} \tag{3.22}$$

Using the equations (3.2), (3.4) and (3.6), it's easy to see that

$$\begin{aligned}|I_4| &\leqslant \frac{C}{|\beta|}\|U\|_{\mathcal{H}_j}\left(\int_{l_0-\delta}^{l_0+\delta} s_1|\varphi_x+\psi+lw|^2\,dx\right)^{1/2} + \frac{C}{|\beta|}\|\theta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j}\\ &\quad+C\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j},\end{aligned} \tag{3.23}$$

and

$$|\operatorname{Re} I_3| \leqslant \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2}\|F\|^2_{\mathcal{H}_j}, \tag{3.24}$$

for some constant $C > 0$ and $j = 1, 2$. Therefore, taking the real part of (3.21), using (3.23), (3.24) and Young's inequality, we have

$$\begin{aligned}\int_{l_0-\delta}^{l_0+\delta} s_1|\Phi|^2\,dx &\leqslant C\int_{l_0-\delta}^{l_0+\delta} s_1|\varphi_x+\psi+lw|^2\,dx + \frac{C}{|\beta|}\|\theta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j}\\ &\quad+C\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j},\end{aligned} \tag{3.25}$$

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for some constant $C > 0$ and $j = 1, 2$. Now, using the Lemma 3.3 and Young's inequality, there exists a constant $C > 0$ such that

$$\int_{l_0-\delta}^{l_0+\delta} s_1|\Phi|^2\,dx \leqslant C\|\theta_x\|_{\mathcal{H}_j}^2 + \frac{C}{|\beta|}\|\theta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + C\|F\|_{\mathcal{H}_j}^2 + \frac{C}{|\beta|^2}\|U\|_{\mathcal{H}_j}^2, \tag{3.26}$$

for $j = 1, 2$. Finally, from the Lemma 3.2, we can conclude (3.20). □

Corollary 3.5. *Under the above notations, there exists a constant $C > 0$ such that*

$$\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |\varphi_x + \psi + lw|^2\,dx \leqslant \frac{C}{|\beta|}\|\theta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + C\|F\|_{\mathcal{H}_j}^2, \tag{3.27}$$

for $j = 1, 2$.

Proof. Just combine Lemmas 3.3 and 3.4, and use the condition (3.12) on s_1. □

Lemma 3.6. *Under the above notations, there exists a constant $C > 0$ such that*

$$\int_{l_0-\delta}^{l_0+\delta} s_1|\psi_x|^2\,dx \leqslant C\|\vartheta_x\|_2\left(\int_{l_0-\delta}^{l_0+\delta} s_1|\Psi|^2\,dx\right)^{1/2} + \frac{C}{|\beta|}\|\vartheta_x\|_2\|U\|_{\mathcal{H}_j} + \frac{C}{|\beta|}\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + \frac{C}{|\beta|}\|F\|_{\mathcal{H}_j}^2, \tag{3.28}$$

for $j = 1, 2$.

Lemma 3.7. *Under the above notations, there exists a constant $C > 0$ such that*

$$\int_{l_0-\delta}^{l_0+\delta} s_1|\Psi|^2\,dx \leqslant \frac{C}{|\beta|}\|\vartheta_x\|_2\|U\|_{\mathcal{H}_j} + \frac{C}{|\beta|}\|\theta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + C\|F\|_{\mathcal{H}_j}^2 + \frac{C}{|\beta|^2}\|U\|_{\mathcal{H}_j}^2, \tag{3.29}$$

for $j = 1, 2$.

Proof. Observing that we have the addition of the thermal coupling in the bending moment here and in [2], the estimate for the parts ψ_x and Ψ follows in a totally analogous way to that performed in [2, Lemma 3.12 and Lemma 3.13]. However, we must pay attention to the presence of two thermal components in the estimate of Ψ (unlike the estimate obtained in Lemma 3.13 in [2]) due to the presence

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of both terms in (3.5). In the remaining proof, all computations follow similarly to the ones previously proved in [2, Lemma 3.12 and Lemma 3.13]. □

Corollary 3.8. *Under the above notations, there exists a constant $C > 0$ such that*

$$\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} \left(|\psi_x|^2 + |\Psi|^2\right) dx \leqslant \frac{C}{|\beta|}\|\vartheta_x\|_2\|U\|_{\mathcal{H}_j} + \frac{C}{|\beta|}\|\theta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j}, \quad \textit{for } j = 1, 2.$$

Proof. Just combine Lemmas 3.6 and 3.7, and use the condition (3.12) on s_1. □

Now, we consider another auxiliary cut-off function $s_2 \in C^2(0, L)$ satisfying

$$\operatorname{supp} s_2 \subset (l_0 - \delta/2, l_0 + \delta/2), \quad 0 \leqslant s_2(x) \leqslant 1, x \in (0, L), \tag{3.30}$$

and

$$s_2(x) = 1 \quad \text{for } x \in [l_0 - \delta/3, l_0 + \delta/3]. \tag{3.31}$$

A prototype of such a function can be considered in a similar way as done in [2, Remark 3.2]. See also [4, Fig. 1].

Lemma 3.9. *Under the above notations, there exists a constant $C > 0$ such that*

$$\begin{aligned}\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2\left(|w_x - l\varphi|^2 + |W|^2\right) dx &\leqslant C|\beta||k - k_0| \int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|\varphi_x + \psi + lw|^2\, dx + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j} \\ &\quad + C\|U\|_{\mathcal{H}_j}\left(\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |\varphi_x + \psi + lw|^2\, dx\right)^{1/2} + C\|F\|^2_{\mathcal{H}_j} \\ &\quad + C\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} \left(|\varphi_x + \psi + lw|^2 + |\Psi|^2\right) dx \\ &\quad + C\|\theta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j}, \quad \textit{for } j = 1, 2.\end{aligned} \tag{3.32}$$

Proof. Multiplying (3.7) by $s_2\frac{lk_0}{\rho_1}\overline{w}$ and integrating on $(0, L)$ we get

$$\begin{aligned}-\frac{k_0^2 l}{\rho_1}\int_0^L s_2(w_x - l\varphi)_x\overline{w}\, dx &= -i\beta k_0 l\int_0^L s_2 W\overline{w}\, dx - \frac{kl^2k_0}{\rho_1}\int_0^L s_2(\varphi_x + \psi + lw)\overline{w}\, dx \\ &\quad + \frac{k_0 l}{\rho_1}\int_0^L s_2[lk_1\theta + \rho_1 f_6]\overline{w}\, dx.\end{aligned} \tag{3.33}$$

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Now, integrating by parts and adjusting some terms, we have

$$\frac{k_0^2 l}{\rho_1}\int_0^L s_2|w_x - l\varphi|^2\,dx = \underbrace{-i\beta k_0 l\int_0^L s_2 W\overline{w}\,dx}_{:=I_5} - \frac{kl^2k_0}{\rho_1}\int_0^L s_2(\varphi_x+\psi+lw)\overline{w}\,dx$$
$$+\frac{k_0 l}{\rho_1}\int_0^L s_2[lk_1\theta+\rho_1 f_6]\overline{w}\,dx - \frac{k_0^2 l}{\rho_1}\int_0^L s_2' w_x\overline{w}\,dx$$
$$+\frac{k_0^2 l^2}{\rho_1}\int_0^L s_2'\varphi\overline{w}\,dx - \frac{k_0^2 l^2}{\rho_1}\int_0^L s_2(w_x - l\varphi)\overline{\varphi}\,dx. \tag{3.34}$$

Note that, using (3.2), (3.4) and (3.6) can be write I_5 as

$$I_5 = i\beta k_0\int_0^L s_2(\varphi_x+\psi+lw)\overline{W}\,dx - k_0\int_0^L s_2\Phi_x\overline{W}\,dx - k_0\int_0^L s_2\Psi\overline{W}\,dx$$
$$-k_0\int_0^L s_2(f_{1,x}+f_3)\overline{W}\,dx - k_0 l\int_0^L s_2 f_5\overline{W}\,dx + k_0 l\int_0^L s_2 W\overline{f_5}\,dx. \tag{3.35}$$

On the other hand, deriving (3.3), multiplying by $\frac{k_0}{\rho_1}s_2\overline{w}$ and integrating in $(0, L)$, we have

$$-\frac{k_0^2 l}{\rho_1}\int_0^L s_2(w_x - l\varphi)_x\overline{w}\,dx = -i\beta k_0\int_0^L s_2\Phi_x\overline{w}\,dx + \frac{kk_0}{\rho_1}\int_0^L s_2(\varphi_x+\psi+lw)_{xx}\overline{w}\,dx$$
$$-\frac{k_1k_0}{\rho_1}\int_0^L s_2\theta_{xx}\overline{w}\,dx + k_0\int_0^L s_2 f_{4,x}\overline{w}\,dx,$$

or yet, integrating by parts and using (3.6) and (3.7), we get

$$k_0 l\int_0^L s_2|W|^2\,dx$$
$$= k_0\int_0^L s_2\Phi_x\overline{W}\,dx - \underbrace{\frac{kk_0}{\rho_1}\int_0^L s_2(\varphi_x+\psi+lw)_x\overline{w_x}\,dx}_{:=I_6}$$
$$-\frac{kk_0}{\rho_1}\int_0^L s_1'(\varphi_x+\psi+lw)_x\overline{w}\,dx + \frac{k_1k_0}{\rho_1}\int_0^L s_2\theta_x\overline{w_x}\,dx + \frac{k_1k_0}{\rho_1}\int_0^L s_2'\theta_x\overline{w}\,dx$$
$$-k_0\int_0^L s_2 f_4\overline{w_x}\,dx - k_0\int_0^L s_2' f_4\overline{w_x}\,dx + k_0\int_0^L s_2\Phi_x\overline{f_5}\,dx$$
$$-\frac{k_0 l}{\rho_1}\int_0^L s_2\big[kl(\varphi_x+\psi+lw) - k_1 l\theta - \rho_1 f_6\big]\overline{w}\,dx + k_0 l\int_0^L s_2 W\overline{f_5}\,dx. \tag{3.36}$$

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Note that, integrating by parts, adjusting some terms and using the equation (3.7), can be write I_6 as

$$\begin{aligned} I_6 = &-i\beta k \int_0^L s_2(\varphi_x + \psi + lw)\overline{W}\,dx + \frac{k^2 l}{\rho_1}\int_0^L s_2|\varphi_x + \psi + lw|^2\,dx \\ &+ \frac{kk_0 l}{\rho_1}\int_0^L s_2(\varphi_x + \psi + lw)\overline{\varphi_x}\,dx + \frac{kk_0}{\rho_1}\int_0^L s_2'(\varphi_x + \psi + lw)\overline{w_x}\,dx \\ &+ \frac{k}{\rho_1}\int_0^L s_2(\varphi_x + \psi + lw)\overline{(-k_1 l\theta - \rho_1 f_6)}\,dx. \end{aligned} \tag{3.37}$$

Therefore, adding (3.34) with (3.36) and replacing the new expressions (3.35) and (3.37), we have

$$\begin{aligned} &\frac{k_0^2 l}{\rho_1}\int_0^L s_2|w_x - l\varphi|^2\,dx + k_0 l\int_0^L s_2|W|^2\,dx \\ &\quad = i\beta(k_0 - k)\int_0^L s_2(\varphi_x + \psi + lw)\overline{W}\,dx \\ &\qquad - k_0\int_0^L s_2\Psi\overline{W}\,dx - \frac{k_0^2 l^2}{\rho_1}\int_0^L s_2(w_x - l\varphi)\overline{\varphi}\,dx \\ &\qquad \underbrace{- \frac{k_0^2 l}{\rho_1}\int_0^L s_2' w_x\overline{w}\,dx}_{:=I_7} + I_8, \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} I_8 = &-k_0\int_0^L s_2(f_{1,x} + f_3)\overline{W}\,dx - k_0 l\int_0^L s_2 f_5\overline{W}\,dx + k_0 l\int_0^L s_2 W\overline{f_5}\,dx + \frac{k_0^2 l^2}{\rho_1}\int_0^L s_2'\varphi\overline{w}\,dx \\ &- \frac{kl^2 k_0}{\rho_1}\int_0^L s_2(\varphi_x + \psi + lw)\overline{w}\,dx + \frac{k_0 l}{\rho_1}\int_0^L s_2[lk_1\theta + \rho_1 f_6]\overline{w}\,dx + \frac{k_1 k_0}{\rho_1}\int_0^L s_2\theta_x\overline{w_x}\,dx \\ &+ \frac{k^2 l}{\rho_1}\int_0^L s_2|\varphi_x + \psi + lw|^2\,dx + \frac{kk_0 l}{\rho_1}\int_0^L s_2(\varphi_x + \psi + lw)\overline{\varphi_x}\,dx + k_0 l\int_0^L s_2 W\overline{f_5}\,dx \\ &+ \frac{kk_0}{\rho_1}\int_0^L s_2'(\varphi_x + \psi + lw)\overline{w_x}\,dx + \frac{k}{\rho_1}\int_0^L s_2(\varphi_x + \psi + lw)\overline{(-k_1 l\theta - \rho_1 f_6)}\,dx \\ &- \frac{kk_0}{\rho_1}\int_0^L s_2'(\varphi_x + \psi + lw)_x\overline{w}\,dx + \frac{k_1 k_0}{\rho_1}\int_0^L s_2'\theta_x\overline{w}\,dx - k_0\int_0^L s_2 f_4\overline{w_x}\,dx \\ &- k_0\int_0^L s_2' f_4\overline{w_x}\,dx + k_0\int_0^L s_2\Phi_x\overline{f_5}\,dx \\ &- \frac{k_0 l}{\rho_1}\int_0^L s_2\big[kl(\varphi_x + \psi + lw) - k_1 l\theta - \rho_1 f_6\big]\overline{w}\,dx. \end{aligned}$$

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It's easy to see that

$$|I_8| \leqslant \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\left(\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |\varphi_x + \psi + lw|^2\,dx\right)^{1/2} + C\|F\|^2_{\mathcal{H}_j}$$
$$+ C\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |\varphi_x + \psi + lw|^2\,dx + C\|\theta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j}, \tag{3.39}$$

and

$$|\operatorname{Re} I_7| \leqslant \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2}\|F\|^2_{\mathcal{H}_j}, \tag{3.40}$$

for some constant $C > 0$ and $j = 1, 2$.

Now, going back to (3.38), taking its real part, using (3.39) and (3.40), we conclude

$$\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2\big(|w_x - l\varphi|^2 + |W|^2\big)\,dx$$
$$\leqslant C|\beta||k_0 - k|\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|\varphi_x + \psi + lw||W|\,dx + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j}$$
$$+ C\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|\Psi||W|\,dx + C\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|w_x - l\varphi||\varphi|\,dx$$
$$+ C\|U\|_{\mathcal{H}_j}\left(\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |\varphi_x + \psi + lw|^2\,dx\right)^{1/2} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j}$$
$$+ C\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |\varphi_x + \psi + lw|^2\,dx + C\|\theta_x\|_2\|U\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j},$$

for some constant $C > 0$ and $j = 1, 2$. Finally, using Young's inequality and the equation (3.2) we obtain (3.32). □

With Lemma 3.9 in hand, we are able to conclude the following result depending on the parameter χ_0 defined in (2.14). More precisely, we have:

Corollary 3.10. *Under the above notations and considering $\varepsilon > 0$, we claim:*

(i) *If $\chi_0 \neq 0$, then there exists a constant $C_\varepsilon > 0$ such that*

$$\int_{l_0-\frac{\delta}{3}}^{l_0+\frac{\delta}{3}} \big(|w_x - l\varphi|^2 + |W|^2\big)\,dx \leqslant \varepsilon\|U\|^2_{\mathcal{H}_j} + C_\varepsilon|\beta|^4\|F\|^2_{\mathcal{H}_j}. \tag{3.41}$$

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(ii) *If* $\chi_0 = 0$, *then there exists a constant* $C_\varepsilon > 0$ *such that*

$$\int_{l_0-\frac{\delta}{3}}^{l_0+\frac{\delta}{3}} \left(|w_x - l\varphi|^2 + |W|^2\right) dx \leqslant \varepsilon \|U\|^2_{\mathcal{H}_j} + C_\varepsilon \|F\|^2_{\mathcal{H}_j}. \tag{3.42}$$

Proof of (i). Since $\chi_0 \neq 0$, then $k_0 - k \neq 0$. Using the Lemma 3.9 with $|\beta| > 1$ large enough, the Corollary 3.5, the Lemma 3.2, the Corollary 3.8 and Young's inequality with $\varepsilon > 0$, we have

$$\int_{l_0-\frac{\delta}{3}}^{l_0+\frac{\delta}{3}} \left(|w_x - l\varphi|^2 + |W|^2\right) dx \leqslant \varepsilon \|U\|^2_{\mathcal{H}_j} + C_\varepsilon \|F\|^2_{\mathcal{H}_j} + C|\beta|^2 \left(\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |\varphi_x + \psi + lw|^2 \, dx \right)$$
$$+ C_\varepsilon \int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |\varphi_x + \psi + lw|^2 \, dx,$$

for some constants $C, C_\varepsilon > 0$ and $j = 1, 2$. In addition, using the Corollary 3.5, proper Young's inequalities with $\varepsilon > 0$ and Lemma 3.2, we conclude

$$C_\varepsilon \int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |\varphi_x + \psi + lw|^2 \, dx \leqslant \frac{\varepsilon}{|\beta|^2} \|U\|^2_{\mathcal{H}_j} + C_\varepsilon |\beta|^2 \|F\|^2_{\mathcal{H}_j},$$

from where one can conclude (3.41). □

Proof of (ii). In this case, $\chi_0 = 0$ implies that $k_0 - k = 0$. Therefore, the desired estimate (3.42) follows from Lemma 3.9 with $|\beta| > 1$ large enough, Corollary (3.5), Lemma 3.2, Corollary 3.8 and Young's inequality with $\varepsilon > 0$. □

3.2. *Completion of the proofs*

From the previous sections, we have finally gathered all ingredients to conclude the proofs of Theorems 2.2, 2.3, and 2.5. For the sake of logistic, we are going to conclude initially Theorems 2.2 and 2.5, and then Theorem 2.3.

3.2.1. *Proof of Theorem 2.2*

Let $\chi_0 \neq 0$ and $\varepsilon > 0$ given. From the Lemma 3.4, the Corollaries 3.5, 3.8 and the estimate (3.41), using Young's Inequality, we conclude that

$$\mathcal{I}_{\frac{\delta}{3}} \leqslant \varepsilon \|U\|^2_{\mathcal{H}_j} + C_\varepsilon |\beta|^4 \|F\|^2_{\mathcal{H}_j} := \Lambda, \tag{3.43}$$

where

$$\mathcal{I}_{\frac{\delta}{3}} := \int_{l_0-\frac{\delta}{3}}^{l_0+\frac{\delta}{3}} \left(|\varphi_x + \psi + lw|^2 + |\Phi|^2 + |\psi_x|^2 + |\Psi|^2 + |w_x - l\varphi|^2 + |W|^2\right) dx, \tag{3.44}$$

for some constant $C_\varepsilon > 0$ and $j = 1, 2$. Now, from the resolvent equations (3.2)–(3.9), we see that $V := (\varphi, \Phi, \psi, \Psi, w, W)$ is a solution of (3.2)–(3.7) and, therefore, solution of the resolvent equation

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of the conservative non-homogeneous Bresse system (A.1)–(A.6) with $G := (g_1, g_2, g_3, g_4, g_5, g_6)$ such that

$$g_1 := f_1, \qquad g_2 := \rho_1 f_2 - k_1\theta_x, \qquad g_3 := f_3,$$
$$g_4 := \rho_2 f_4 + k_1\theta - k_2\vartheta_x, \qquad g_5 := f_5, \qquad g_6 := \rho_1 f_6 + k_1 l\theta.$$

In addition, consider

$$b_1 := l_0 - \delta/3 \quad \text{and} \quad b_2 := l_0 + \delta/3.$$

Therefore, from Corollary A.2, Lemma 3.2 and Young's inequality, we get

$$\int_0^L \left(|\varphi_x + \psi + lw|^2 + |\Phi|^2 + |\psi_x|^2 + |\Psi|^2 + |w_x - l\varphi|^2 + |W|^2\right) dx \leqslant \varepsilon C\|U\|^2_{\mathcal{H}_j} + C_\varepsilon|\beta|^4\|F\|^2_{\mathcal{H}_j},$$

for some constants $C, C_\varepsilon > 0$ and $j = 1, 2$. From (3.10), we have

$$\|U\|^2_{\mathcal{H}_j} \leqslant \varepsilon C\|U\|^2_{\mathcal{H}_j} + C_\varepsilon|\beta|^4\|F\|^2_{\mathcal{H}_j}.$$

Then, choosing $\varepsilon > 0$ small enough and regarding the resolvent equation (3.1), we finally obtain

$$\left\|(i\beta I_d - \mathcal{A}_j)^{-1}F\right\|_{\mathcal{H}_j} \leqslant C|\beta|^2\|F\|_{\mathcal{H}_j}, \quad |\beta| \to +\infty, \tag{3.45}$$

for some constant $C > 0$. From Lemma 3.1 and using (3.45), we conclude from Borichev–Tomilov Theorem, see [3, Theorem 2.4], that

$$\left\|U(t)\right\|_{\mathcal{H}_j} \leqslant \frac{C}{t^{1/2}}\|U_0\|_{D(\mathcal{A}_j)}, \quad t \to +\infty,$$

for $U_0 \in D(\mathcal{A}_j)$, $j = 1, 2$, which proves (2.2) for $n = 1$. The other decay rates in (2.2) follow by using induction over $n \geqslant 2$.

3.2.2. Proof of Theorem 2.5

Let $\varepsilon > 0$ be given. Since $\chi_0 = 0$ we can apply Corollary 3.10, estimate (3.42). In this case, from Lemma 3.4 and Corollaries 3.5 and 3.8, we have in particular

$$\mathcal{I}_{\frac{\delta}{3}} \leqslant \varepsilon\|U\|^2_{\mathcal{H}_j} + C_\varepsilon\|F\|^2_{\mathcal{H}_j} := \Lambda, \tag{3.46}$$

for some constant $C_\varepsilon > 0$ and $j = 1, 2$, where $\mathcal{I}_{\frac{\delta}{3}}$ is given in (3.44). Proceeding similarly as above, namely, from Corollary A.2, Lemma 3.2 and Young's inequality, we deduce

$$\int_0^L \left(|\varphi_x + \psi + lw|^2 + |\Phi|^2 + |\psi_x|^2 + |\Psi|^2 + |w_x - l\varphi|^2 + |W|^2\right) dx \leqslant \varepsilon C\|U\|^2_{\mathcal{H}_j} + C_\varepsilon\|F\|^2_{\mathcal{H}_j}, \tag{3.47}$$

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for some constants $C, C_\varepsilon > 0$ and $j = 1, 2$. Combining Lemma 3.2 and (3.47), we obtain

$$\|U\|^2_{\mathcal{H}_j} \leqslant \varepsilon C \|U\|^2_{\mathcal{H}_j} + C_\varepsilon \|F\|^2_{\mathcal{H}_j},$$

and taking $\varepsilon > 0$ small enough, we conclude

$$\left\|(i\beta I_d - \mathcal{A}_j)^{-1} F\right\|_{\mathcal{H}_j} \leqslant C\|F\|_{\mathcal{H}_j}, \quad |\beta| \to +\infty, j = 1, 2. \tag{3.48}$$

Therefore, using once again Lemma 3.1 and also the estimate (3.48), we conclude the exponential decay (2.18) according to general result for linear semigroups on Hilbert spaces, see Prüss Theorem [22].

3.2.3. Proof of Theorem 2.3

Let us consider $\chi_0 \neq 0$ and fix $U_0 \in D(\mathcal{A}_2)$. In order to prove the desired optimality, we shall argue by contraction.

Indeed, let us suppose that there exists a constant $\nu_0 > 0$ such that (2.16) holds true. Therefore, by taking $\nu = 2 - \frac{2}{1+2\nu_0} \in (0, 2)$, we get the following equivalent (to (2.16)) estimate

$$\left\|U(t)\right\|_{\mathcal{H}_2} \leqslant \frac{C}{t^{\frac{1}{2-\nu}}} \|U_0\|_{D(\mathcal{A}_2)}, \quad t \to +\infty.$$

From this and equivalence coming from the Borichev–Tomilov Theorem, cf. [3, Theorem 2.4], there exists a constant $C > 0$ such that

$$\frac{1}{|\beta|^{2-\nu}} \left\|(i\beta I_d - \mathcal{A}_2)^{-1}\right\|_{\mathcal{L}(\mathcal{H}_2)} \leqslant C, \quad |\beta| \to +\infty. \tag{3.49}$$

On the other hand, if given a bounded sequence $(F_\mu)_{\mu\in\mathbb{N}} \subset \mathcal{H}_2$, we can find a real sequence $(\beta_\mu)_{\mu\in\mathbb{N}} \subset \mathbb{R}^+$, satisfying $\lim_{\mu\to\infty} \beta_\mu = +\infty$, such that

$$\lim_{\mu\to+\infty} \frac{1}{|\beta_\mu|^{2-\nu}} \left\|(i\beta_\mu I_d - \mathcal{A}_2)^{-1} F_\mu\right\|_{\mathcal{H}_2} = +\infty, \tag{3.50}$$

we conclude the desired contradiction with (3.49).

To show (3.50), we assume (without loss of generality) that $L = \pi$ and consider $F_\mu \in \mathcal{H}_2$ as

$$F_\mu(x) = \left(0, 0, 0, 0, 0, \frac{1}{\rho_2}\cos(\mu x), 0, 0\right).$$

It is easy to verify that $\|F_\mu\|_{\mathcal{H}_2} \leqslant C$ for some constant $C > 0$. In addition, the corresponding resolvent equation

$$(i\beta_\mu I_d - \mathcal{A}_2)U_\mu = F_\mu \quad \Leftrightarrow \quad U_\mu = (i\beta_\mu I_d - \mathcal{A}_2)^{-1} F_\mu \tag{3.51}$$

can be rewritten in terms of its components as follows

$$i\beta\varphi - \Phi = 0,$$

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$$
\begin{aligned}
&i\beta\rho_1\Phi - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + k_1\theta_x = 0,\\
&i\beta\psi - \Psi = 0,\\
&i\beta\rho_2\Psi - b\psi_{xx} + k(\varphi_x + \psi + lw) - k_1\theta + k_2\vartheta_x = 0,\\
&i\beta w - W = 0,\\
&i\beta\rho_1 W - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) - k_1 l\theta = \cos(\mu x),\\
&i\beta\rho_3\theta - \gamma_1\theta_{xx} + k_1(\Phi_x + \Psi + lW) = 0,\\
&i\beta\rho_4\vartheta - \gamma_2\vartheta_{xx} + k_2\Psi_x = 0,
\end{aligned}
$$

where we still denote $U_\mu := (\varphi, \Phi, \psi, \Psi, w, W, \theta, \vartheta)$ and $\beta_\mu := \beta$ to simplify the notation. From the first, third and fifth equation of the above system, the reduced system is obtained in terms of φ, ψ, w, θ and ϑ

$$
\begin{aligned}
&-\beta^2\rho_1\varphi - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + k_1\theta_x = 0,\\
&-\beta^2\rho_2\psi - b\psi_{xx} + k(\varphi_x + \psi + lw) - k_1\theta + k_2\vartheta_x = 0,\\
&-\beta^2\rho_1 w - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) - k_1 l\theta = \cos(\mu x),\\
&i\beta\rho_3\theta - \gamma_1\theta_{xx} + i\beta k_1\varphi_x + i\beta k_1\psi + i\beta k_1 lw = 0,\\
&i\beta\rho_4\vartheta - \gamma_2\vartheta_{xx} + i\beta k_2\psi_x = 0,
\end{aligned}
\tag{3.52}
$$

which allows for a solution of the type

$$
\varphi = A\sin(\mu x),\ \psi = B\cos(\mu x),\ w = C\cos(\mu x),\ \theta = D\cos(\mu x),\ \vartheta = E\sin(\mu x), \quad x \in [0, \pi],
$$

where $A = A_\mu$, $B = B_\mu$, $C = C_\mu$, $D = D_\mu$ and $E = E_\mu$ will be determined later. In this way, to solve (3.52) is equivalent to find a solution (A, B, C, D, E) for the algebraic system

$$
\begin{aligned}
&\left(-\beta^2\rho_1 + k\mu^2 + k_0 l^2\right)A + k\mu B + (k + k_0)l\mu C - k_1\mu D = 0,\\
&k\mu A + \left(-\beta^2\rho_2 + b\mu^2 + k\right)B + klC - k_1 D + k_2\mu E = 0,\\
&(k + k_0)l\mu A + klB + \left(-\beta^2\rho_1 + k_0\mu^2 + kl^2\right)C - k_1 lD = 1,\\
&i\beta k_1\mu A + i\beta k_1 B + i\beta k_1 l + \left(i\beta\rho_3 + \gamma_1\mu^2\right)D = 0,\\
&-i\beta k_2\mu B + \left(i\beta\rho_4 + \gamma_2\mu^2\right)E = 0.
\end{aligned}
\tag{3.53}
$$

We denote the matrix of coefficients in (3.53) by

$$
M = \begin{pmatrix}
P_1 & k\mu & (k + k_0)l\mu & -k_1\mu & 0\\
k\mu & P_2 & kl & -k_1 & k_2\mu\\
(k + k_0)l\mu & kl & P_3 & -k_1 l & 0\\
i\beta k_1\mu & i\beta k_1 & i\beta k_1 l & P_4 & 0\\
0 & -i\beta k_2\mu & 0 & 0 & P_5
\end{pmatrix}
\tag{3.54}
$$

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where

$$\begin{cases} P_1 = -\beta^2\rho_1 + k\mu^2 + k_0 l^2, \\ P_2 = -\beta^2\rho_2 + b\mu^2 + k, \\ P_3 = -\beta^2\rho_1 + k_0\mu^2 + kl^2, \\ P_4 = i\beta\rho_3 + \gamma_1\mu^2, \\ P_5 = i\beta\rho_4 + \gamma_2\mu^2, \end{cases} \tag{3.55}$$

are functions in the variable β.

Using Cramer's Rule we can determine C from the following expression

$$C = \frac{\det M_3}{\det M},$$

where

$$M_3 = \begin{pmatrix} P_1 & k\mu & 0 & -k_1\mu & 0 \\ k\mu & P_2 & 0 & -k_1 & k_2\mu \\ (k+k_0)l\mu & kl & 1 & -k_1 l & 0 \\ i\beta k_1\mu & i\beta k_1 & 0 & P_4 & 0 \\ 0 & -i\beta k_2\mu & 0 & 0 & P_5 \end{pmatrix}.$$

Then, a simple calculation shows that

$$\begin{aligned} \det M = {} & \underbrace{\left(P_1P_3 - l^2(k+k_0)^2\mu^2\right)}_{:=I_{12}} P_2P_4P_5 + \left(l^2(k+k_0)^2\mu^2 - P_1P_3\right) i\beta k_2^2\mu^2 P_4 \\ & - 2i\beta k_1^2 l^2(k+k_0)\mu^2 P_2P_5 + i\beta k_1^2\mu^2 P_2P_3P_5 - 2i\beta k_1^2 k\mu^2 P_3P_5 \\ & + i\beta k_1^2 l^2 P_1P_2P_5 - k^2l^2 P_1P_4P_5 - 2k_1^2k_2^2l^2(k+k_0)\beta^2\mu^4 + k_1^2k_2^2l^2\beta^2\mu^2 P_1 \\ & + 4i\beta k_1^2 kl^2(k+k_0)\mu^2 P_5 - i\beta k_1^2 l^2(k+k_0)^2\mu^2 P_5 + 2k^2l^2(k+k_0)\mu^2 P_4P_5 \\ & + k_1^2k_2^2\beta^2\mu^4 P_3 + i\beta k_1^2 P_3P_1P_5 - k^2\mu^2 P_4P_3P_5 - 2i\beta k_1^2 l^2 k P_1P_5, \end{aligned} \tag{3.56}$$

and

$$\begin{aligned} \det M_3 = {} & P_1P_2P_4P_5 + i\beta k_2^2\mu^2 P_1P_4 - k_2^2k_1^2\beta^2\mu^4 - k^2\mu^2 P_4P_5 - i\beta k_1^2 k\mu^2 - i\beta k_1^2 k\mu^2 P_5 \\ & + i\beta k_1^2\mu^2 P_2P_5 + i\beta k_1^2 P_1P_5. \end{aligned}$$

It's simple to check that

$$\det M \neq 0 \quad \text{and} \quad \det M_3 \neq 0,$$

for μ large enough. In what follows we choose a sequence $\beta = \beta_\mu$ such that

$$\beta_\mu = \sqrt{\frac{k_0\mu^2 + kl^2}{\rho_1} - \Xi}, \tag{3.57}$$

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where Ξ is a constant, which is obtained so that I_{12} behaves as a constant that is independent of μ, this is

$$\Xi = \frac{l^2(k+k_0)^2}{\rho_1(k-k_0)}, \tag{3.58}$$

and so we conclude that

$$I_{12} = P_1 P_3 - l^2(k+k_0)^2\mu^2 = \frac{l^4(k+k_0)^4}{(k-k_0)^2} + l^4(k_0+k)^2,$$

and still

$$P_1 = (k-k_0)\mu^2 + (k_0-k)l^2 + \frac{l^2(k+k_0)^2}{k-k_0},$$

$$P_2 = \left(b - \frac{k_0\rho_2}{\rho_1}\right)\mu^2 + k - \frac{kl^2\rho_2}{\rho_1} + \frac{l^2(k+k_0)^2\rho_2}{(k-k_0)\rho_1},$$

$$P_3 = \frac{l^2(k+k_0)^2}{k-k_0}.$$

Note that (3.58) is well-defined since we have $\chi_0 \neq 0$.

Therefore, under these conditions, we can conclude that

(i) If $\rho_1 b \neq k_0\rho_2$, then $|C_\mu| \approx \sigma_0\mu$, when $\mu \to +\infty$, $\sigma_0 > 0$.
(ii) If $\rho_1 b = k_0\rho_2$, then $|C_\mu| \approx \sigma_1\mu$, when $\mu \to +\infty$, $\sigma_1 > 0$.

Besides, from the choice of β_μ in (3.57) one sees that $\beta_\mu \approx \sigma_2\mu$, $\sigma_2 > 0$, when $\mu \to +\infty$, and still keeping in mind that $W(x) = i\beta_\mu w(x) = i\beta_\mu C_\mu \cos(\mu x)$, $x \in [0,\pi]$, then

$$\|U_\mu\|^2_{\mathcal{H}_2} \geqslant \rho_1 \int_0^\pi \left|W(x)\right|^2 dx = \rho_1|\beta_\mu|^2|C_\mu|^2 \int_0^\pi \cos^2(\mu x)\,dx = \frac{\pi}{2}\rho_1|\beta_\mu|^2|C_\mu|^2, \tag{3.59}$$

which implies

$$|\beta_\mu|^{\nu-2}\|U_\mu\|_{\mathcal{H}_2} \geqslant \sqrt{\frac{\pi}{2}}\rho_1|\beta_\mu|^{\nu-1}|C_\mu| \approx \sigma_3\mu^\nu, \quad \sigma_3 > 0, \text{ when } \mu \to +\infty. \tag{3.60}$$

Therefore, from (3.51) and (3.60),

$$\lim_{\mu\to+\infty} \frac{1}{|\beta_\mu|^{2-\nu}} \left\|(i\beta_\mu I_d - \mathcal{A}_2)^{-1}F_\mu\right\|_{\mathcal{H}_2} = \lim_{\mu\to+\infty} |\beta_\mu|^{\nu-2}\|U_\mu\|_{\mathcal{H}_2} = +\infty,$$

which proves (3.50). Hence, the optimality follows.

In particular, from (3.51) and (3.59) we also see that

$$\lim_{\mu\to+\infty} \left\|(i\beta_\mu I_d - \mathcal{A}_2)^{-1}F_\mu\right\|_{\mathcal{H}_2} = \lim_{\mu\to+\infty} \|U_\mu\|_{\mathcal{H}_2} = +\infty,$$

and from Theorem 2.5 the semigroup $\{e^{\mathcal{A}_2 t}\}$ is not exponentially stable on $\mathcal{H}_2$.

This concludes the proof of Theorem 2.3.

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4. Final considerations

Let us consider some final remarks and comments on the main results stated in Section 2.2 and comparing these results with the existing literature.

I. Polynomial stability. The semi-uniform polynomial decay rate

$$\left(\frac{1}{t}\right)^{n/2}, \quad n \in \mathbb{N}, \tag{4.1}$$

achieved in (2.15) for $U_0 \in D(\mathcal{A}_j{}^n)$ and $\chi_0 \neq 0$ is the same independently of the boundary conditions (2.7) ($j = 1$) or (2.8) ($j = 2$). This invariance with respect to the boundary conditions was obtained using the result of observability inequality (see Appendix A) together with the cut-off multipliers, aiming *good* estimates through the resolvent equation, without needing the 'trace' theorem for the one-dimensional case in order to evaluate boundary point-wise terms. Moreover, we still note that the same pattern of stability will be kept for any other boundary conditions whose existence result is ensured.

II. Optimality. Proceeding similarly to [2, Theorem 3.3], we obtain an optimality of the polynomial decay rate $\frac{1}{t^{1/2}}$ for the boundary condition (2.8) when $\chi_0 \neq 0$. Unfortunately, since the technique employed in the proof of Theorem 2.3 requires compatibility between the symmetry of the system and boundary conditions, the optimality only works for the mixed boundary condition (2.8). An analogous approach does not work well for (2.7). However, due to the conservative nature of both boundary conditions, one might expect the optimality in case (2.7). This fact is still open.

III. Exponential stability. This fact could be expected in a first contact with system (2.1)–(2.5) when $k = k_0$. For the sake of completeness of the studies carried out on such a system initially proposed, together with the idea of complying the results obtained here with [2], we prove the exponential stability of the system (2.1)–(2.5) for both boundary conditions when $k = k_0$, by taking the advantage of the estimates performed for the polynomial decay rate when $k \neq k_0$.

IV. Pattern of stability. As previously mentioned in [2] (see Remark V therein), with the results obtained in this work (Theorems 2.2, 2.3 and 2.5), we ratify the invariance of the stability results with respect to the boundary conditions, and we also notice that there is a pattern of stability regarding the addition of thermal couplings in two displacements of the system. More precisely, here we consider thermal couplings inserted in the shear force and bending moment whereas in the pioneer work of this trilogy [2] we consider such couplings in the axial force and in the bending moment coming from [19]. However, using a new way for the multiplier technique along with cut-off functions and the observability inequality, it was possible to obtain the same results for both systems.

An interesting fact to note is that the order of obtaining the estimates is changed according to the presence of thermal coupling in the specific force. The diagrams (1.8) and (1.9) clarify this fact and, still, in the present lemmas of the Section 3.1 we can see the path of proofs clearer. The importance of equal wave speeds $k = k_0$ appears to differentiate exponential and polynomial stability results (see Lemma 3.9).

To conclude the trilogy initially proposed, it remains to prove the invariance of these results in a coming work, adding thermal couplings in the axial and shear forces, where a complete physical modeling will be also considered; as well as (possibly) some numerical results in future works on the subject can be addressed.

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Appendix A. Observability inequality for arched beams

To make this article more self-contained as possible, we still carry out this short appendix that brings the important result on inverse and direct inequalities of observability type for elastic Bresse systems, which has been proved in the first work of the trilogy. More precisely, we state the observability inequality for Bresse-type systems in a static general framework. It constitutes a fundamental result for extending localized estimates to the entire bounded domain under consideration. The complete proof of the results can be found in [2, Section 2].

We start by considering the following system:

$$i\beta\varphi - \Phi = g_1 \quad \text{in } (0, L), \tag{A.1}$$

$$i\beta\rho_1\Phi - \big(k(\varphi_x + \psi + lw)\big)_x - k_0 l(w_x - l\varphi) = g_2 \quad \text{in } (0, L), \tag{A.2}$$

$$i\beta\psi - \Psi = g_3 \quad \text{in } (0, L), \tag{A.3}$$

$$i\beta\rho_2\Psi - (b\psi_x)_x + k(\varphi_x + \psi + lw) = g_4 \quad \text{in } (0, L), \tag{A.4}$$

$$i\beta w - W = g_5 \quad \text{in } (0, L), \tag{A.5}$$

$$i\beta\rho_1 W - \big(k_0(w_x - l\varphi)\big)_x + kl(\varphi_x + \psi + lw) = g_6 \quad \text{in } (0, L), \tag{A.6}$$

where $(g_1, g_2, g_3, g_4, g_5, g_6) \in \mathscr{H}_i$, $i = 1, 2$, with

$$\mathscr{H}_1 = H_0^1 \times L^2 \times H_0^1 \times L^2 \times H_0^1 \times L^2 \quad \text{and} \quad \mathscr{H}_2 = H_0^1 \times L^2 \times H_*^1 \times L_*^2 \times H_*^1 \times L_*^2.$$

For a vector-valued function $V = (\varphi, \Phi, \psi, \Psi, w, W)$ and $0 \leqslant a_1 < a_2 \leqslant L$, we use the notation $\|\cdot\|_{a_1,a_2}$ to stand for

$$\|V\|_{a_1,a_2}^2 := \int_{a_1}^{a_2} \big(|\varphi_x + \psi + lw|^2 + |\Phi|^2 + |\psi_x|^2 + |\Psi|^2 + |w_x - l\varphi|^2 + |W|^2\big)\, dx.$$

Proposition A.1 ([2, Proposition 2.2]). *Let $V = (\varphi, \Phi, \psi, \Psi, w, W)$ be a solution of (A.1)–(A.6). Then, for any numbers $0 \leqslant a_1 < a_2 \leqslant L$, there exist universal constants $C_0, C_1 > 0$ (depending only on ρ_1, ρ_2, k, k_0, b, l) such that*

$$I(a_j) \leqslant C_0\|V\|_{a_1,a_2}^2 + C_0\|G\|_{0,L}^2, \quad j = 1, 2, \tag{A.7}$$

$$\|V\|_{a_1,a_2}^2 \leqslant C_1 I(a_j) + C_1\|G\|_{0,L}^2, \quad j = 1, 2, \tag{A.8}$$

by taking $|\beta| > 1$ large enough.

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An important consequence of Proposition A.1 is the next corollary, which is the precise result we have used in the present paper.

Corollary A.2 ([2, Corollary 2.3]). *Let $V = (\varphi, \Phi, \psi, \Psi, w, W)$ be a solution of (A.1)–(A.6). If for some sub-interval $(b_1, b_2) \subset (0, L)$ one has*

$$\|V\|^2_{b_1,b_2} \leqslant \Lambda, \quad \textit{for some parameter } \Lambda = \Lambda(V, G, \beta), \tag{A.9}$$

then there exists a (universal) constant $C > 0$ such that

$$\|V\|^2_{0,L} \leqslant C\Lambda + C\|G\|^2_{0,L}. \tag{A.10}$$

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