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# Memory Effects on the Stability of Viscoelastic Timoshenko Systems in the Whole $1D$-Space

By

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**Abstract.** In this paper, we investigate new classes of viscoelastic Timoshenko-Ehrenfest systems under the presence of full or partial memory effects. Our achievements rely on recent approaches to the theory of dissipative structure for systems of differential equations, by featuring optimal pointwise estimates in the Fourier space, $L^2$-estimates for the solutions, and explicit energy decay rates depending on the viscoelastic damping coupling. Therefore, under a complete stability analysis, original results as well as improvements of previous work in the literature are our main findings.

*Key Words and Phrases.* Timoshenko system, Fourier transform, Decay rate estimates, Stability.

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## 1. Introduction

In this article, we present new stability results concerning the following viscoelastic beam model

$$
\begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x+\psi)_x + \kappa\kappa_0(g_1 * (\phi_x+\psi)_x) = 0 \quad \text{in } (0,\infty)\times \boldsymbol{R}, \\ \rho_2\psi_{tt} - b\psi_{xx} + bb_0(g_2 * \psi_{xx}) + \kappa(\phi_x+\psi) \\ \qquad - \kappa\kappa_0(g_1 * (\phi_x+\psi)) = 0 \quad \text{in } (0,\infty)\times \boldsymbol{R}, \end{cases} \tag{1.1}
$$

with initial data

$$
(\phi,\phi_t,\psi,\psi_t)(0,x) = (\phi_0,\phi_1,\psi_0,\psi_1)(x), \qquad x \in \boldsymbol{R}, \tag{1.2}
$$

where $\rho_1,\rho_2,\kappa,b>0$, $\kappa_0,b_0 \ge 0$, and $*$ stands for the usual convolution

$$
(g * f)(t) := \int_0^t g(t-\tau)f(\tau)d\tau, \qquad t>0.
$$

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With respect to the memory kernels $g_1$ and $g_2$, the following assumption is taken into account.

**Assumption 1.1.** *The relaxation functions $g_1, g_2 \in C^2 \cap L^1$ satisfy the following conditions*

$$h_1^* := 1 - \kappa_0 \int_0^\infty g_1(\tau)d\tau > 0, \qquad h_2^* := 1 - b_0 \int_0^\infty g_2(\tau)d\tau > 0, \tag{1.3}$$

$$g_j(0) > 0, \qquad -C_j g_j(t) \le g_j'(t) \le -c_j g_j(t), \qquad |g_j''(t)| \le \tilde{C}_j g_j(t), \tag{1.4}$$

*for $j = 1, 2$, and $t \ge 0$; where $c_j$, $C_j$, $\tilde{C}_j$ are positive constants independent of time $t$.*

As explained in the ending Appendix A, (1.1) is a valid model related to viscoelastic beams of Timoshenko-Ehrenfest type, here posed on unbounded domains. All physical meanings, as well as the deduction of (1.1), shall be presented in detail throughout Appendix A. We advance here that (1.1) may represent three classes of viscoelastic Timoshenko-Ehrenfest beams with different features. Each case and its respective novelty shall be highlighted further up.

To analyze the dissipative structure for (1.1)–(1.2), we introduce the new variables

$$v := \kappa(\phi_x + \psi), \qquad w := \phi_t, \qquad z := b\psi_x, \qquad y := \psi_t,$$

and set the vector-valued function $u := (v, w, z, y)^\top$. Thus, problem (1.1)–(1.2) can be abstractly reformulated as

$$A^0 u_t + Au_x + Lu + M_1 g_1 * u_x + M_2 g_2 * u_x + N g_1 * u = 0, \tag{1.5}$$

$$u(0, x) = (v_0, w_0, z_0, y_0)(x) := u_0(x), \tag{1.6}$$

where we denote $v_0 := \kappa(\phi_{0x} + \psi_0)$, $w_0 := \phi_1$, $z_0 := b\psi_{0x}$, $y_0 := \psi_1$, and the matrix coefficients

$$A^0 = \begin{pmatrix} 1/\kappa & 0 & 0 & 0 \\ 0 & \rho_1 & 0 & 0 \\ 0 & 0 & 1/b & 0 \\ 0 & 0 & 0 & \rho_2 \end{pmatrix}, \qquad A = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{1.7}$$

$$L = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \kappa_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\kappa_0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, along the next sections our main results feature estimates and stability properties with respect to both problems (1.1)–(1.2) and (1.5)–(1.6). To this purpose, three possible cases come into play as pointed out below.

### 1.1. Full viscoelastic coupling: case $\kappa_0 > 0$ and $b_0 > 0$

For $\kappa_0 > 0$ and $b_0 > 0$, (1.1) represents a fully damped viscoelastic Timoshenko-Ehrenfest system with memory coupling on both the bending moment and shear force. To our best knowledge, problems (1.1)–(1.2) and (1.5)–(1.6) have not been studied in literature so far. Only on bounded domains, namely, replacing $\boldsymbol{R}$ by $[0, L]$, $L > 0$, there is a slightly modified semilinear version of (1.1) presented with past history, cf. [7], where the asymptotic behavior of solutions was studied with exponential kernels $g_1$, $g_2$. But the latter as well as its results are not comparable to our case due to the character of the systems here and there addressed.

Here, we present for the first time optimal stability results concerning system (1.1), by following recent developments, cf. [16, 17, 18], adopted to our problem.

Our main results in this case feature the following novelties:

- In Section 2, Theorem 2.2 provides pointwise estimates for the solution of (1.5)–(1.6) in the Fourier space when $\kappa_0 > 0$ and $b_0 > 0$, and the optimality of such pointwise estimates is evaluated in Section 3 for suitable choices of the exponential kernels $g_1$, $g_2$, by analyzing the expansion for the corresponding eigenvalues.
- Proposition 4.5 delivers $L^2$-estimates for the solution of problem (1.5)–(1.6) by means of the fundamental solution obtained in Section 4.
- Theorems 5.1 and 5.4, in Section 5, bring out the energy and decay rate estimates, respectively, with respect to the solution of problem (1.5)–(1.6) and, consequently, (1.1)–(1.2).

### 1.2. Partially viscoelastic coupling: case $\kappa_0 > 0$ and $b_0 = 0$

For $\kappa_0 > 0$ and $b_0 = 0$, (1.1) means the Timoshenko-Ehrenfest system with a viscoelastic coupling on the shear force only. In this case, it reduces into the following system

$$(1.8)\quad \begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x + \psi)_x + \kappa\kappa_0(g_1 * (\phi_x + \psi)_x) = 0 & \text{in } (0, \infty) \times \boldsymbol{R}, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\phi_x + \psi) - \kappa\kappa_0(g_1 * (\phi_x + \psi)) = 0 & \text{in } (0, \infty) \times \boldsymbol{R}. \end{cases}$$

As far as we know, system (1.8) has not been addressed in the literature. Only in a bounded interval, we mean on $[0, L]$, $L > 0$, it has been treated very recently, cf. [1], where the authors studied the uniform general stability of the energy. Two main points in computations are regarded in [1], namely, Poincaré's inequality and estimates with bounds $1/L$, which do not hold true in the present work. Therefore, we rely on estimates as in the previous case that are not considered so far in the literature for this model on $\boldsymbol{R}$.

The highlights with respect to (1.8) and (1.5)–$(1.6)_{b_0=0}$ are given as follows:

- Theorem 2.3 displays pointwise estimates for the solution of (1.5)–$(1.6)_{b_0=0}$ in the Fourier space, and the optimality of such pointwise estimates is also assessed in Section 3 for an explicit exponential choice for $g_1$.
- Proposition 4.6 exhibits $L^2$-estimates for the solution of problem (1.5)–$(1.6)_{b_0=0}$ through the fundamental solution, still obtained in Section 4 for this case.
- Theorems 5.2 and 5.5 present the energy and decay rate estimates, respectively, in what concerns the solution of problem (1.5)–$(1.6)_{b_0=0}$ and, therefore, (1.8) with initial data (1.2).

### 1.3. Partially viscoelastic coupling: case $\kappa_0 = 0$ and $b_0 > 0$

For $\kappa_0 = 0$ and $b_0 > 0$, (1.1) stands for the Timoshenko-Ehrenfest system with viscoelastic coupling on the bending moment only. In this case, it turns into the classical viscoelastic system

$$(1.9)\quad \begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x + \psi)_x = 0 & \text{in } (0, \infty) \times \boldsymbol{R}, \\ \rho_2\psi_{tt} - b\psi_{xx} + bb_0(g_2 * \psi_{xx}) + \kappa(\phi_x + \psi) = 0 & \text{in } (0, \infty) \times \boldsymbol{R}. \end{cases}$$

In bounded domains like $[0, L]$, $L > 0$, it has been firstly introduced in [2] with an exponentially decaying positive kernel $g_2$. The authors proved that the related system is exponentially stable if and only if the equal wave speeds condition $\kappa/\rho_1 = b/\rho_2$ is satisfied. After that, several works have treated related problems on bounded intervals, see e.g. [1] where it is provided a survey of references containing more recent generalized results still in bounded intervals.

On the other hand, there are just a few papers dealing with problem (1.9), say on unbounded domains, as e.g. the real line $\boldsymbol{R}$. We quote the pioneering work by Liu-Kawashima [9] where it is presented the decay property of solutions to the Timoshenko system with memory-type dissipation on the bending moment equivalent to (1.9). Furthermore, Mori [11] improved the decay estimate derived by [9]. In this third (and complementary) case, our main purpose is to recall the known result obtained in [11], and derive the same decay estimate for (1.9). Indeed, Liu-Kawashima [9] and Mori [11] considered (1.9) with $\rho_1 = \rho_2 = \kappa = b_0 = 1$, and they did not focus on the relation to the physical parameters. We will pay attention to the physical parameters and derive the desired decay structures. By following the same lines as in the two aforementioned cases, our main results concerning problems (1.9) and (1.5)–$(1.6)_{\kappa_0=0}$ are stated in Theorem 2.4, Proposition 4.7, and Theorems 5.3, 5.6.

## 2. Fourier analysis

By means of the Fourier transform applied to (1.5)–(1.6), we obtain the problem

$$\begin{cases} A^0\hat{u}_t + i\xi A\hat{u} + L\hat{u} + i\xi M_1 g_1 * \hat{u} + i\xi M_2 g_2 * \hat{u} + N g_1 * \hat{u} = 0, \\ \hat{u}(0,\xi) = (\hat{v}_0, \hat{w}_0, \hat{z}_0, \hat{y}_0)(\xi) := \hat{u}_0(\xi), \end{cases} \tag{2.1}$$

where the matrix coefficients are the same as in (1.7). Besides, system (2.1) can be written in terms of its components as follows

$$\begin{cases} \hat{v}_t - \kappa i\xi\hat{w} - \kappa\hat{y} = 0, \\ \rho_1\hat{w}_t - i\xi\hat{v} + \kappa_0 i\xi(g_1 * \hat{v}) = 0, \\ \hat{z}_t - bi\xi\hat{y} = 0, \\ \rho_2\hat{y}_t - i\xi\hat{z} + b_0 i\xi(g_2 * \hat{z}) + \hat{v} - \kappa_0(g_1 * \hat{v}) = 0. \end{cases} \tag{2.2}$$

In what follows, we are going to provide pointwise estimates in the Fourier space for the solution $u$ of problem (1.5)–(1.6), namely, through the solution $\hat{u}$ of system (2.1) and its components equations (2.2). The existence of solution to (1.5)–(1.6) will be addressed in Section 4.

Before proceeding with the main results, let us first prepare the basic energy identity. We first observe that some properties of the memory effect are useful to our approach. Indeed, for any complex-valued functions $g$ and $f$, we define

$$(g \diamond f)(t) := \int_0^t g(t-\tau)(f(\tau) - f(t))d\tau,$$

$$(g \square f)(t) := \int_0^t g(t-\tau)|f(t) - f(\tau)|^2 d\tau.$$

Then, we have the following lemma, whose proof is referred to [2, 10].

**Lemma 2.1.** *Let $g$ be a real-valued function. Then, for any complex-valued function $\varphi$, the following identities hold*

$$(g * \varphi)(t) = \left(\int_0^t g(\tau)d\tau\right)\varphi(t) + (g \diamond \varphi)(t),$$

$$\operatorname{Re}\{(g * \varphi)(t)\bar{\varphi}_t(t)\} = \frac{1}{2}(g' \Box \varphi)(t) - \frac{1}{2}g(t)|\varphi(t)|^2 - \frac{1}{2}\frac{d}{dt}\left\{(g\Box\varphi)(t) - \left(\int_0^t g(\tau)d\tau\right)|\varphi(t)|^2\right\},$$

$$|(g \diamond \varphi)(t)|^2 \le \left(\int_0^t |g(\tau)|d\tau\right)(|g|\Box\varphi)(t).$$

Now, taking the inner product of (2.1) with $\hat{u}$, and taking the real part in the resulting expression, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{\kappa}|\hat{v}|^2 + \rho_1|\hat{w}|^2 + \frac{1}{b}|\hat{z}|^2 + \rho_2|\hat{y}|^2\right) - \frac{\kappa_0}{\kappa}\operatorname{Re}((g_1 * \hat{v})\bar{\hat{v}}_t) - \frac{b_0}{b}\operatorname{Re}((g_2 * \hat{z})\bar{\hat{z}}_t) = 0.$$

To control the memory terms, we employ the second equation in Lemma 2.1, and obtain the energy relation

$$\frac{\partial}{\partial t}E_0 + \frac{\kappa_0}{\kappa}g_1(t)|\hat{v}|^2 - \frac{\kappa_0}{\kappa}(g_1'\Box\hat{v}) + \frac{b_0}{b}g_2(t)|\hat{z}|^2 - \frac{b_0}{b}(g_2'\Box\hat{z}) = 0, \tag{2.3}$$

where

$$E_0 := \frac{1}{\kappa}h_1(t)|\hat{v}|^2 + \rho_1|\hat{w}|^2 + \frac{1}{b}h_2(t)|\hat{z}|^2 + \rho_2|\hat{y}|^2 + \frac{\kappa_0}{\kappa}(g_1\Box\hat{v}) + \frac{b_0}{b}(g_2\Box\hat{z}),$$

and

$$h_1(t) := 1 - \kappa_0\int_0^t g_1(\tau)d\tau, \qquad h_2(t) := 1 - b_0\int_0^t g_2(\tau)d\tau.$$

From (1.3), we have $h_1(t) \ge h_1^*$ and $h_2(t) \ge h_2^*$ for $t \ge 0$. Equation (2.3) means that $E_0$ is conservative if $\kappa_0 = b_0 = 0$. This implies that the stability analysis must be done when at least one of the coefficients $\kappa_0$ or $b_0$ is positive. For completeness, we consider all possibilities below.

### 2.1. Pointwise estimate for $\kappa_0 > 0$ and $b_0 > 0$

**Theorem 2.2.** *Let $u$ be a solution of problem* (1.5)–(1.6) *with $\kappa_0 > 0$ and $b_0 > 0$. Then $\hat{u}$ satisfies the following pointwise estimates in the*

*Fourier space*:

$$|\hat{u}(t,\xi)| \le Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|, \qquad \text{where } \rho(\xi) = \frac{\xi^4}{(1+\xi^2)^2} \tag{2.4}$$

*for some constants* $C, c > 0$.

*Proof.* For the next computations, let us regard the component equations in (2.2).

We initially multiply the first and second equations in (2.2) by $\rho_1 i\xi\bar{\hat{w}}$ and $-i\xi\bar{\hat{v}}$, respectively, and add them up. Then, taking the real part, we get

$$\begin{aligned}\rho_1\xi\frac{\partial}{\partial t}\,\mathrm{Re}(i\hat{v}\bar{\hat{w}}) + \rho_1\kappa\xi^2|\hat{w}|^2 - \xi^2|\hat{v}|^2 - \rho_1\kappa\xi\,\mathrm{Re}(i\hat{y}\bar{\hat{w}})\\ + \kappa_0\xi^2\,\mathrm{Re}((g_1 * \hat{v})\bar{\hat{v}}) = 0.\end{aligned} \tag{2.5}$$

Similarly, using the third and fourth equations in (2.2), we obtain

$$\begin{aligned}\rho_2\xi\frac{\partial}{\partial t}\,\mathrm{Re}(i\hat{y}\bar{\hat{z}}) + \xi^2|\hat{z}|^2 - \rho_2 b\xi^2|\hat{y}|^2 + \xi\,\mathrm{Re}(i\hat{v}\bar{\hat{z}})\\ - \kappa_0\xi\,\mathrm{Re}(i(g_1 * \hat{v})\bar{\hat{z}}) - b_0\xi^2\,\mathrm{Re}((g_2 * \hat{z})\bar{\hat{z}}) = 0.\end{aligned} \tag{2.6}$$

Furthermore, the second and third equations in (2.2) give

$$\rho_1\frac{\partial}{\partial t}\,\mathrm{Re}(\hat{w}\bar{\hat{z}}) - \xi\,\mathrm{Re}(i\hat{v}\bar{\hat{z}}) + \rho_1 b\xi\,\mathrm{Re}(i\hat{w}\bar{\hat{y}}) + \kappa_0\xi\,\mathrm{Re}(i(g_1 * \hat{v})\bar{\hat{z}}) = 0. \tag{2.7}$$

Then, to eliminate the interaction terms, we combine (2.6) and (2.7), and obtain

$$\begin{aligned}\frac{\partial}{\partial t}\{\rho_2\xi\,\mathrm{Re}(i\hat{y}\bar{\hat{z}}) + \rho_1\,\mathrm{Re}(\hat{w}\bar{\hat{z}})\} + \xi^2|\hat{z}|^2 - \rho_2 b\xi^2|\hat{y}|^2 + \rho_1 b\xi\,\mathrm{Re}(i\hat{w}\bar{\hat{y}})\\ - b_0\xi^2\,\mathrm{Re}((g_2 * \hat{z})\bar{\hat{z}}) = 0.\end{aligned} \tag{2.8}$$

Therefore, summing up (2.5) and (2.8), we get

$$\begin{aligned}\frac{\partial}{\partial t}\{\rho_1\xi\,\mathrm{Re}(i\hat{v}\bar{\hat{w}}) + \rho_2\xi\,\mathrm{Re}(i\hat{y}\bar{\hat{z}}) + \rho_1\,\mathrm{Re}(\hat{w}\bar{\hat{z}})\} + \xi^2(\rho_1\kappa|\hat{w}|^2 + |\hat{z}|^2)\\ - \xi^2(|\hat{v}|^2 + \rho_2 b|\hat{y}|^2) + \rho_1(b+\kappa)\xi\,\mathrm{Re}(i\hat{w}\bar{\hat{y}}) + \kappa_0\xi^2\,\mathrm{Re}((g_1 * \hat{v})\bar{\hat{v}})\\ - b_0\xi^2\,\mathrm{Re}((g_2 * \hat{z})\bar{\hat{z}}) = 0.\end{aligned} \tag{2.9}$$

On the other hand, multiplying the first and fourth equations in (2.2) by $\rho_2\bar{\hat{y}}$ and $\bar{\hat{v}}$, respectively, taking the real part, and combining the resultant equations, we get

$$\rho_2 \frac{\partial}{\partial t} \operatorname{Re}(\hat{v}\bar{\hat{y}}) + |\hat{v}|^2 - \rho_2\kappa|\hat{y}|^2 - \rho_2\kappa\xi \operatorname{Re}(i\hat{w}\bar{\hat{y}})$$
$$+ \xi \operatorname{Re}(i\hat{v}\bar{\hat{z}}) + b_0\xi \operatorname{Re}(i(g_2 * \hat{z})\bar{\hat{v}}) - \kappa_0 \operatorname{Re}((g_1 * \hat{v})\bar{\hat{v}}) = 0.$$

The second and fourth equations in (2.2) give

$$-\rho_1 b_0 \operatorname{Re}((g_2 * \hat{z})\bar{\hat{w}}_t) - \rho_2\kappa_0 \operatorname{Re}((g_1 * \hat{v})\bar{\hat{y}}_t) + \kappa_0^2|g_1 * \hat{v}|^2$$
$$- b_0\xi \operatorname{Re}(i(g_2 * \hat{z})\bar{\hat{v}}) - \kappa_0\xi \operatorname{Re}(i(g_1 * \hat{v})\bar{\hat{z}}) - \kappa_0 \operatorname{Re}((g_1 * \hat{v})\bar{\hat{v}}) = 0.$$

Then, adding the above last two equations, it yields

$$\rho_2 \frac{\partial}{\partial t} \operatorname{Re}(\hat{v}\bar{\hat{y}}) + |\hat{v} - \kappa_0(g_1 * \hat{v})|^2 - \rho_2\kappa|\hat{y}|^2 - \rho_2\kappa\xi \operatorname{Re}(i\hat{w}\bar{\hat{y}}) + \xi \operatorname{Re}(i\hat{v}\bar{\hat{z}})$$
$$- \kappa_0\xi \operatorname{Re}(i(g_1 * \hat{v})\bar{\hat{z}}) - \rho_1 b_0 \operatorname{Re}((g_2 * \hat{z})\bar{\hat{w}}_t) - \rho_2\kappa_0 \operatorname{Re}((g_1 * \hat{v})\bar{\hat{y}}_t) = 0.$$

Furthermore, combining (2.7) and this equation, we obtain

$$\frac{\partial}{\partial t}\{\rho_1 \operatorname{Re}(\hat{w}\bar{\hat{z}}) + \rho_2 \operatorname{Re}(\hat{v}\bar{\hat{y}}) - \rho_1 b_0 \operatorname{Re}((g_2 * \hat{z})\bar{\hat{w}}) - \rho_2\kappa_0 \operatorname{Re}((g_1 * \hat{v})\bar{\hat{y}})\}$$
$$+ |\hat{v} - \kappa_0(g_1 * \hat{v})|^2 - \rho_2\kappa|\hat{y}|^2 + (\rho_1 b - \rho_2\kappa)\xi \operatorname{Re}(i\hat{w}\bar{\hat{y}})$$
$$- \rho_1 b_0 \operatorname{Re}(\bar{\hat{w}}(g_2 * \hat{z})_t) + \rho_2\kappa_0 \operatorname{Re}(\bar{\hat{y}}(g_1 * \hat{v})_t) = 0, \tag{2.10}$$

and

$$-\rho_1\rho_2 \frac{\partial}{\partial t}\{b \operatorname{Re}(\hat{v}\bar{\hat{y}}) + \kappa \operatorname{Re}(\hat{w}\bar{\hat{z}}) - \kappa_0 b \operatorname{Re}((g_1 * \hat{v})\bar{\hat{y}})\} + \rho_1\rho_2\kappa b|\hat{y}|^2$$
$$- \rho_1 b|\hat{v} - \kappa_0(g_1 * \hat{v})|^2 - (\rho_1 b - \rho_2\kappa)\xi \operatorname{Re}(i\hat{v}\bar{\hat{z}})$$
$$+ \kappa_0(\rho_1 b - \rho_2\kappa)\xi \operatorname{Re}(i(g_1 * \hat{v})\bar{\hat{z}}) - \rho_1\rho_2\kappa_0 b \operatorname{Re}(\bar{\hat{y}}(g_1 * \hat{v})_t)$$
$$+ \rho_1^2 b b_0 \operatorname{Re}((g_2 * \hat{z})\bar{\hat{w}}_t) = 0. \tag{2.11}$$

To capture the memory effect, multiplying the second equation in (2.2) by $-i\xi(g_1 * \bar{\hat{v}})_t$ and taking the real part, we have

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}\kappa_0\xi^2|g_1 * \hat{v}|^2 - \rho_1\xi \operatorname{Re}(i\hat{w}(g_1 * \bar{\hat{v}})_t)\right\} + \rho_1\kappa g_1(0)\xi^2|\hat{w}|^2$$
$$+ \rho_1\kappa g_1(0)\xi \operatorname{Re}(i\hat{w}\bar{\hat{y}}) + \rho_1\xi \operatorname{Re}(i\hat{w}(g_1' * \bar{\hat{v}})_t) - \xi^2 \operatorname{Re}(\hat{v}(g_1 * \bar{\hat{v}})_t) = 0.$$

To eliminate $\mathrm{Re}(i\hat{w}\bar{\hat{y}})$, we combine (2.5) and this equation. This yields

$$\frac{\partial}{\partial t}E_v + \xi^2|\hat{v}|^2 - \kappa_0\xi^2\,\mathrm{Re}((g_1 * \hat{v})\bar{\hat{v}}) + \frac{\rho_1}{g_1(0)}\xi\,\mathrm{Re}(i\hat{w}(g_1' * \bar{\hat{v}})_t) - \frac{1}{g_1(0)}\xi^2\,\mathrm{Re}(\hat{v}(g_1 * \bar{\hat{v}})_t) = 0, \tag{2.12}$$

where

$$E_v := \frac{1}{g_1(0)}\left\{\frac{1}{2}\kappa_0\xi^2|g_1 * \hat{v}|^2 - \rho_1\xi\,\mathrm{Re}(i\hat{w}(g_1 * \bar{\hat{v}})_t) - \rho_1 g_1(0)\xi\,\mathrm{Re}(i\hat{v}\bar{\hat{w}})\right\}.$$

Similarly, multiplying the fourth equation in (2.2) by $-i\xi(g_2 * \bar{\hat{z}})_t$ and taking the real part, we also obtain

$$\frac{\partial}{\partial t}E_y + \rho_2 b\xi^2|\hat{y}|^2 + \frac{\rho_2}{g_2(0)}\xi\,\mathrm{Re}(i\hat{y}(g_2' * \bar{\hat{z}})_t) - \frac{1}{g_2(0)}\xi^2\,\mathrm{Re}(\hat{z}(g_2 * \bar{\hat{z}})_t) - \frac{1}{g_2(0)}\xi\,\mathrm{Re}(i\hat{v}(g_2 * \bar{\hat{z}})_t) + \frac{\kappa_0}{g_2(0)}\xi\,\mathrm{Re}(i(g_1 * \hat{v})(g_2 * \bar{\hat{z}})_t) = 0, \tag{2.13}$$

where

$$E_y := \frac{1}{g_2(0)}\left\{\frac{1}{2}b_0\xi^2|g_2 * \hat{z}|^2 - \rho_2\xi\,\mathrm{Re}(i\hat{y}(g_2 * \bar{\hat{z}})_t)\right\}.$$

The above identities are enough to achieve the energy estimate. Indeed, computing $(2.9) \times \xi^2 + (2.10) \times \xi^2 + (2.12) \times 2\xi^2$, we obtain

$$\begin{aligned}
&\xi^2\frac{\partial}{\partial t}E_1 + \xi^4(|\hat{v}|^2 + \rho_1\kappa|\hat{w}|^2 + |\hat{z}|^2) + \xi^2|\hat{v} - \kappa_0(g_1 * \hat{v})|^2 - \rho_2(b\xi^2 + \kappa)\xi^2|\hat{y}|^2 \\
&\quad + \{\rho_1(b+\kappa) + (\rho_1 b - \rho_2\kappa)\}\xi^3\,\mathrm{Re}(i\hat{w}\bar{\hat{y}}) - \kappa_0\xi^4\,\mathrm{Re}((g_1 * \hat{v})\bar{\hat{v}}) \\
&\quad - b_0\xi^4\,\mathrm{Re}((g_2 * \hat{z})\bar{\hat{z}}) + \frac{2\rho_1}{g_1(0)}\xi^3\,\mathrm{Re}(i\hat{w}(g_1' * \bar{\hat{v}})_t) - \frac{2}{g_1(0)}\xi^4\,\mathrm{Re}(\hat{v}(g_1 * \bar{\hat{v}})_t) \\
&\quad - \rho_1 b_0\xi^2\,\mathrm{Re}(\bar{\hat{w}}(g_2 * \hat{z})_t) + \rho_2\kappa_0\xi^2\,\mathrm{Re}(\bar{\hat{y}}(g_1 * \hat{v})_t) = 0,
\end{aligned}$$

where

$$\begin{aligned}
E_1 &:= 2E_v + \rho_1\xi\,\mathrm{Re}(i\hat{v}\bar{\hat{w}}) + \rho_2\xi\,\mathrm{Re}(i\hat{y}\bar{\hat{z}}) + 2\rho_1\,\mathrm{Re}(\hat{w}\bar{\hat{z}}) + \rho_2\,\mathrm{Re}(\hat{v}\bar{\hat{y}}) \\
&\quad - \rho_1 b_0\,\mathrm{Re}((g_2 * \hat{z})\bar{\hat{w}}) - \rho_2\kappa_0\,\mathrm{Re}((g_1 * \hat{v})\bar{\hat{y}}).
\end{aligned}$$

Using the Hölder inequality and the fact that $(g * \varphi)_t = g(0)\varphi + (g' * \varphi) = g(t)\varphi + (g' \diamond \varphi)$, we estimate

$$\xi^2 \frac{\partial}{\partial t} E_1 + \frac{h_1(t)}{2}(\xi^2 + h_1(t))\xi^2|\hat{v}|^2 + \frac{\rho_1 \kappa}{2}\xi^4|\hat{w}|^2 + \frac{h_2(t)}{2}\xi^4|\hat{z}|^2 \tag{2.14}$$
$$\leq \left\{\rho_2 b\xi^2 + \rho_2(\kappa + \kappa_0) + \frac{(2\rho_1 b + \kappa(\rho_1 - \rho_2))^2}{\rho_1 \kappa}\right\}\xi^2|\hat{y}|^2$$
$$+ \frac{16\rho_1|g_1'(t)|^2}{\kappa g_1^2(0)}\xi^2|\hat{v}|^2 + \left(\frac{2}{g_1(0)}\xi^2 + \frac{\rho_2\kappa_0 g_1(t)}{2}\right)g_1(t)\xi^2|\hat{v}|^2$$
$$+ \kappa_0^2\left(\frac{1}{h_1(t)}\xi^2 + 1\right)\xi^2|(g_1 \diamond \hat{v})|^2$$
$$+ \left(\frac{4}{g_1^2(0)h_1(t)}\xi^2 + \frac{\rho_2\kappa_0}{2}\right)\xi^2|(g_1' \diamond \hat{v})|^2 + \frac{16\rho_1}{\kappa g_1^2(0)}\xi^2|(g_1'' \diamond \bar{\hat{v}})|^2$$
$$+ \frac{4\rho_1 b_0^2 g_2^2(t)}{\kappa}|\hat{z}|^2 + \frac{b_0^2}{2}\xi^4|(g_2 \diamond \hat{z})|^2 + \frac{4\rho_1 b_0^2}{\kappa}|(g_2' \diamond \hat{z})|^2.$$

On the other hand, the equation (2.13) is estimated as

$$\frac{\partial}{\partial t}E_y + \frac{\rho_2 b}{2}\xi^2|\hat{y}|^2 \leq \frac{\rho_2|g_2'(t)|^2}{bg_2^2(0)}|\hat{z}|^2 + \frac{\rho_2}{bg_2^2(0)}|(g_2'' \diamond \hat{z})|^2 \tag{2.15}$$
$$+ \frac{1}{g_2(0)}\xi^2 \operatorname{Re}(\hat{z}(g_2 * \bar{\hat{z}})_t) + \frac{1}{g_2(0)}\xi \operatorname{Re}(i\hat{v}(g_2 * \bar{\hat{z}})_t)$$
$$- \frac{\kappa_0}{g_2(0)}\xi \operatorname{Re}(i(g_1 * \hat{v})(g_2 * \bar{\hat{z}})_t).$$

Therefore, calculating $(2.15) \times (\xi^2 + 1) + (2.14) \times \alpha_1$, we obtain

$$\frac{\partial}{\partial t}\{(\xi^2 + 1)\mathscr{E}_y + \alpha_1\xi^2\mathscr{E}_1\} + \frac{\alpha_1 h_1(t)}{2}\left(\xi^2 + \frac{h_1(t)}{2}\right)\xi^2|\hat{v}|^2 + \frac{\alpha_1\rho_1\kappa}{2}\xi^4|\hat{w}|^2$$
$$+ \frac{\alpha_1 h_2(t)}{4}\xi^4|\hat{z}|^2 + \frac{\rho_2 b}{2}\left\{(1 - 2\alpha_1)\xi^2 + 1\right.$$
$$\left. - \frac{2\alpha_1}{b}\left(\kappa + \kappa_0 + \frac{(2\rho_1 b + \kappa(\rho_1 - \rho_2))^2}{\rho_1\rho_2\kappa}\right)\right\}\xi^2|\hat{y}|^2 \leq J_v + J_z.$$

Here, we take

$$\alpha_1 = \frac{1}{4}\min\left\{1, b\left(\kappa + \kappa_0 + \frac{(2\rho_1 b + \kappa(\rho_1 - \rho_2))^2}{\rho_1\rho_2\kappa}\right)^{-1}\right\}$$

and estimate

$$
\begin{aligned}
J_v :=\ & \alpha_1\left(\frac{2}{g_1(0)}\xi^2+\frac{\rho_2\kappa_0 g_1(t)}{2}\right)g_1(t)\xi^2|\hat{v}|^2+\frac{16\alpha_1\rho_1|g_1'(t)|^2}{\kappa g_1^2(0)}\xi^2|\hat{v}|^2 \\
& +\frac{16\alpha_1\rho_1}{\kappa g_1^2(0)}\xi^2|(g_1''\diamond\bar{\hat{v}})|^2+\alpha_1\kappa_0^2\left(\frac{1}{h_1(t)}\xi^2+2\right)\xi^2|(g_1\diamond\hat{v})|^2 \\
& +\alpha_1\left(\frac{4}{g_1^2(0)h_1(t)}\xi^2+\frac{\rho_2\kappa_0}{2}\right)\xi^2|(g_1'\diamond\hat{v})|^2 \\
\le\ & \frac{\kappa_0}{\kappa}\left(\frac{\kappa}{2\kappa_0 g_1(0)}\xi^2+\frac{4\rho_1 C_1^2}{\kappa_0 g_1(0)}+\frac{\rho_2\kappa g_1(0)}{8}\right)g_1(t)\xi^2|\hat{v}|^2 \\
& +\frac{\kappa_0}{\kappa}\left\{\frac{\kappa}{h_1(t)}\left(\frac{1}{4}+\frac{C_1}{\kappa_0 g_1(0)}\right)\xi^2+\frac{\kappa}{2}+\frac{\rho_2\kappa C_1^2}{8\kappa_0}+\frac{4\rho_1\tilde{C}_1^2}{\kappa_0^2 g_1^2(0)}\right\}\xi^2(g_1\Box\hat{v}),
\end{aligned}
$$

$$
\begin{aligned}
J_z :=\ & \left\{\frac{1}{g_2(0)}(\xi^2+1)\left(\xi^2+\frac{5}{2\alpha_1}(\xi^2+1)\right)+\frac{4\alpha_1\rho_1 b_0^2 g_2(t)}{\kappa}\right\}g_2(t)|\hat{z}|^2 \\
& +\frac{\rho_2|g_2'(t)|^2}{bg_2^2(0)}(\xi^2+1)|\hat{z}|^2+\frac{\alpha_1 b_0^2}{2}\xi^4|(g_2\diamond\hat{z})|^2 \\
& +\left\{\frac{1}{\alpha_1 g_2^2(0)}\left(\frac{5}{2}+\frac{1}{h_2(t)}\right)(\xi^2+1)^2+\frac{4\alpha_1\rho_1 b_0^2}{\kappa}\right\}|(g_2'\diamond\hat{z})|^2 \\
& +\frac{\rho_2}{bg_2^2(0)}(\xi^2+1)|(g_2''\diamond\hat{z})|^2 \\
\le\ & \frac{b_0}{b}\left\{\frac{b}{b_0 g_2(0)}(\xi^2+1)\left(\left(\frac{5}{2\alpha_1}+1\right)\xi^2+\frac{5}{2\alpha_1}+\frac{\rho_2 C_2^2}{b}\right)+\frac{\rho_1 b b_0 g_2(t)}{\kappa}\right\}g_2(t)|\hat{z}|^2 \\
& +\frac{b_0}{b}\left\{\frac{b}{8}\xi^4+\frac{bC_2}{\alpha_1 b_0 g_2(0)}\left(\frac{5}{2}+\frac{1}{h_2(t)}\right)(\xi^2+1)^2\right. \\
& \qquad\left.+\frac{\rho_2\tilde{C}_2^2}{b_0^2 g_2^2(0)}(\xi^2+1)+\frac{\rho_1 b C_2^2}{\kappa}\right\}(g_2\Box\hat{v})
\end{aligned}
$$

derived by

$$
|(g_j\diamond\hat{v})|^2\le\left(\int_0^t g_j(\tau)d\tau\right)(g_j\Box\hat{v})\le\frac{1}{\sigma_j}(g_j\Box\hat{v}), \tag{2.16}
$$

$$
|(g_j'\diamond\hat{v})|^2\le\left(\int_0^t(-g_j'(\tau))d\tau\right)(-g_j'\Box\hat{v})\le\begin{cases}\dfrac{C_j^2}{\sigma_j}(g_j\Box\hat{v}),\\ g_j(0)C_j(g_j\Box\hat{v}),\end{cases}
$$

$$
|(g_j''\diamond\hat{v})|^2\le\left(\int_0^t|g_j''(\tau)|d\tau\right)(|g_j''|\Box\hat{v})\le\frac{\tilde{C}_j^2}{\sigma_j}(g_j\Box\hat{v}),
$$

for $j = 1, 2$, where $\sigma_1 := \kappa_0$ and $\sigma_2 := b_0$. Namely, this yields

$$\begin{aligned}(2.17)\qquad & \frac{\partial}{\partial t}\{(\xi^2+1)E_y + \alpha_1\xi^2 E_1\} + \frac{\alpha_1 h_1^*}{2}\left(\xi^2 + \frac{h_1^*}{2}\right)\xi^2|\hat{v}|^2 + \frac{\alpha_1\rho_1\kappa}{2}\xi^4|\hat{w}|^2 \\ & + \frac{\alpha_1 h_2^*}{4}\xi^4|\hat{z}|^2 + \frac{\rho_2 b}{4}(\xi^2+1)\xi^2|\hat{y}|^2 \\ & \leq \frac{\kappa_0}{\kappa}C_v^*(\xi^2+1)\xi^2(g_1(t)|\hat{v}|^2 + (g_1\Box\hat{v})) \\ & + \frac{b_0}{b}C_z^*(\xi^2+1)^2(g_2(t)|\hat{z}|^2 + (g_2\Box\hat{z})),\end{aligned}$$

where

$$\begin{aligned}C_v^* &:= \max\left\{\frac{\kappa}{2\kappa_0 g_1(0)}, \frac{4\rho_1 C_1^2}{\kappa_0 g_1(0)} + \frac{\rho_2\kappa g_1(0)}{8}, \frac{\kappa}{h_1^*}\left(\frac{1}{4} + \frac{C_1}{\kappa_0 g_1(0)}\right),\right. \\ &\qquad \left.\frac{\kappa}{2} + \frac{\rho_2\kappa C_1^2}{8\kappa_0} + \frac{4\rho_1\tilde{C}_1^2}{\kappa_0^2 g_1^2(0)}\right\}, \\ C_z^* &:= \max\left\{\frac{b}{b_0 g_2(0)}\left(\frac{5}{2\alpha_1}+1\right), \frac{b}{b_0 g_2(0)}\left(\frac{5}{2\alpha_1} + \frac{\rho_2 C_2^2}{b}\right) + \frac{\rho_1 b b_0 g_2(0)}{\kappa},\right. \\ &\qquad \left.\frac{bC_2}{\alpha_1 b_0 g_2(0)}\left(\frac{5}{2}+\frac{1}{h_2^*}\right) + \frac{b}{8}, \frac{bC_2}{\alpha_1 b_0 g_2(0)}\left(\frac{5}{2}+\frac{1}{h_2^*}\right) + \frac{\rho_2\tilde{C}_2^2}{b_0^2 g_2^2(0)} + \frac{\rho_1 b C_2^2}{\kappa}\right\}.\end{aligned}$$

Consequently, making the combination $(2.3)\times(\xi^2+1)^2 + (2.17)\times\alpha_0$, we arrive at

$$\begin{aligned}(2.18)\qquad & \frac{\partial}{\partial t}\mathscr{E}_0 + \frac{\alpha_0\alpha_1 h_1^*}{2}\left(\xi^2+\frac{h_1^*}{2}\right)\xi^2|\hat{v}|^2 + \frac{\alpha_0\alpha_1\rho_1\kappa}{2}\xi^4|\hat{w}|^2 + \frac{\alpha_0\alpha_1 h_2^*}{4}\xi^4|\hat{z}|^2 \\ & + \frac{\alpha_0\rho_2 b}{4}(\xi^2+1)\xi^2|\hat{y}|^2 + \frac{1}{2}(\xi^2+1)^2(g_1(t)|\hat{v}|^2 + c_1(g_1\Box\hat{v})) \\ & + \frac{1}{2}(\xi^2+1)^2(g_2(t)|\hat{z}|^2 + c_2(g_2\Box\hat{z})) \leq 0,\end{aligned}$$

where $\mathscr{E}_0 := (\xi^2+1)^2 E_0 + \alpha_0(\xi^2+1)E_y + \alpha_0\alpha_1\xi^2 E_1$ and $\alpha_0$ is chosen as

$$(2.19)\qquad \alpha_0 \leq \frac{1}{2}\min\left\{\frac{1}{C_v^*}, \frac{c_1}{C_v^*}, \frac{1}{C_z^*}, \frac{c_2}{C_z^*}\right\}.$$

We estimate the energy $\mathscr{E}_0$. Because of Lemma 2.1 and (2.16), we have

$$
\begin{aligned}
(2.20)\quad |E_v| &\leq \left(1+\frac{1}{\kappa_0 g_1(0)}\right)\xi^2|\hat{v}|^2+\frac{3\rho_1^2}{2}|\hat{w}|^2+\frac{\kappa_0}{g_1(0)}\xi^2|g_1\diamond\hat{v}|^2 \\
&\quad+\frac{1}{2g_1^2(0)}\xi^2|g_1'\diamond\hat{v}|^2 \\
&\leq \left(1+\frac{1}{\kappa_0 g_1(0)}\right)\xi^2|\hat{v}|^2+\frac{3\rho_1^2}{2}|\hat{w}|^2+\frac{1}{g_1(0)}\left(1+\frac{C_1}{2}\right)\xi^2(g_1\Box\hat{v}), \\
|E_y| &\leq \left(\frac{1}{2}+\frac{1}{b_0 g_2(0)}\right)\xi^2|\hat{z}|^2+\rho_2^2|\hat{y}|^2+\frac{b_0}{g_2(0)}\xi^2|g_2\diamond\hat{z}|^2+\frac{1}{2g_2^2(0)}\xi^2|g_2'\diamond\hat{z}|^2 \\
&\leq \left(\frac{1}{2}+\frac{1}{b_0 g_2(0)}\right)\xi^2|\hat{z}|^2+\rho_2^2|\hat{y}|^2+\frac{1}{g_2(0)}\left(1+\frac{C_2}{2}\right)\xi^2(g_2\Box\hat{z}),
\end{aligned}
$$

and

$$
\begin{aligned}
|E_1| &\leq 2|E_v|+\frac{1}{2}(\xi^2+h_1^2(t))|\hat{v}|^2+2\rho_1^2|\hat{w}|^2+\frac{3\rho_2^2}{2}|\hat{y}|^2 \\
&\quad+\frac{1}{2}(\xi^2+1+h_2^2(t))|\hat{z}|^2+\frac{\kappa_0^2}{2}|g_1\diamond\hat{v}|^2+\frac{b_0^2}{2}|g_2\diamond\hat{z}|^2 \\
&\leq \frac{1}{2}\left\{\left(5+\frac{4}{\kappa_0 g_1(0)}\right)\xi^2+h_1^2(t)\right\}|\hat{v}|^2+5\rho_1^2|\hat{w}|^2+\frac{1}{2}(\xi^2+1+h_2^2(t))|\hat{z}|^2 \\
&\quad+\frac{3\rho_2^2}{2}|\hat{y}|^2+\left(\frac{2+C_1}{g_1(0)}\xi^2+\frac{\kappa_0}{2}\right)(g_1\Box\hat{v})+\frac{b_0}{2}(g_2\Box\hat{v}).
\end{aligned}
$$

Thus, we estimate

$$
\begin{aligned}
(\xi^2+1)|E_y|+\alpha_1\xi^2|E_1| &\leq \left(\frac{5}{8}+\frac{1}{2\kappa_0 g_1(0)}\right)(\xi^2+1)\xi^2|\hat{v}|^2+\frac{5\rho_1^2}{4}\xi^2|\hat{w}|^2 \\
&\quad+\left(\frac{3}{4}+\frac{1}{b_0 g_2(0)}\right)(\xi^2+1)\xi^2|\hat{z}|^2+\frac{11\rho_2^2}{8}(\xi^2+1)|\hat{y}|^2 \\
&\quad+\left(\frac{2+C_1}{4g_1(0)}+\frac{\kappa_0}{8}\right)(\xi^2+1)\xi^2(g_1\Box\hat{v}) \\
&\quad+\left\{\frac{1}{g_2(0)}\left(1+\frac{C_2}{2}\right)+\frac{b_0}{8}\right\}(\xi^2+1)\xi^2(g_2\Box\hat{v}),
\end{aligned}
$$

and this gives

$$
\begin{aligned}
(2.21)\qquad \mathscr{E}_0 &\geq \frac{3h_1^*}{4\kappa}|\hat{v}|^2 + \frac{\rho_1}{2}|\hat{w}|^2 + \frac{3h_2^*}{4b}|\hat{z}|^2 + \frac{\rho_2}{2}|\hat{y}|^2 + \frac{\kappa_0}{2\kappa}(g_1\Box\hat{v}) + \frac{b_0}{2b}(g_2\Box\hat{z}) \\
&\geq \frac{h_1^*}{2\kappa}|\hat{v}|^2 + \frac{\rho_1}{2}|\hat{w}|^2 + \frac{h_2^*}{2b}|\hat{z}|^2 + \frac{\rho_2}{2}|\hat{y}|^2 + \frac{h_1^* g_1(t)}{4\kappa g_1(0)}|\hat{v}|^2 + \frac{\kappa_0}{2\kappa}(g_1\Box\hat{v}) \\
&\quad + \frac{h_2^* g_2(t)}{4b g_2(0)}|\hat{z}|^2 + \frac{b_0}{2b}(g_2\Box\hat{z}) \\
&\geq m_0(|\hat{u}|^2 + g_1(t)|\hat{v}|^2 + (g_1\Box\hat{v}) + g_2(t)|\hat{z}|^2 + (g_2\Box\hat{z})), \\
\mathscr{E}_0 &\leq M_0(|\hat{u}|^2 + g_1(t)|\hat{v}|^2 + (g_1\Box\hat{v}) + g_2(t)|\hat{z}|^2 + (g_2\Box\hat{z})),
\end{aligned}
$$

where

$$
m_0 := \min\left\{\frac{h_1^*}{2\kappa}, \frac{\rho_1}{2}, \frac{h_2^*}{2b}, \frac{\rho_2}{2}, \frac{h_1^*}{4\kappa g_1(0)}, \frac{\kappa_0}{2\kappa}, \frac{h_2^*}{4b g_2(0)}, \frac{b_0}{2b}\right\},
$$

$$
\begin{aligned}
M_0 := \max\Bigg\{&1, \frac{1}{\kappa} + \left(\frac{5}{8} + \frac{1}{2\kappa_0 g_1(0)}\right), \rho_1\left(1 + \frac{5\rho_1}{4}\right), \frac{1}{b} + \left(\frac{3}{4} + \frac{1}{b_0 g_2(0)}\right), \\
&\rho_2\left(1 + \frac{11\rho_2}{8}\right), \frac{\kappa_0}{\kappa} + \left(\frac{2 + C_1}{4g_1(0)} + \frac{\kappa_0}{8}\right), \frac{b_0}{b} + \left\{\frac{1}{g_2(0)}\left(1 + \frac{C_2}{2}\right) + \frac{b_0}{8}\right\}\Bigg\},
\end{aligned}
$$

and we take $\alpha_0$ satisfying (2.19) and

$$
\begin{aligned}
\alpha_0 \leq \frac{1}{2}\min\Bigg\{&2, \frac{h_1^*}{2\kappa}\left(\frac{5}{8} + \frac{1}{2\kappa_0 g_1(0)}\right)^{-1}, \frac{4}{5\rho_1}, \frac{h_2^*}{2b}\left(\frac{3}{4} + \frac{1}{b_0 g_2(0)}\right)^{-1}, \frac{8}{11\rho_2}, \\
&\frac{\kappa_0}{\kappa}\left(\frac{2 + C_1}{4g_1(0)} + \frac{\kappa_0}{8}\right)^{-1}, \frac{b_0}{b}\left\{\frac{1}{g_2(0)}\left(1 + \frac{C_2}{2}\right) + \frac{b_0}{8}\right\}^{-1}\Bigg\}.
\end{aligned}
$$

Therefore, using (2.18) and (2.21), we get

$$
\begin{aligned}
(2.22)\qquad |\hat{u}|^2 + U[\hat{v},\hat{z}] + \frac{n_0}{m_0}\int_0^t \Bigg\{&\frac{\xi^2}{\xi^2+1}(|\hat{v}|^2 + |\hat{y}|^2) \\
&+ \frac{\xi^4}{(\xi^2+1)^2}(|\hat{w}|^2 + |\hat{z}|^2) + U[\hat{v},\hat{z}]\Bigg\} d\tau \\
\leq \frac{M_0}{m_0}(|\hat{u}|^2 + U[\hat{v},\hat{z}]),&
\end{aligned}
$$

where $U[\hat{v},\hat{z}] := g_1(t)|\hat{v}|^2 + c_1(g_1\Box\hat{v}) + g_2(t)|\hat{z}|^2 + c_2(g_2\Box\hat{z})$ and

$$
n_0 := \frac{1}{2}\min\left\{1, \frac{\alpha_0\alpha_1(h_1^*)^2}{2 + h_1^*}, \alpha_0\alpha_1\rho_1\kappa, \frac{\alpha_0\alpha_1 h_2^*}{2}, \frac{\alpha_0\rho_2 b}{2}\right\}.
$$

Therefore, the pointwise estimate (2.4) is finally obtained by using (2.18) and (2.21), and the proof of Theorem 2.2 is complete. $\square$

### 2.2. Pointwise estimate for $\kappa_0 > 0$ and $b_0 = 0$

**Theorem 2.3.** *Let $u$ be a solution of problem (1.5)–(1.6) with $\kappa_0 > 0$ and $b_0 = 0$. Then $\hat{u}$ satisfies the following pointwise estimates in the Fourier space*:

$$|\hat{u}(t,\xi)| \le Ce^{-c\eta(\xi)t}|\hat{u}_0(\xi)| \tag{2.23}$$

*for some constants $C, c > 0$, where*

$$\eta(\xi) = \begin{cases} \dfrac{\xi^6}{(1+\xi^2)^4} & \text{if } \dfrac{\kappa}{\rho_1} \neq \dfrac{b}{\rho_2}, \\ \dfrac{\xi^6}{(1+\xi^2)^3} & \text{if } \dfrac{\kappa}{\rho_1} = \dfrac{b}{\rho_2}. \end{cases} \tag{2.24}$$

*Proof.* Here, we take advantage of the computations provided in the proof of Theorem 2.2 with $b_0 = 0$. From (2.7), the equations (2.5), (2.9), and (2.10), in case $b_0 = 0$, can be rewritten as

$$\begin{aligned} &\rho_1 \frac{\partial}{\partial t}\left\{\xi \operatorname{Re}(i\hat{v}\bar{\hat{w}}) - \frac{\kappa}{b}\operatorname{Re}(\hat{w}\bar{\hat{z}})\right\} + \rho_1\kappa\xi^2|\hat{w}|^2 - \xi^2|\hat{v}|^2 \\ &\qquad + \frac{\kappa}{b}\xi \operatorname{Re}(i\hat{v}\bar{\hat{z}}) + \kappa_0\xi^2 \operatorname{Re}((g_1 * \hat{v})\bar{\hat{v}}) - \frac{\kappa\kappa_0}{b}\xi \operatorname{Re}(i(g_1*\hat{v})\bar{\hat{z}}) = 0, \end{aligned} \tag{2.25}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\left\{\rho_1\xi \operatorname{Re}(i\hat{v}\bar{\hat{w}}) + \rho_2\xi \operatorname{Re}(i\hat{y}\bar{\hat{z}}) - \frac{\rho_1\kappa}{b}\operatorname{Re}(\hat{w}\bar{\hat{z}})\right\} + \xi^2(\rho_1\kappa|\hat{w}|^2 + |\hat{z}|^2) \\ &\qquad - \xi^2(|\hat{v}|^2 + \rho_2 b|\hat{y}|^2) + \frac{\kappa+b}{b}\xi \operatorname{Re}(i\hat{v}\bar{\hat{z}}) - \frac{\kappa_0(\kappa+b)}{b}\xi \operatorname{Re}(i(g_1*\hat{v})\bar{\hat{z}}) \\ &\qquad + \kappa_0\xi^2 \operatorname{Re}((g_1*\hat{v})\bar{\hat{v}}) = 0, \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} &-\rho_2\frac{\partial}{\partial t}\left\{\frac{b}{\kappa}\operatorname{Re}(\hat{v}\bar{\hat{y}}) + \operatorname{Re}(\hat{w}\bar{\hat{z}}) - \frac{\kappa_0 b}{\kappa}\operatorname{Re}((g_1*\hat{v})\bar{\hat{y}})\right\} + \rho_2 b|\hat{y}|^2 \\ &\qquad - \frac{b}{\kappa}|\hat{v} - \kappa_0(g_1*\hat{v})|^2 - \frac{\rho_1 b - \rho_2\kappa}{\rho_1\kappa}\xi \operatorname{Re}(i\hat{v}\bar{\hat{z}}) \\ &\qquad + \frac{\kappa_0(\rho_1 b - \rho_2\kappa)}{\rho_1\kappa}\xi \operatorname{Re}(i(g_1*\hat{v})\bar{\hat{z}}) - \frac{\rho_2\kappa_0 b}{\kappa}\operatorname{Re}(\bar{\hat{y}}(g_1*\hat{v})_t) = 0. \end{aligned} \tag{2.27}$$

Then, combining (2.26) and (2.27), we obtain

$$
\begin{aligned}
(2.28)\qquad & \frac{\partial}{\partial t}E_2 + \xi^2(\rho_1\kappa|\hat{w}|^2 + |\hat{z}|^2) + \rho_2 b(\xi^2+1)|\hat{y}|^2 - \xi^2|\hat{v}|^2 \\
& - \frac{b}{\kappa}(2\xi^2+1)|\hat{v} - \kappa_0(g_1 * \hat{v})|^2 + \kappa_0\xi^2 \operatorname{Re}((g_1 * \hat{v})\bar{\hat{v}}) + \frac{\kappa+b}{b}\xi \operatorname{Re}(i\hat{v}\bar{\hat{z}}) \\
& - \frac{\kappa_0(\kappa+b)}{b}\xi \operatorname{Re}(i(g_1 * \hat{v})\bar{\hat{z}}) - \frac{(\rho_1 b - \rho_2\kappa)}{\rho_1\kappa}(2\xi^2+1)\xi \operatorname{Re}(i\hat{v}\bar{\hat{z}}) \\
& + \frac{\kappa_0(\rho_1 b - \rho_2\kappa)}{\rho_1\kappa}(2\xi^2+1)\xi \operatorname{Re}(i(g_1 * \hat{v})\bar{\hat{z}}) \\
& - \frac{\rho_2\kappa_0 b}{\kappa}(2\xi^2+1) \operatorname{Re}(\bar{\hat{y}}(g_1 * \hat{v})_t) = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
E_2 := {} & \rho_1\xi \operatorname{Re}(i\hat{v}\bar{\hat{w}}) + \rho_2\xi \operatorname{Re}(i\hat{y}\bar{\hat{z}}) - \frac{\rho_1\kappa}{b} \operatorname{Re}(\hat{w}\bar{\hat{z}}) \\
& - \rho_2(2\xi^2+1)\left\{\frac{b}{\kappa} \operatorname{Re}(\hat{v}\bar{\hat{y}}) + \operatorname{Re}(\hat{w}\bar{\hat{z}}) - \frac{\kappa_0 b}{\kappa} \operatorname{Re}((g_1 * \hat{v})\bar{\hat{y}})\right\}.
\end{aligned}
$$

<u>*Case* $\rho_1 b \neq \rho_2\kappa$.</u> In the case of different wave speeds, we combine (2.25) and (2.28) to obtain

$$
\begin{aligned}
& \frac{\partial}{\partial t}\left\{E_2 + \rho_1\xi^2\left(\xi \operatorname{Re}(i\hat{v}\bar{\hat{w}}) - \frac{\kappa}{b} \operatorname{Re}(\hat{w}\bar{\hat{z}})\right)\right\} + \rho_1\kappa(\xi^2+1)\xi^2|\hat{w}|^2 + \xi^2|\hat{z}|^2 \\
& + \rho_2 b(\xi^2+1)|\hat{y}|^2 - (\xi^2+1)\xi^2|\hat{v}|^2 - \frac{b}{\kappa}(2\xi^2+1)|\hat{v} - \kappa_0(g_1 * \hat{v})|^2 \\
& + \kappa_0(\xi^2+1)\xi^2 \operatorname{Re}((g_1 * \hat{v})\bar{\hat{v}}) \\
& + \left\{\frac{1}{b}(\kappa\xi^2 + \kappa + b) - \frac{(\rho_1 b - \rho_2\kappa)}{\rho_1\kappa}(2\xi^2+1)\right\}\xi\{\operatorname{Re}(i\hat{v}\bar{\hat{z}}) - \kappa_0 \operatorname{Re}(i(g_1 * \hat{v})\bar{\hat{z}})\} \\
& - \frac{\rho_2\kappa_0 b}{\kappa}(2\xi^2+1) \operatorname{Re}(\bar{\hat{y}}(g_1 * \hat{v})_t) = 0.
\end{aligned}
$$

Thus, applying the Hölder inequality to this equation, we estimate

$$
\begin{aligned}
(2.29)\qquad & \frac{\partial}{\partial t}\left\{E_2 + \rho_1\xi^2\left(\xi \operatorname{Re}(i\hat{v}\bar{\hat{w}}) - \frac{\kappa}{b} \operatorname{Re}(\hat{w}\bar{\hat{z}})\right)\right\} + \rho_1\kappa(\xi^2+1)\xi^2|\hat{w}|^2 \\
& \quad + \frac{1}{2}\xi^2|\hat{z}|^2 + \rho_2 b(\xi^2+1)|\hat{y}|^2 \\
& \leq h_1(t)(\xi^2+1)\xi^2|\hat{v}|^2 + \frac{b}{\kappa}(2\xi^2+1)|h_1(t)\hat{v} - \kappa_0(g_1 \diamond \hat{v})|^2 \\
& \quad + \kappa_0(\xi^2+1)\xi^2|\hat{v}|\,|(g_1 \diamond \hat{v})|
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{\kappa+b}{b}+\frac{2|\rho_1 b-\rho_2\kappa|}{\rho_1\kappa}\right)^2(\xi^2+1)^2(h_1^2(t)|\hat{v}|^2+\kappa_0^2|(g_1\diamond\hat{v})|^2)\\
&+\frac{\rho_2\kappa_0 b}{\kappa}(2\xi^2+1)\operatorname{Re}(\bar{\hat{y}}(g_1*\hat{v})_t)\\
\le{}& C_v(\xi^2+1)^2(h_1^2(t)|\hat{v}|^2+\kappa_0^2|(g_1\diamond\hat{v})|^2)\\
&+\frac{\rho_2\kappa_0 b}{\kappa}(2\xi^2+1)\operatorname{Re}(\bar{\hat{y}}(g_1*\hat{v})_t),
\end{aligned}
$$

where

$$
C_v:=\left(\frac{\kappa+b}{b}+\frac{2|\rho_1 b-\rho_2\kappa|}{\rho_1\kappa}\right)^2+\frac{4b}{\kappa}+\frac{\sqrt{2}+1}{2h_1^*}.
$$

Calculating $(2.12)\times(\xi^2+1)^2\xi^2+(2.29)\times\xi^4/(2C_v)$ and using (2.16), we get

$$
\begin{aligned}
(2.30)\quad &\xi^2\frac{\partial}{\partial t}\left\{(\xi^2+1)^2E_v+\frac{1}{2C_v}\xi^2E_2+\frac{1}{2C_v}\rho_1\xi^4\left(\xi\operatorname{Re}(i\hat{v}\bar{\hat{w}})-\frac{\kappa}{b}\operatorname{Re}(\hat{w}\bar{\hat{z}})\right)\right\}\\
&+\frac{h_1(t)}{4}(\xi^2+1)^2\xi^4|\hat{v}|^2+\frac{\rho_1\kappa}{4C_v}(\xi^2+1)\xi^6|\hat{w}|^2\\
&+\frac{1}{4C_v}\xi^6|\hat{z}|^2+\frac{\rho_2 b}{4C_v}(\xi^2+1)\xi^4|\hat{y}|^2\\
\le{}&\frac{g_1(t)}{g_1(0)}(\xi^2+1)^2\xi^4|\hat{v}|^2+\frac{2\rho_2\kappa_0^2bg_1^2(t)}{\kappa^2C_v}(\xi^2+1)\xi^4|\hat{v}|^2\\
&+\frac{2\rho_1C_v|g_1'(t)|^2}{\kappa g_1^2(0)}(\xi^2+1)^3|\hat{v}|^2+\kappa_0^2\left(\frac{1}{2}+\frac{2}{h_1(t)}\right)(\xi^2+1)^2\xi^4|(g_1\diamond\hat{v})|^2\\
&+\frac{\rho_2\kappa_0^2b}{\kappa^2C_v}(\xi^2+1)\xi^4|(g_1'\diamond\hat{v})|^2+\frac{2}{g_1^2(0)h_1(t)}(\xi^2+1)^2\xi^4|(g_1'\diamond\hat{v})|^2\\
&+\frac{2\rho_1C_v}{\kappa g_1^2(0)}(\xi^2+1)^3|(g_1''\diamond\hat{v})|^2\\
\le{}&\frac{\kappa_0}{\kappa}\left(\frac{\kappa}{\kappa_0g_1(0)}+\frac{2\rho_1C_vC_1^2}{\kappa_0g_1(0)}+\frac{2\rho_2\kappa_0bg_1(0)}{\kappa C_v}\right)g_1(t)(\xi^2+1)^4|\hat{v}|^2\\
&+\frac{\kappa_0}{\kappa}\left(\frac{\kappa}{2}+\frac{2\kappa}{h_1^*}+\frac{\rho_2bC_1^2}{\kappa C_v}+\frac{2\kappa C_1}{\kappa_0g_1(0)h_1^*}+\frac{2\rho_1C_v\tilde{C}_1^2}{\kappa_0^2g_1^2(0)}\right)(\xi^2+1)^4(g_1\Box\hat{v}).
\end{aligned}
$$

Consequently, making the combination $(2.31)\times(\xi^2+1)^4+(2.30)\times\beta_0$, we arrive at

$$\frac{\partial}{\partial t}\mathscr{E}_1 + \frac{\beta_0 h_1^*}{4}(\xi^2+1)^2\xi^4|\hat{v}|^2 + \frac{\beta_0\rho_1\kappa}{4C_v}(\xi^2+1)\xi^6|\hat{w}|^2 + \frac{\beta_0}{4C_v}\xi^6|\hat{z}|^2 + \frac{\beta_0\rho_2 b}{4C_v}(\xi^2+1)\xi^4|\hat{y}|^2 + \frac{\kappa_0 g_1(t)}{2\kappa}(\xi^2+1)^4|\hat{v}|^2 + \frac{\kappa_0 c_1}{2\kappa}(\xi^2+1)^4(g_1\Box\hat{v}) \le 0, \tag{2.31}$$

where

$$\mathscr{E}_1 := (\xi^2+1)^4E_0 + \beta_0\xi^2\left\{(\xi^2+1)^2E_v + \frac{1}{2C_v}\xi^2E_2 + \frac{1}{2C_v}\rho_1\xi^4\left(\xi\,\mathrm{Re}(i\hat{v}\bar{\hat{w}}) - \frac{\kappa}{b}\,\mathrm{Re}(\hat{w}\bar{\hat{z}})\right)\right\},$$

and $\beta_0$ is taken as

$$\beta_0 \le \frac{1}{2}\min\left\{\left(\frac{\kappa}{\kappa_0 g_1(0)} + \frac{2\rho_1 C_v C_1^2}{\kappa_0 g_1(0)} + \frac{2\rho_2\kappa_0 b g_1(0)}{\kappa C_v}\right)^{-1}, c_1\left(\frac{\kappa}{2} + \frac{2\kappa}{h_1^*} + \frac{\rho_2 b C_1^2}{\kappa C_v} + \frac{2\kappa C_1}{\kappa_0 g_1(0)h_1^*} + \frac{2\rho_1 C_v\tilde{C}_1^2}{\kappa_0^2 g_1^2(0)}\right)^{-1}\right\}. \tag{2.32}$$

Next, we estimate the energy $\mathscr{E}_1$. From (2.20) and

$$\begin{aligned}\left|E_2 + \rho_1\xi^2\left(\xi\,\mathrm{Re}(i\hat{v}\bar{\hat{w}}) - \frac{\kappa}{b}\,\mathrm{Re}(\hat{w}\bar{\hat{z}})\right)\right| &\le \left(\frac{1}{2} + \frac{b^2h_1^2(t)}{\kappa^2}\right)(\xi^2+1)^2|\hat{v}|^2 + (\rho_1^2+\rho_2^2)(\xi^2+1)|\hat{w}|^2 \\ &\quad + \frac{1}{2}\left(3 + \frac{\kappa^2}{b^2}\right)(\xi^2+1)|\hat{z}|^2 + \frac{5\rho_2^2}{2}|\hat{y}|^2 + \frac{\kappa_0 b^2}{\kappa^2}(\xi^2+1)^2(g_1\Box\hat{v}),\end{aligned}$$

we estimate

$$\begin{aligned}&(\xi^2+1)^2|E_v| + \frac{1}{2C_v}\xi^2\left|E_2 + \rho_1\xi^2\left(\xi\,\mathrm{Re}(i\hat{v}\bar{\hat{w}}) - \frac{\kappa}{b}\,\mathrm{Re}(\hat{w}\bar{\hat{z}})\right)\right| \\ &\le \left(1 + \frac{1}{\kappa_0 g_1(0)} + \frac{1}{4C_v} + \frac{b^2h_1^2(t)}{2\kappa^2C_v}\right)(\xi^2+1)^2\xi^2|\hat{v}|^2 \\ &\quad + \frac{1}{2}\left(3\rho_1^2 + \frac{\rho_1^2+\rho_2^2}{C_v}\right)(\xi^2+1)^2|\hat{w}|^2 + \frac{1}{4C_v}\left(3 + \frac{\kappa^2}{b^2}\right)(\xi^2+1)\xi^2|\hat{z}|^2 \\ &\quad + \frac{5\rho_2^2}{4C_v}\xi^2|\hat{y}|^2 + \left(\frac{1}{g_1(0)} + \frac{C_1}{2g_1(0)} + \frac{\kappa_0 b^2}{2\kappa^2C_v}\right)(\xi^2+1)^2\xi^2(g_1\Box\hat{v}),\end{aligned}$$

and this gives

$$
\begin{aligned}
(2.33)\qquad \mathscr{E}_1 &\geq \frac{3h_1^*}{4\kappa}|\hat{v}|^2 + \frac{\rho_1}{2}|\hat{w}|^2 + \frac{1}{2b}|\hat{z}|^2 + \frac{\rho_2}{2}|\hat{y}|^2 + \frac{\kappa_0}{2\kappa}(g_1\Box\hat{v}) \\
&\geq \frac{h_1^*}{2\kappa}|\hat{v}|^2 + \frac{\rho_1}{2}|\hat{w}|^2 + \frac{1}{2b}|\hat{z}|^2 + \frac{\rho_2}{2}|\hat{y}|^2 + \frac{h_1^* g_1(t)}{4\kappa g_1(0)}|\hat{v}|^2 + \frac{\kappa_0}{2\kappa}(g_1\Box\hat{v}) \\
&\geq m_1(|\hat{u}|^2 + g_1(t)|\hat{v}|^2 + (g_1\Box\hat{v})), \\
\mathscr{E}_1 &\leq M_1(|\hat{u}|^2 + g_1(t)|\hat{v}|^2 + (g_1\Box\hat{v})),
\end{aligned}
$$

where

$$
\begin{aligned}
m_1 &:= \frac{1}{2}\min\left\{\frac{h_1^*}{\kappa}, \rho_1, \frac{1}{b}, \rho_2, \frac{h_1^*}{2\kappa g_1(0)}, \frac{\kappa_0}{\kappa}\right\}, \\
M_1 &:= \max\left\{\frac{1}{\kappa} + \left(1 + \frac{1}{\kappa_0 g_1(0)} + \frac{1}{4C_v} + \frac{b^2 h_1^2(t)}{2\kappa^2 C_v}\right), \rho_1 + \frac{1}{2}(3\rho_1^2) + \frac{\rho_1^2+\rho_2^2}{2C_v},\right. \\
&\qquad \left.\frac{1}{b} + \frac{1}{4C_v}\left(3 + \frac{\kappa^2}{b^2}\right), \rho_2\left(1 + \frac{5\rho_2}{4C_v}\right), \frac{\kappa_0}{\kappa} + \left(\frac{1}{g_1(0)} + \frac{C_1}{2g_1(0)} + \frac{\kappa_0 b^2}{2\kappa^2 C_v}\right)\right\},
\end{aligned}
$$

where we take $\beta_0$ satisfying (2.32) and

$$
\begin{aligned}
\beta_0 \leq \frac{1}{2}\min\Bigg\{&2, \frac{h_1^*}{2\kappa}\left(1 + \frac{1}{\kappa_0 g_1(0)} + \frac{1}{4C_v} + \frac{b^2}{2\kappa^2 C_v}\right)^{-1}, 2\left(3\rho_1 + \frac{\rho_1^2+\rho_2^2}{\rho_1 C_v}\right)^{-1}, \\
&\frac{4C_v}{b}\left(3 + \frac{\kappa^2}{b^2}\right)^{-1}, \frac{4C_v}{5\rho_2}, \frac{b_0}{b}\left(\frac{1}{g_1(0)} + \frac{C_1}{2g_1(0)} + \frac{\kappa_0 b^2}{2\kappa^2 C_v}\right)^{-1}\Bigg\}.
\end{aligned}
$$

Thus, using (2.31) and (2.33), we get

$$
\begin{aligned}
(2.34)\qquad &|\hat{u}|^2 + U_1[\hat{v}] + \frac{n_1}{m_1}\int_0^t\left\{\frac{\xi^4}{(\xi^2+1)^2}|\hat{v}|^2 + \frac{\xi^6}{(\xi^2+1)^3}|\hat{w}|^2 + \frac{\xi^6}{(\xi^2+1)^4}|\hat{z}|^2\right. \\
&\qquad\qquad \left. + \frac{\xi^4}{(\xi^2+1)^3}|\hat{y}|^2 + U_1[\hat{v}]\right\}d\tau \\
&\leq \frac{M_1}{m_1}(|\hat{u}|^2 + U_1[\hat{v}]),
\end{aligned}
$$

where $U_1[\hat{v}] := g_1(t)|\hat{v}|^2 + c_1(g_1\Box\hat{v})$ and

$$
n_1 := \frac{1}{2}\min\left\{\frac{\beta_0 h_1^*}{2}, \frac{\beta_0\rho_1\kappa}{2C_v}, \frac{\beta_0}{2C_v}, \frac{\beta_0\rho_2 b}{2C_v}, \frac{\kappa_0}{\kappa}, \frac{\kappa_0 c_1}{\kappa}\right\}.
$$

Therefore, by virtue of (2.31) and (2.33), one can conclude the pointwise estimate (2.23) with $\eta$ given in (2.24) for $\rho_1 b \neq \rho_2 \kappa$.

*Case $\rho_1 b = \rho_2 \kappa$.* In the case of equal wave speeds, equation (2.28) gives

$$\begin{aligned}(2.35)\quad & \frac{\partial}{\partial t}E_2 + \rho_1\kappa\xi^2|\hat{w}|^2 + \frac{1}{2}\xi^2|\hat{z}|^2 + \rho_2 b(\xi^2+1)|\hat{y}|^2 \\ & \leq h_1(t)\xi^2|\hat{v}|^2 + \kappa_0\xi^2|(g_1\diamond\hat{v})|\,|\hat{v}| + \frac{4b}{\kappa}(\xi^2+1)(h_1^2(t)|\hat{v}|^2 + \kappa_0^2|(g_1\diamond\hat{v})|^2) \\ & \quad + \frac{(\kappa+b)^2}{b^2}(h_1^2(t)|\hat{v}|^2 + \kappa_0^2|(g_1\diamond\hat{v})|^2) + \frac{\rho_2\kappa_0 b}{\kappa}(2\xi^2+1)\,\mathrm{Re}(\bar{\hat{y}}(g_1*\hat{v})_t) \\ & \leq \tilde{C}_v(\xi^2+1)(h_1^2(t)|\hat{v}|^2 + \kappa_0^2|(g_1\diamond\hat{v})|^2) \\ & \quad + \frac{\rho_2\kappa_0 b}{\kappa}(2\xi^2+1)\,\mathrm{Re}(\bar{\hat{y}}(g_1*\hat{v})_t),\end{aligned}$$

where

$$\tilde{C}_v := \left(\frac{\kappa+b}{b}\right)^2 + \frac{4b}{\kappa} + \frac{\sqrt{2}+1}{2h_1^*}.$$

Calculating $(2.12)\times(\xi^2+1)\xi^2 + (2.35)\times\xi^4/(2\tilde{C}_v)$ and employing (2.16), we get

$$\begin{aligned}(2.36)\quad & \xi^2\frac{\partial}{\partial t}\left\{(\xi^2+1)\mathscr{E}_v + \frac{1}{2\tilde{C}_v}\xi^2 E_2\right\} + \frac{h_1(t)}{4}(\xi^2+1)\xi^4|\hat{v}|^2 + \frac{\rho_1\kappa}{4\tilde{C}_v}\xi^6|\hat{w}|^2 \\ & \quad + \frac{1}{4\tilde{C}_v}\xi^6|\hat{z}|^2 + \frac{\rho_2 b}{4\tilde{C}_v}(\xi^2+1)\xi^4|\hat{y}|^2 \\ & \leq \left(\frac{1}{g_1(0)} + \frac{2\rho_2\kappa_0^2 b g_1(t)}{\kappa^2\tilde{C}_v}\right)g_1(t)(\xi^2+1)\xi^4|\hat{v}|^2 \\ & \quad + \frac{2\rho_1\tilde{C}_v|g_1'(t)|^2}{\kappa g_1^2(0)}(\xi^2+1)^2|\hat{v}|^2 + \kappa_0^2\left(\frac{1}{2}+\frac{2}{h_1(t)}\right)(\xi^2+1)\xi^4|(g_1\diamond\hat{v})|^2 \\ & \quad + \frac{2\rho_1\tilde{C}_v}{\kappa g_1^2(0)}(\xi^2+1)^2|(g_1''\diamond\hat{v})|^2 \\ & \quad + 2\left(\frac{\rho_2\kappa_0^2 b}{\kappa^2\tilde{C}_v} + \frac{1}{g_1^2(0)h_1(t)}\right)(\xi^2+1)\xi^4|(g_1'\diamond\hat{v})|^2 \\ & \leq \frac{\kappa_0}{\kappa}\left(\frac{\kappa}{\kappa_0 g_1(0)} + \frac{2\rho_1\tilde{C}_v C_1^2}{\kappa_0 g_1(0)} + \frac{2\rho_2\kappa_0 b g_1(0)}{\kappa\tilde{C}_v}\right)g_1(t)(\xi^2+1)^3|\hat{v}|^2 \\ & \quad + \frac{\kappa_0}{\kappa}\left(\frac{\kappa}{2} + \frac{2\kappa}{h_1^*} + \frac{2\rho_2 b C_1^2}{\kappa\tilde{C}_v} + \frac{2\kappa C_1}{\kappa_0 g_1(0)h_1^*} + \frac{2\rho_1\tilde{C}_v\tilde{C}_1^2}{\kappa_0^2 g_1^2(0)}\right)(\xi^2+1)^3(g_1\square\hat{v}).\end{aligned}$$

Consequently, making the combination $(2.31) \times (\xi^2+1)^3 + (2.36) \times \tilde{\beta}_0$, we arrive at

$$(2.37) \qquad \frac{\partial}{\partial t}\tilde{\mathscr{E}}_1 + \frac{\tilde{\beta}_0 h_1^*}{4}(\xi^2+1)\xi^4|\hat{v}|^2 + \frac{\tilde{\beta}_0\rho_1\kappa}{4C_v}\xi^6|\hat{w}|^2 + \frac{\tilde{\beta}_0}{4C_v}\xi^6|\hat{z}|^2 + \frac{\tilde{\beta}_0\rho_2 b}{4C_v}(\xi^2+1)\xi^4|\hat{y}|^2 + \frac{\kappa_0 g_1(t)}{2\kappa}(\xi^2+1)^3|\hat{v}|^2 + \frac{\kappa_0 c_1}{2\kappa}(\xi^2+1)^3(g_1\square\hat{v}) \le 0,$$

where

$$\tilde{\mathscr{E}}_1 := (\xi^2+1)^3 E_0 + \tilde{\beta}_0\xi^2\left\{(\xi^2+1)E_v + \frac{1}{2\tilde{C}_v}\xi^2 E_2\right\}$$

and $\tilde{\beta}_0$ is taken as

$$(2.38) \qquad \tilde{\beta}_0 \le \frac{1}{2}\min\left\{\left(\frac{\kappa}{\kappa_0 g_1(0)} + \frac{2\rho_1\tilde{C}_v C_1^2}{\kappa_0 g_1(0)} + \frac{2\rho_2\kappa_0 b g_1(0)}{\kappa\tilde{C}_v}\right)^{-1}, \; c_1\left(\frac{\kappa}{2} + \frac{2\kappa}{h_1^*} + \frac{\rho_2 b C_1^2}{\kappa\tilde{C}_v} + \frac{2\kappa C_1}{\kappa_0 g_1(0)h_1^*} + \frac{2\rho_1\tilde{C}_v\tilde{C}_1^2}{\kappa_0^2 g_1^2(0)}\right)^{-1}\right\}.$$

We also estimate the energy $\tilde{\mathscr{E}}$. Because of

$$|\mathscr{E}_2| \le \left(\frac{1}{2} + \frac{b^2h_1^2(t)}{\kappa^2}\right)(\xi^2+1)|\hat{v}|^2 + (\rho_1^2+\rho_2^2)(\xi^2+1)|\hat{w}|^2 + \frac{1}{2}\left(3+\frac{\kappa^2}{b^2}\right)(\xi^2+1)|\hat{z}|^2 + \frac{5\rho_2^2}{2}(\xi^2+1)|\hat{y}|^2 + \frac{\kappa_0 b^2}{\kappa^2}(\xi^2+1)(g_1\square\hat{v}),$$

we have

$$\begin{aligned}(\xi^2+1)|E_v| + \frac{1}{2C_v}\xi^2|E_2| &\le \left(1 + \frac{1}{\kappa_0 g_1(0)} + \frac{1}{4C_v} + \frac{b^2h_1^2(t)}{2\kappa^2 C_v}\right)(\xi^2+1)\xi^2|\hat{v}|^2 \\ &\quad + \frac{1}{2}\left(3\rho_1^2 + \frac{\rho_1^2+\rho_2^2}{C_v}\right)(\xi^2+1)^2|\hat{w}|^2 + \frac{1}{4C_v}\left(3+\frac{\kappa^2}{b^2}\right)(\xi^2+1)\xi^2|\hat{z}|^2 \\ &\quad + \frac{5\rho_2^2}{4C_v}(\xi^2+1)\xi^2|\hat{y}|^2 + \left(\frac{1}{g_1(0)} + \frac{C_1}{2g_1(0)} + \frac{\kappa_0 b^2}{2\kappa^2 C_v}\right)(\xi^2+1)\xi^2(g_1\square\hat{v}),\end{aligned}$$

and this gives

$$
\begin{aligned}
(2.39) \qquad \tilde{\mathscr{E}}_1 &\geq (\xi^2+1)^3\left\{\frac{3h_1^*}{4\kappa}|\hat{v}|^2+\frac{\rho_1}{2}|\hat{w}|^2+\frac{1}{2b}|\hat{z}|^2+\frac{\rho_2}{2}|\hat{y}|^2+\frac{\kappa_0}{2\kappa}(g_1\square\hat{v})\right\} \\
&\geq (\xi^2+1)^3\left\{\frac{h_1^*}{2\kappa}|\hat{v}|^2+\frac{\rho_1}{2}|\hat{w}|^2+\frac{1}{2b}|\hat{z}|^2+\frac{\rho_2}{2}|\hat{y}|^2\right. \\
&\qquad\left.+\frac{h_1^*g_1(t)}{4\kappa g_1(0)}|\hat{v}|^2+\frac{\kappa_0}{2\kappa}(g_1\square\hat{v})\right\} \\
&\geq \tilde{m}_1(\xi^2+1)^3(|\hat{u}|^2+g_1(t)|\hat{v}|^2+(g_1\square\hat{v})), \\
\tilde{\mathscr{E}}_1 &\leq \tilde{M}_1(\xi^2+1)^3(|\hat{u}|^2+g_1(t)|\hat{v}|^2+(g_1\square\hat{v})),
\end{aligned}
$$

where

$$
\tilde{m}_1 := \frac{1}{2}\min\left\{\frac{h_1^*}{\kappa},\rho_1,\frac{1}{b},\rho_2,\frac{h_1^*}{2\kappa g_1(0)},\frac{\kappa_0}{\kappa}\right\},
$$

$$
\begin{aligned}
\tilde{M}_1 := \max\Bigg\{&\frac{1}{\kappa}+\left(1+\frac{1}{\kappa_0 g_1(0)}+\frac{1}{4C_v}+\frac{b^2}{2\kappa^2C_v}\right),\rho_1\left(1+\frac{3\rho_1}{2}\right)+\frac{\rho_1^2+\rho_2^2}{2C_v}, \\
&\frac{1}{b}+\frac{1}{4C_v}\left(3+\frac{\kappa^2}{b^2}\right),\rho_2\left(1+\frac{5\rho_2}{4C_v}\right),\frac{\kappa_0}{\kappa}+\left(\frac{1}{g_1(0)}+\frac{C_1}{2g_1(0)}+\frac{\kappa_0b^2}{2\kappa^2C_v}\right)\Bigg\},
\end{aligned}
$$

where we take $\tilde{\beta}_0$ satisfying (2.38) and

$$
\begin{aligned}
\tilde{\beta}_0 \leq \frac{1}{2}\min\Bigg\{&2,\frac{h_1^*}{2\kappa}\left(1+\frac{1}{\kappa_0g_1(0)}+\frac{1}{4C_v}+\frac{b^2}{2\kappa^2C_v}\right)^{-1},2\left(3\rho_1+\frac{\rho_1^2+\rho_2^2}{\rho_1C_v}\right)^{-1}, \\
&\frac{4C_v}{b}\left(3+\frac{\kappa^2}{b^2}\right)^{-1},\frac{4C_v}{5\rho_2},\frac{b_0}{b}\left(\frac{1}{g_1(0)}+\frac{C_1}{2g_1(0)}+\frac{\kappa_0b^2}{2\kappa^2C_v}\right)^{-1}\Bigg\}.
\end{aligned}
$$

Therefore, using (2.31) and (2.39), we get

$$
\begin{aligned}
(2.40) \qquad |\hat{u}|^2+U_1[\hat{v}]+\frac{\tilde{n}_1}{\tilde{m}_1}\int_0^t\Bigg\{&\frac{\xi^4}{(\xi^2+1)^2}(|\hat{v}|^2+|\hat{y}|^2) \\
&+\frac{\xi^6}{(\xi^2+1)^3}(|\hat{w}|^2+|\hat{z}|^2)+U_1[\hat{v}]\Bigg\}d\tau \\
\leq \frac{\tilde{M}_1}{\tilde{m}_1}(|\hat{u}|^2+U_1[\hat{v}]),&
\end{aligned}
$$

where

$$\tilde{n}_1 := \frac{1}{2}\min\left\{\frac{\beta_0 h_1^*}{2}, \frac{\beta_0\rho_1\kappa}{2C_v}, \frac{\beta_0}{2C_v}, \frac{\beta_0\rho_2 b}{2C_v}, \frac{\kappa_0}{\kappa}, \frac{\kappa_0 c_1}{\kappa}\right\}.$$

Finally, the pointwise estimate (2.23), with $\eta$ given in (2.24) for $\rho_1 b = \rho_2\kappa$, is also obtained by virtue of (2.31) and (2.39). This completes the proof of Theorem 2.3. □

### 2.3. Pointwise estimate for $\kappa_0 = 0$ and $b_0 > 0$

**Theorem 2.4.** *Let $u$ be a solution of problem* (1.5)–(1.6) *with* $\kappa_0 = 0$ *and* $b_0 > 0$. *Then $\hat{u}$ satisfies the following pointwise estimates in the Fourier space*:

$$|\hat{u}(t,\xi)| \le Ce^{-c\zeta(\xi)t}|\hat{u}_0(\xi)| \tag{2.41}$$

*for some constants* $C, c > 0$, *where*

$$\zeta(\xi) = \begin{cases} \dfrac{\xi^4}{(1+\xi^2)^3} & \text{if } \dfrac{\kappa}{\rho_1} \neq \dfrac{b}{\rho_2}, \\ \dfrac{\xi^4}{(1+\xi^2)^2} & \text{if } \dfrac{\kappa}{\rho_1} = \dfrac{b}{\rho_2}. \end{cases} \tag{2.42}$$

*Proof.* We also employ here the equations derived in the proof of Theorem 2.2, but now regarding that $\kappa_0 = 0$.

Combining (2.9) and (2.10) for $\kappa_0 = 0$, we firstly have

$$\begin{aligned} &\frac{\partial}{\partial t}E_3 + \xi^2(\rho_1\kappa|\hat{w}|^2 + |\hat{z}|^2) + (\xi^2+1)|\hat{v}|^2 - \rho_2((2\kappa+b)\xi^2+\kappa)|\hat{y}|^2 \\ &\quad + \rho_1(b+\kappa)\xi\,\mathrm{Re}(i\hat{w}\bar{\hat{y}}) + (\rho_1 b - \rho_2\kappa)(2\xi^2+1)\xi\,\mathrm{Re}(i\hat{w}\bar{\hat{y}}) \\ &\quad - b_0\xi^2\,\mathrm{Re}((g_2 * \hat{z})\bar{\hat{z}}) - \rho_1 b_0(2\xi^2+1)\,\mathrm{Re}(\bar{\hat{w}}(g_2*\hat{z})_t) = 0, \end{aligned} \tag{2.43}$$

where

$$\begin{aligned} E_3 := {} & \rho_1\xi\,\mathrm{Re}(i\hat{v}\bar{\hat{w}}) + \rho_2\xi\,\mathrm{Re}(i\hat{y}\bar{\hat{z}}) + \rho_1\,\mathrm{Re}(\hat{w}\bar{\hat{z}}) \\ & + (2\xi^2+1)\{\rho_1\,\mathrm{Re}(\hat{w}\bar{\hat{z}}) + \rho_2\,\mathrm{Re}(\hat{v}\bar{\hat{y}}) - \rho_1 b_0\,\mathrm{Re}((g_2*\hat{z})\bar{\hat{w}})\}. \end{aligned}$$

As before, we separate again the next proofs in two cases concerning the wave speeds.

*Case $\rho_1 b \neq \rho_2\kappa$.* In the case $\rho_1 b \neq \rho_2\kappa$, we combine (2.8) and (2.43) to obtain

$$\frac{\partial}{\partial t}\{E_3 + \xi^2(\rho_2\xi \operatorname{Re}(i\hat{y}\bar{\hat{z}}) + \rho_1 \operatorname{Re}(\hat{w}\bar{\hat{z}}))\} + \rho_1\kappa\xi^2|\hat{w}|^2 + (\xi^2+1)\xi^2|\hat{z}|^2 + (\xi^2+1)|\hat{v}|^2$$
$$= \rho_2(b(\xi^2+1)\xi^2 + \kappa(2\xi^2+1))|\hat{y}|^2 - \rho_1(b\xi^2 + b + \kappa)\xi \operatorname{Re}(i\hat{w}\bar{\hat{y}})$$
$$- (\rho_1 b - \rho_2\kappa)(2\xi^2+1)\xi \operatorname{Re}(i\hat{w}\bar{\hat{y}}) + b_0(\xi^2+1)\xi^2 \operatorname{Re}((g_2 * \hat{z})\bar{\hat{z}})$$
$$+ \rho_1 b_0(2\xi^2+1) \operatorname{Re}(\bar{\hat{w}}(g_2 * \hat{z})_t).$$

Therefore, applying the Hölder inequality in the right-hand side of this identity, we estimate

$$\frac{\partial}{\partial t}\{E_3 + \xi^2(\rho_2\xi \operatorname{Re}(i\hat{y}\bar{\hat{z}}) + \rho_1 \operatorname{Re}(\hat{w}\bar{\hat{z}}))\} + \frac{\rho_1\kappa}{2}\xi^2|\hat{w}|^2$$
$$+ \frac{h_2(t)}{2}(\xi^2+1)\xi^2|\hat{z}|^2 + (\xi^2+1)|\hat{v}|^2$$
$$\le \rho_2 b C_y(\xi^2+1)^2|\hat{y}|^2 + \frac{b_0^2}{2h_2(t)}(\xi^2+1)\xi^2|(g_2 \diamond \hat{z})|^2$$
$$+ \rho_1 b_0(2\xi^2+1) \operatorname{Re}(\bar{\hat{w}}(g_2 * \hat{z})_t), \tag{2.44}$$

where

$$C_y := \frac{1}{\rho_2 b}\left\{\rho_2(2\kappa + b) + \rho_1\frac{(b+\kappa)^2}{\kappa} + \frac{4(\rho_1 b - \rho_2\kappa)^2}{\rho_1\kappa}\right\}.$$

Then, calculating $(2.13) \times (\xi^2+1)^2 + (2.44) \times \xi^2/(2C_y)$ and using (2.16), we get

$$\frac{\partial}{\partial t}\left\{(\xi^2+1)^2 E_y + \frac{1}{2C_y}\xi^2(E_3 + \xi^2(\rho_2\xi \operatorname{Re}(i\hat{y}\bar{\hat{z}}) + \rho_1 \operatorname{Re}(\hat{w}\bar{\hat{z}})))\right\}$$
$$+ \frac{1}{4C_y}(\xi^2+1)\xi^2|\hat{v}|^2 + \frac{\rho_1\kappa}{8C_y}\xi^4|\hat{w}|^2 + \frac{h_2(t)}{8C_y}(\xi^2+1)\xi^4|\hat{z}|^2$$
$$+ \frac{\rho_2 b}{4}(\xi^2+1)^2\xi^2|\hat{y}|^2$$
$$\le \frac{g_2(t)}{g_2(0)}(\xi^2+1)^2\xi^2|\hat{z}|^2 + \frac{b_0^2}{4h_2(t)C_y}(\xi^2+1)\xi^4|(g_2 \diamond \hat{z})|^2$$
$$+ \frac{2C_y}{g_2^2(0)h_2(t)}(\xi^2+1)^3|g_2' \diamond \hat{z}|^2$$
$$+ \frac{4\rho_1 b_0^2}{\kappa C_y}(\xi^2+1)^2(g_2^2(t)|\hat{z}|^2 + |g_2' \diamond \hat{z}|^2) \tag{2.45}$$

$$
\begin{aligned}
&+\frac{2C_y}{g_2^2(0)}(\xi^2+1)^3(g_2^2(t)|\hat{z}|^2+|g_2'\diamond\hat{z}|^2)\\
&+\frac{2\rho_2}{bg_2^2(0)}(\xi^2+1)^2(|g_2'(t)|^2|\hat{z}|^2+|g_2''\diamond\hat{z}|^2)\\
\le{}&\frac{b_0}{b}\left(\frac{b}{b_0g_2(0)}+\frac{4\rho_1bb_0g_2(0)}{\kappa C_y}+\frac{2bC_y}{b_0g_2(0)}+\frac{2\rho_2C_2^2}{b_0g_2(0)}\right)g_2(t)(\xi^2+1)^3|\hat{z}|^2\\
&+\frac{b_0}{b}\left(\frac{b}{C_y}\left(\frac{4\rho_1C_2^2}{\kappa}+\frac{1}{4h_2(t)}\right)+\frac{2bC_yC_2}{b_0g_2(0)}\left(1+\frac{1}{h_2(t)}\right)\right.\\
&\qquad\left.+\frac{2\rho_2\tilde{C}_2^2}{b_0^2g_2^2(0)}\right)(\xi^2+1)^3(g_2\square\hat{z}).
\end{aligned}
$$

Consequently, making the combination $(2.31)\times(\xi^2+1)^3+(2.45)\times\gamma_0$, we arrive at

$$
\begin{aligned}
\text{(2.46)}\qquad&\frac{\partial}{\partial t}\mathscr{E}_2+\frac{\gamma_0}{4C_y}(\xi^2+1)\xi^2|\hat{v}|^2+\frac{\gamma_0\rho_1\kappa}{8C_y}\xi^4|\hat{w}|^2+\frac{\gamma_0h_2(t)}{8C_y}(\xi^2+1)\xi^4|\hat{z}|^2\\
&+\frac{\gamma_0\rho_2b}{4}(\xi^2+1)^2\xi^2|\hat{y}|^2+\frac{b_0g_2(t)}{2b}(\xi^2+1)^3|\hat{z}|^2\\
&+\frac{b_0c_2}{2b}(\xi^2+1)^3(g_2\square\hat{z})\le 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathscr{E}_2:={}&(\xi^2+1)^3E_0+\gamma_0\Big\{(\xi^2+1)^2E_y\\
&+\frac{1}{2C_y}\xi^2(E_3+\xi^2(\rho_2\xi\,\mathrm{Re}(i\hat{y}\bar{\hat{z}})+\rho_1\,\mathrm{Re}(\hat{w}\bar{\hat{z}})))\Big\},
\end{aligned}
$$

and $\gamma_0$ is taken as

$$
\begin{aligned}
\text{(2.47)}\qquad\gamma_0\le\frac{1}{2}\min\Bigg\{&\left(\frac{b}{b_0g_2(0)}+\frac{4\rho_1bb_0g_2(0)}{\kappa C_y}+\frac{2bC_y}{b_0g_2(0)}+\frac{2\rho_2C_2^2}{b_0g_2(0)}\right)^{-1},\\
&c_2\left(\frac{b}{C_y}\left(\frac{4\rho_1C_2^2}{\kappa}+\frac{1}{4h_2(t)}\right)+\frac{2bC_yC_2}{b_0g_2(0)}\left(1+\frac{1}{h_2(t)}\right)\right.\\
&\left.+\gamma_0\frac{2\rho_2\tilde{C}_2^2}{b_0^2g_2^2(0)}\right)^{-1}\Bigg\}.
\end{aligned}
$$

Let us estimate the energy $\mathscr{E}_2$. Using (2.20) and

$$\begin{aligned}
&|E_3 + \xi^2(\rho_2\xi\ \mathrm{Re}(i\hat{y}\bar{\hat{z}}) + \rho_1\ \mathrm{Re}(\hat{w}\bar{\hat{z}}))| \\
&\quad \le \frac{3}{2}(\xi^2+1)|\hat{v}|^2 + 3\rho_1^2(\xi^2+1)|\hat{w}|^2 + \rho_2^2(\xi^2+1)^2|\hat{y}|^2 \\
&\qquad + (1+h_2(t)^2)(\xi^2+1)|\hat{z}|^2 + b_0(\xi^2+1)(g_2\square\hat{z}),
\end{aligned}$$

we have

$$\begin{aligned}
&(\xi^2+1)^2|E_y| + \frac{1}{2C_y}\xi^2|E_3 + \xi^2(\rho_2\xi\ \mathrm{Re}(i\hat{y}\bar{\hat{z}}) + \rho_1\ \mathrm{Re}(\hat{w}\bar{\hat{z}}))| \\
&\quad \le \frac{3}{4C_y}(\xi^2+1)\xi^2|\hat{v}|^2 + \frac{3\rho_1^2}{2C_y}(\xi^2+1)\xi^2|\hat{w}|^2 + \rho_2^2\left(\frac{1}{2C_y}+1\right)(\xi^2+1)^3|\hat{y}|^2 \\
&\qquad + \left(\frac{1}{2} + \frac{1}{b_0 g_2(0)} + \frac{1+h_2(t)^2}{2C_y}\right)(\xi^2+1)\xi^2|\hat{z}|^2 \\
&\qquad + \left(\frac{1}{g_2(0)}\left(1+\frac{C_2}{2}\right) + \frac{b_0}{2C_y}\right)(\xi^2+1)^2\xi^2(g_2\square\hat{z}),
\end{aligned}$$

and this gives

$$\begin{aligned}
\mathscr{E}_2 &\ge (\xi^2+1)^3\left(\frac{1}{2\kappa}|\hat{v}|^2 + \frac{\rho_1}{2}|\hat{w}|^2 + \frac{3h_2^*}{4b}|\hat{z}|^2 + \frac{\rho_2}{2}|\hat{y}|^2 + \frac{b_0}{2b}(g_2\square\hat{z})\right) \qquad (2.48)\\
&\ge (\xi^2+1)^3\left(\frac{1}{2\kappa}|\hat{v}|^2 + \frac{\rho_1}{2}|\hat{w}|^2 + \frac{h_2^*}{2b}|\hat{z}|^2 + \frac{\rho_2}{2}|\hat{y}|^2\right. \\
&\qquad\qquad \left. + \frac{h_2^* g_2(t)}{4b g_2(0)}|\hat{z}|^2 + \frac{b_0}{2b}(g_2\square\hat{z})\right) \\
&\ge m_2(\xi^2+1)^3(|\hat{u}|^2 + g_2(t)|\hat{z}|^2 + (g_2\square\hat{z})), \\
\mathscr{E}_2 &\le M_2(\xi^2+1)^3(|\hat{u}|^2 + g_2(t)|\hat{z}|^2 + (g_2\square\hat{z})),
\end{aligned}$$

where

$$\begin{aligned}
m_2 &:= \frac{1}{2}\ \min\left\{\frac{1}{\kappa}, \rho_1, \frac{h_2^*}{b}, \rho_2, \frac{h_1^*}{2b g_2(0)}, \frac{b_0}{b}\right\}, \\
M_2 &:= \max\left\{\frac{1}{\kappa} + \frac{3}{4C_y}, \rho_1 + \frac{3\rho_1^2}{2C_y}, \frac{1}{b} + \frac{1}{2} + \frac{1}{b_0 g_2(0)} + \frac{1}{C_y},\right. \\
&\qquad \left. \rho_2 + \rho_2^2\left(\frac{1}{2C_y}+1\right), \frac{b_0}{b} + \left(\frac{1}{g_2(0)}\left(1+\frac{C_2}{2}\right) + \frac{b_0}{2C_y}\right)\right\},
\end{aligned}$$

where we take $\gamma_0$ satisfying (2.47) and

$$\gamma_0 \le \min\left\{1, \frac{2C_y}{3\kappa}, \frac{C_y}{3\rho_1}, \frac{h_2^*}{4b}\left(\frac{1}{2} + \frac{1}{b_0 g_2(0)} + \frac{1 + (h_2^*)^2}{2C_y}\right)^{-1},\right.$$
$$\left.\frac{1}{2\rho_2}\left(\frac{1}{2C_y} + 1\right)^{-1}, \frac{b_0}{2b}\left(\frac{1}{g_2(0)}\left(1 + \frac{C_2}{2}\right) + \frac{b_0}{2C_y}\right)^{-1}\right\}.$$

Then, using (2.46) and (2.48), we conclude

$$\begin{aligned}(2.49)\qquad |\hat{u}|^2 + U_2[\hat{z}] + \frac{n_2}{m_2}\int_0^t \left\{ \frac{\xi^2}{(\xi^2+1)^2}|\hat{v}|^2 + \frac{\xi^4}{(\xi^2+1)^3}|\hat{w}|^2 + \frac{\xi^4}{(\xi^2+1)^2}|\hat{z}|^2 \right. \\ \left. + \frac{\xi^2}{\xi^2+1}|\hat{y}|^2 + U_2[\hat{z}] \right\} d\tau \\ \le \frac{M_2}{m_2}(|\hat{u}|^2 + U_2[\hat{z}]),\end{aligned}$$

where $U_2[\hat{z}] := g_2(t)|\hat{z}|^2 + c_2(g_2 \Box \hat{z})$ and

$$n_2 := \frac{1}{2}\min\left\{\frac{\gamma_0}{2C_y}, \frac{\gamma_0\rho_1\kappa}{4C_y}, \frac{\gamma_0 h_2^*}{4C_y}, \frac{\gamma_0 b}{2}, \frac{b_0}{b}, \frac{b_0 c_2}{b}\right\}.$$

Hence, the pointwise estimate (2.41), with $\zeta$ given in (2.42) for $\rho_1 b \neq \rho_2\kappa$, can be conclude from (2.46) and (2.48).

*Case $\rho_1 b = \rho_2\kappa$.* In the case $\rho_1 b = \rho_2\kappa$, the equation (2.28) (with $\kappa_0 = 0$) provides

$$\begin{aligned}(2.50)\qquad \frac{\partial}{\partial t}E_3 + \frac{\rho_1\kappa}{2}\xi^2|\hat{w}|^2 + \frac{h_2(t)}{2}\xi^2|\hat{z}|^2 + (\xi^2+1)|\hat{v}|^2 \\ \le \rho_2 b\tilde{C}_y(\xi^2+1)|\hat{y}|^2 + \frac{b_0^2}{2h_2(t)}\xi^2|g_2 \diamond \hat{z}|^2 \\ + \rho_1 b_0(2\xi^2+1)\,\mathrm{Re}(\bar{\hat{w}}(g_2 * \hat{z})_t),\end{aligned}$$

where

$$\tilde{C}_y := \frac{1}{\rho_2 b}\left\{\rho_2(2\kappa + b) + \rho_2\kappa + \frac{\rho_1(b+\kappa)^2}{2\kappa}\right\}.$$

Then, calculating $(2.13) \times (\xi^2 + 1) + (2.50) \times \xi^2/(2\tilde{C}_y)$ and employing (2.16), we get

$$\begin{aligned}
(2.51)\quad &\frac{\partial}{\partial t}\left\{(\xi^2+1)E_y+\frac{1}{2\tilde{C}_y}\xi^2E_3\right\}+\frac{1}{4\tilde{C}_y}(\xi^2+1)\xi^2|\hat{v}|^2+\frac{\rho_1\kappa}{8\tilde{C}_y}\xi^4|\hat{w}|^2\\
&\quad+\frac{h_2(t)}{8\tilde{C}_y}\xi^4|\hat{z}|^2+\frac{\rho_2 b}{4}(\xi^2+1)\xi^2|\hat{y}|^2\\
&\le\frac{g_2(t)}{g_2(0)}(\xi^2+1)\xi^2|\hat{z}|^2+\frac{b_0^2}{4\tilde{C}_yh_2(t)}\xi^4|g_2\diamond\hat{z}|^2\\
&\quad+\frac{2\tilde{C}_y}{g_2^2(0)h_2(t)}(\xi^2+1)^2|g_2'\diamond\hat{z}|^2\\
&\quad+\frac{4\rho_1b_0^2}{\kappa\tilde{C}_y}(\xi^2+1)^2(g_2^2(t)|\hat{z}|^2+|g_2'\diamond\hat{z}|^2)\\
&\quad+\frac{2\rho_2}{bg_2^2(0)}(\xi^2+1)^2(|g_2'(t)|^2|\hat{z}|^2+|g_2''\diamond\hat{z}|^2)\\
&\quad+\frac{2\tilde{C}_y}{g_2^2(0)}(\xi^2+1)(g_2^2(t)|\hat{z}|^2+|g_2'\diamond\hat{z}|^2)\\
&\le\frac{b_0}{b}\left(\frac{b}{b_0g_2(0)}+\frac{4\rho_1bb_0g_2(0)}{\kappa\tilde{C}_y}\right.\\
&\qquad\left.+\frac{2}{b_0g_2(0)}(\rho_2C_2^2+b\tilde{C}_y)\right)g_2(t)(\xi^2+1)^2|\hat{z}|^2\\
&\quad+\frac{b_0}{b}\left(\frac{b}{\tilde{C}_y}\left(\frac{4\rho_1C_2^2}{\kappa}+\frac{1}{4h_2(t)}\right)+\frac{2b\tilde{C}_yC_2}{b_0g_2(0)}\left(1+\frac{1}{h_2(t)}\right)\right.\\
&\qquad\left.+\frac{2\rho_2\tilde{C}_2^2}{b_0^2g_2^2(0)}\right)(\xi^2+1)^2(g_2\Box\hat{z}).
\end{aligned}$$

Consequently, combining $(2.31)\times(\xi^2+1)^2+(2.51)\times\tilde{\gamma}_0$, we obtain

$$\begin{aligned}
(2.52)\quad &\frac{\partial}{\partial t}\tilde{\mathscr{E}}_2+\frac{\tilde{\gamma}_0}{4\tilde{C}_y}(\xi^2+1)\xi^2|\hat{v}|^2+\frac{\tilde{\gamma}_0\rho_1\kappa}{8\tilde{C}_y}\xi^4|\hat{w}|^2+\frac{\tilde{\gamma}_0h_2(t)}{8\tilde{C}_y}\xi^4|\hat{z}|^2\\
&\quad+\frac{\tilde{\gamma}_0\rho_2b}{4}(\xi^2+1)\xi^2|\hat{y}|^2+\frac{b_0g_2(t)}{2b}(\xi^2+1)^2|\hat{z}|^2\\
&\quad+\frac{b_0c_2}{2b}(\xi^2+1)^2(g_2\Box\hat{z})\le 0,
\end{aligned}$$

where

$$\tilde{\mathscr{E}}_2 := (\xi^2+1)^2 E_0 + \tilde{\gamma}_0 \left\{ (\xi^2+1) E_y + \frac{1}{2\tilde{C}_y} \xi^2 E_3 \right\},$$

and $\tilde{\gamma}_0$ is taken as

$$\begin{aligned} (2.53) \quad \tilde{\gamma}_0 \leq \frac{1}{2} \min \Bigg\{ &\left( \frac{b}{b_0 g_2(0)} + \frac{4\rho_1 b b_0 g_2(0)}{\kappa \tilde{C}_y} + \frac{2}{b_0 g_2(0)} (\rho_2 C_2^2 + b\tilde{C}_y) \right)^{-1}, \\ &c_2 \left( \frac{b}{\tilde{C}_y} \left( \frac{4\rho_1 C_2^2}{\kappa} + \frac{1}{4h_2^*} \right) + \frac{2b\tilde{C}_y C_2}{b_0 g_2(0)} \left( 1 + \frac{1}{h_2^*} \right) + \frac{2\rho_2 \tilde{C}_2^2}{b_0^2 g_2^2(0)} \right)^{-1} \Bigg\}. \end{aligned}$$

We also estimate the energy $\tilde{\mathscr{E}}_2$ as follows. Because of (2.20) and

$$\begin{aligned} |E_3| \leq{}& \frac{3}{2}(\xi^2+1)|\hat{v}|^2 + 3\rho_1^2(\xi^2+1)|\hat{w}|^2 + (1+h_2^2(t))(\xi^2+1)|\hat{z}|^2 \\ &+ \frac{3\rho_2^2}{2}(\xi^2+1)|\hat{y}|^2 + b_0(\xi^2+1)(g_2 \Box \hat{z}), \end{aligned}$$

we have

$$\begin{aligned} (\xi^2+1)|E_y| + \frac{1}{2C_y}\xi^2|E_3| \leq{}& \frac{3}{4C_y}(\xi^2+1)\xi^2|\hat{v}|^2 + \frac{3\rho_1^2}{2C_y}(\xi^2+1)\xi^2|\hat{w}|^2 \\ &+ \left( \frac{1}{2} + \frac{1}{b_0 g_2(0)} + \frac{1+h_2^2(t)}{2C_y} \right)(\xi^2+1)\xi^2|\hat{z}|^2 \\ &+ \rho_2^2\left(1 + \frac{3}{4C_y}\right)(\xi^2+1)^2|\hat{y}|^2 \\ &+ \left( \frac{b_0}{2C_y} + \frac{1}{g_2(0)}\left(1 + \frac{C_2}{2}\right)\right)(\xi^2+1)\xi^2(g_2 \Box \hat{z}) \end{aligned}$$

and this provides

$$\begin{aligned} (2.54) \qquad \tilde{\mathscr{E}}_2 &\geq (\xi^2+1)^2\left( \frac{1}{2\kappa}|\hat{v}|^2 + \frac{\rho_1}{2}|\hat{w}|^2 + \frac{3h_2^*}{4b}|\hat{z}|^2 + \frac{\rho_2}{2}|\hat{y}|^2 + \frac{b_0}{2b}(g_2 \Box \hat{z}) \right) \\ &\geq (\xi^2+1)^2\bigg( \frac{1}{2\kappa}|\hat{v}|^2 + \frac{\rho_1}{2}|\hat{w}|^2 + \frac{h_2^*}{2b}|\hat{z}|^2 + \frac{\rho_2}{2}|\hat{y}|^2 \\ &\qquad + \frac{h_2^* g_2(t)}{4bg_2(0)}|\hat{z}|^2 + \frac{b_0}{2b}(g_2 \Box \hat{z}) \bigg) \\ &\geq \tilde{m}_2(\xi^2+1)^2(|\hat{u}|^2 + g_2(t)|\hat{z}|^2 + (g_2 \Box \hat{z})), \\ \tilde{\mathscr{E}}_2 &\leq \tilde{M}_2(\xi^2+1)^2(|\hat{u}|^2 + g_2(t)|\hat{z}|^2 + (g_2 \Box \hat{z})), \end{aligned}$$

where

$$\tilde{m}_2 := \frac{1}{2}\min\left\{\frac{1}{\kappa}, \rho_1, \frac{h_2^*}{b}, \rho_2, \frac{h_1^*}{2bg_2(0)}, \frac{b_0}{b}\right\},$$

$$\tilde{M}_2 := \max\left\{\frac{1}{\kappa} + \frac{3}{4C_y}, \rho_1 + \frac{3\rho_1^2}{2C_y}, \frac{1}{b} + \frac{1}{2} + \frac{1}{b_0 g_2(0)} + \frac{1 + h_2^2(t)}{2C_y},\right.$$
$$\left.\rho_2 + \rho_2^2\left(1 + \frac{3}{4C_y}\right), \frac{b_0}{b} + \frac{b_0}{2C_y} + \frac{1}{g_2(0)}\left(1 + \frac{C_2}{2}\right)\right\},$$

and $\tilde{\gamma}_0$ is taken as (2.47) and

$$\tilde{\gamma}_0 \le \frac{1}{2}\min\left\{2, \frac{4C_y}{3\kappa}, \frac{2C_y}{3\rho_1}, \frac{h_2^*}{2b}\left(\frac{1}{2} + \frac{1}{b_0 g_2(0)} + \frac{1 + h_2^2(t)}{2C_y}\right)^{-1},\right.$$
$$\left.\frac{1}{\rho_2}\left(1 + \frac{3}{4C_y}\right)^{-1}, \frac{b_0}{b}\left(\frac{b_0}{2C_y} + \frac{1}{g_2(0)}\left(1 + \frac{C_2}{2}\right)\right)^{-1}\right\}.$$

Thus, using (2.52) and (2.54), we finally get

$$(2.55) \qquad |\hat{u}|^2 + U_2[\hat{z}] + \frac{\tilde{n}_2}{\tilde{m}_2}\int_0^t \left\{\frac{\xi^2}{\xi^2 + 1}(|\hat{v}|^2 + |\hat{y}|^2)\right.$$
$$\left. + \frac{\xi^4}{(\xi^2 + 1)^2}(|\hat{w}|^2 + |\hat{z}|^2) + U_2[\hat{z}]\right\} d\tau$$
$$\le \frac{\tilde{M}_2}{\tilde{m}_2}(|\hat{u}|^2 + U_2[\hat{z}]),$$

where

$$n_2 := \frac{1}{2}\min\left\{\frac{\tilde{\gamma}_0}{2\tilde{C}_y}, \frac{\tilde{\gamma}_0\rho_1\kappa}{4\tilde{C}_y}, \frac{\tilde{\gamma}_0 h_2^*}{4\tilde{C}_y}, \frac{\tilde{\gamma}_0\rho_2 b}{2}, \frac{b_0}{b}, \frac{b_0 c_2}{b}\right\}.$$

Therefore, the pointwise estimate (2.41), with $\zeta$ given in (2.42) for $\rho_1 b = \rho_2\kappa$, is obtained by means of (2.52) and (2.54). This finishes the proof of Theorem 2.4. □

## 3. Spectral analysis

In this section, we investigate the optimality of the pointwise estimates achieved in Theorems 2.2, 2.3, and 2.4, for a suitable choice of the memory kernels. To this end, we shall proceed as before by dealing with the three cases concerning the parameters $\kappa_0$ and $b_0$.

### 3.1. Optimality for $\kappa_0 > 0$ and $b_0 > 0$

Let us assume $\kappa_0 > 0$ and $b_0 > 0$ and suppose that $g_j(t) := \varepsilon_j \mu_j e^{-\mu_j t}/\sigma_j$ for $j = 1, 2$, where $\mu_j > 0$, $0 < \varepsilon_j < 1$ and $\sigma_1 = \kappa_0$ and $\sigma_2 = b_0$. Then, it is easy to check that these functions satisfy the conditions (1.3)–(1.4) required in Assumption 1.1, with constants $c_j = C_j = \mu_j$ and $\tilde{C}_j = \mu_j^2$. Under this setting, we introduce the new variables

$$v := \kappa(\phi_x + \psi), \qquad w := \phi_t, \qquad z := b\psi_x, \qquad y := \psi_t.$$

$$p := \kappa(\phi_x + \psi) - \frac{\kappa\kappa_0}{\varepsilon_1}(g_1 * (\phi_x + \psi)), \qquad q := b\psi_x - \frac{bb_0}{\varepsilon_2}(g_2 * \psi_x).$$

Consequently, problem (1.1) can be rewritten as the following symmetric hyperbolic system

$$A^0 u_t + A u_x + L u = 0, \tag{3.1}$$

where $u = (v, w, z, y, p, q)^\top$ and $A^0 = \operatorname{diag}(\tilde{\varepsilon}_1/\kappa \;\; \rho_1 \;\; \tilde{\varepsilon}_2/b \;\; \rho_2 \;\; \varepsilon_1/\kappa \;\; \varepsilon_2/b)$,

$$A = -\begin{pmatrix} 0 & \tilde{\varepsilon}_1 & 0 & 0 & 0 & 0 \\ \tilde{\varepsilon}_1 & 0 & 0 & 0 & \varepsilon_1 & 0 \\ 0 & 0 & 0 & \tilde{\varepsilon}_2 & 0 & 0 \\ 0 & 0 & \tilde{\varepsilon}_2 & 0 & 0 & \varepsilon_2 \\ 0 & \varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_2 & 0 & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & 0 & 0 & -\tilde{\varepsilon}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{\varepsilon}_1 & 0 & 0 & 0 & \varepsilon_1 & 0 \\ 0 & 0 & 0 & -\varepsilon_1 & \varepsilon_1\mu_1/\kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_2\mu_2/b \end{pmatrix},$$

where $\tilde{\varepsilon}_j := 1 - \varepsilon_j$ for $j = 1, 2$. Additionally, the symmetric system (3.1) can be expressed in term of its components as follows

$$\begin{aligned} v_t - \kappa w_x - \kappa y &= 0, \\ \rho_1 w_t - (1-\varepsilon_1)v_x - \varepsilon_1 p_x &= 0, \\ z_t - b y_x &= 0, \\ \rho_2 y_t - (1-\varepsilon_2)z_x - \varepsilon_2 q_x + (1-\varepsilon_1)v + \varepsilon_1 p &= 0, \\ p_t - \kappa w_x - \kappa y + \mu_1 p &= 0, \\ q_t - b y_x + \mu_2 q &= 0. \end{aligned}$$

Also, applying the Fourier transform in (3.1), we obtain

$$A^0 \hat{u}_t + i\xi A\hat{u} + L\hat{u} = 0. \tag{3.2}$$

To obtain the desired property for the solution $\hat{u}$ of the Fourier problem (3.2), we analyze the eigenvalues of the corresponding eigenvalue problem. For detailed arguments on the subject applied to other systems, we refer [16, 17, 18] to readers.

Here, the eigenvalues satisfy the characteristic equation

$$\det(\lambda I - \hat{\Phi}(i\xi)) = 0, \qquad \text{where } \hat{\Phi}(i\xi) := -(A^0)^{-1}(i\xi A + L),$$

that is, the parameter $\lambda$ satisfies the following sixth-order polynomial equation

$$\begin{aligned} &\rho_1\rho_2(\lambda+\mu_1)(\lambda+\mu_2)\lambda^4 + \{\rho_1 b(\lambda+\mu_1)(\lambda+\mu_2\tilde{\varepsilon}_2) \\ &\quad + \rho_2\kappa(\lambda+\mu_2)(\lambda+\mu_1\tilde{\varepsilon}_1)\}\xi^2\lambda^2 + \rho_1\kappa(\lambda+\mu_2)(\lambda+\mu_1\tilde{\varepsilon}_1)\lambda^2 \\ &\quad + \kappa b(\lambda+\mu_1\tilde{\varepsilon}_1)(\lambda+\mu_2\tilde{\varepsilon}_2)\xi^4 = 0. \end{aligned} \tag{3.3}$$

Let us study the asymptotic expansion of $\lambda = \lambda(\xi)$ for $|\xi| \to 0$ and for $|\xi| \to \infty$, once these expansions essentially determine the asymptotic behavior of solutions. We first consider the following asymptotic expansion for $|\xi| \to 0$:

$$\lambda_j(\xi) = \sum_{\ell=0}^{\infty} \lambda_{j,k}\xi^k, \qquad j = 1, \dots, 6. \tag{3.4}$$

Substituting (3.4) in (3.3), we compare the terms of the same order in $\xi$. Doing so, we obtain

$$\begin{aligned} &\lambda_j(\xi) = \omega_j + O(|\xi|), \qquad \lambda_4(\xi) = -\mu_2 + O(|\xi|), \\ &\lambda_k(\xi) = \pm\sqrt{\frac{b\tilde{\varepsilon}_2}{\rho_1}}\, i\xi^2 - \left(\frac{b\varepsilon_2}{2\rho_1\mu_2} \pm \frac{\rho_1 b\tilde{\varepsilon}_2 + \rho_2\kappa\tilde{\varepsilon}_1}{2\rho_1\kappa\tilde{\varepsilon}_1}\sqrt{\frac{b\tilde{\varepsilon}_2}{\rho_1}}\, i\right)\xi^4 + O(|\xi|^5) \end{aligned} \tag{3.5}$$

for $j = 1, 2, 3$ and $k = 5, 6$. Here, $\omega_j$ is a solution for $f(\omega) = 0$ with

$$f(\omega) := \omega^3 + \mu_1\omega^2 + \frac{\kappa}{\rho_2}\omega + \frac{\kappa\mu_1\tilde{\varepsilon}_1}{\rho_2}. \tag{3.6}$$

Remark that these solutions satisfy $\omega_1 + \omega_2 + \omega_3 = -\mu_1$. Since $f(0) = \kappa\mu_1\tilde{\varepsilon}_1/\rho_2 > 0$ and $f(-\mu_1) = -\kappa\mu_1\varepsilon_1/\rho_2 < 0$, we get $\mathrm{Re}(\omega_j) < 0$ for $j = 1, 2, 3$.

Analogously, we consider the asymptotic expansion for $|\xi| \to \infty$. For this purpose, we introduce $v$ by $\lambda = \xi v$, and we get from (3.3) the next identity

$$\begin{aligned}(3.7)\qquad &\rho_1\rho_2(v+\mu_1\xi^{-1})(v+\mu_2\xi^{-1})v^4+\rho_1\kappa(v+\mu_2\xi^{-1})(v+\mu_1\tilde{\varepsilon}_1\xi^{-1})\xi^{-2}v^2\\ &+\{\rho_1 b(v+\mu_1\xi^{-1})(v+\mu_2\tilde{\varepsilon}_2\xi^{-1})+\rho_2\kappa(v+\mu_2\xi^{-1})(v+\mu_1\tilde{\varepsilon}_1\xi^{-1})\}v^2\\ &+\kappa b(v+\mu_1\tilde{\varepsilon}_1\xi^{-1})(v+\mu_2\tilde{\varepsilon}_2\xi^{-1})=0.\end{aligned}$$

Then we make the ansatz

$$(3.8)\qquad v_j(\xi)=\sum_{k=0}^{\infty}v_{j,k}\xi^{-k},\qquad j=1,\dots,6,$$

and replace this expression in (3.7). Thus, we obtain

$$(3.9)\qquad \lambda_1(\xi)=-\mu_1\tilde{\varepsilon}_1+O(|\xi|^{-1}),\qquad \lambda_2(\xi)=-\mu_2\tilde{\varepsilon}_2+O(|\xi|^{-1}),$$

and

$$(3.10)\qquad \begin{aligned}\lambda_j(\xi)&=\pm\sqrt{\frac{\kappa}{\rho_1}}i\xi-\frac{\mu_1\varepsilon_1}{2}+O(|\xi|^{-1}),\\ \lambda_{j+2}(\xi)&=\pm\sqrt{\frac{b}{\rho_2}}i\xi-\frac{\mu_2\varepsilon_2}{2}+O(|\xi|^{-1})\end{aligned}$$

for $j=3,4$ if $\rho_1 b\neq\rho_2\kappa$,

$$(3.11)\qquad \begin{aligned}\lambda_j(\xi)&=\sqrt{\frac{\kappa}{\rho_1}}i\xi-\frac{1}{4}\left(\mu_1\varepsilon_1+\mu_2\varepsilon_2\pm\sqrt{(\mu_1\varepsilon_1-\mu_2\varepsilon_2)^2-\frac{4\kappa}{\rho_2}}\right)+O(|\xi|^{-1}),\\ \lambda_{j+2}(\xi)&=-\sqrt{\frac{\kappa}{\rho_1}}i\xi-\frac{1}{4}\left(\mu_1\varepsilon_1+\mu_2\varepsilon_2\pm\sqrt{(\mu_1\varepsilon_1-\mu_2\varepsilon_2)^2-\frac{4\kappa}{\rho_2}}\right)+O(|\xi|^{-1}),\end{aligned}$$

for $j=3,4$ if $\rho_1 b=\rho_2\kappa$.

In conclusion, the asymptotic expansions (3.5), (3.9), (3.10) and (3.11) reveal us that the pointwise estimate in Theorem 2.2 is optimal.

### 3.2. Optimality for $\kappa_0>0$ and $b_0=0$

In the case $\kappa_0>0$ and $b_0=0$, the Timoshenko system (1.1) is rewritten as (3.1) with $u=(v,w,z,y,p)^\top$ and $A^0=\mathrm{diag}(\tilde{\varepsilon}_1/\kappa\ \ \rho_1\ \ 1/b\ \ \rho_2\ \ \varepsilon_1/\kappa)$,

$$A=-\begin{pmatrix}0&\tilde{\varepsilon}_1&0&0&0\\ \tilde{\varepsilon}_1&0&0&0&\varepsilon_1\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&\varepsilon_1&0&0&0\end{pmatrix},\qquad L=\begin{pmatrix}0&0&0&-\tilde{\varepsilon}_1&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ \tilde{\varepsilon}_1&0&0&0&\varepsilon_1\\ 0&0&0&-\varepsilon_1&\varepsilon_1\mu_1/\kappa\end{pmatrix}.$$

Then, the corresponding characteristic equation is

$$\rho_1\rho_2(\lambda+\mu_1)\lambda^4 + \{\rho_1 b(\lambda+\mu_1) + \rho_2\kappa(\lambda+\mu_1\tilde{\varepsilon}_1)\}\xi^2\lambda^2 + \rho_1\kappa(\lambda+\mu_1\tilde{\varepsilon}_1)\lambda^2 + \kappa b(\lambda+\mu_1\tilde{\varepsilon}_1)\xi^4 = 0. \tag{3.12}$$

We also consider the asymptotic expansion of $\lambda = \lambda(\xi)$ for $|\xi| \to 0$ and $|\xi| \to \infty$. Now, replacing (3.4) in (3.12), we obtain for $|\xi| \to 0$

$$\begin{aligned} \lambda_j(\xi) &= \omega_j + O(|\xi|), \\ \lambda_k(\xi) &= \pm\sqrt{\frac{b}{\rho_1}}i\xi^2 \mp \frac{\rho_1 b + \rho_2\kappa\tilde{\varepsilon}_1}{2\rho_1\kappa\tilde{\varepsilon}_1}\sqrt{\frac{b}{\rho_1}}i\xi^4 \\ &\quad - \frac{1}{2\rho_1\kappa\tilde{\varepsilon}_1}\left\{\frac{b^2\varepsilon_1}{\mu_1\tilde{\varepsilon}_1} \mp \left(\rho_2 b + \frac{3(\rho_1 b + \rho_2\kappa\tilde{\varepsilon}_1)^2}{4\rho_1\kappa\tilde{\varepsilon}_1}\right)\sqrt{\frac{b}{\rho_1}}i\right\}\xi^6 + O(|\xi|^7), \end{aligned} \tag{3.13}$$

for $j = 1, 2, 3$ and $k = 4, 5$, where $\omega_j$ is a solution for $f(\omega) = 0$ with (3.6). On the other hand, employing (3.8) for $|\xi| \to \infty$, we get

$$\lambda_1(\xi) = -\mu_1\tilde{\varepsilon}_1 + O(|\xi|^{-1}), \tag{3.14}$$

and

$$\begin{aligned} \lambda_j(\xi) &= \pm\sqrt{\frac{\kappa}{\rho_1}}i\xi - \frac{\mu_1\varepsilon_1}{2} + O(|\xi|^{-1}), \\ \lambda_{j+2}(\xi) &= \pm\sqrt{\frac{b}{\rho_2}}i\xi \pm \frac{\rho_1\kappa}{2(\rho_1 b - \rho_2\kappa)}\sqrt{\frac{b}{\rho_2}}i\xi^{-1} - \frac{\rho_1^2\kappa b\mu_1\varepsilon_1}{2(\rho_1 b - \rho_2\kappa)^2}\xi^{-2} + O(|\xi|^{-3}), \end{aligned} \tag{3.15}$$

for $j = 2, 3$ if $\rho_1 b \neq \rho_2\kappa$, and

$$\begin{aligned} \lambda_j(\xi) &= \sqrt{\frac{\kappa}{\rho_1}}i\xi - \frac{1}{4}\left(\mu_1\varepsilon_1 \pm \sqrt{(\mu_1\varepsilon_1)^2 - \frac{4\kappa}{\rho_2}}\right) + O(|\xi|^{-1}), \\ \lambda_{j+2}(\xi) &= -\sqrt{\frac{\kappa}{\rho_1}}i\xi - \frac{1}{4}\left(\mu_1\varepsilon_1 \pm \sqrt{(\mu_1\varepsilon_1)^2 - \frac{4\kappa}{\rho_2}}\right) + O(|\xi|^{-1}), \end{aligned} \tag{3.16}$$

for $j = 2, 3$ if $\rho_1 b = \rho_2\kappa$. Eventually, the asymptotic expansions (3.13), (3.14), (3.15) and (3.16) tell us that the pointwise estimate in Theorem 2.3 is also optimal.

### 3.3. Optimality for $\kappa_0 = 0$ and $b_0 > 0$

In the case $\kappa_0 = 0$ and $b_0 > 0$, the Timoshenko system (1.1) is rewritten as (3.1) with $u = (v, w, z, y, q)^\top$ and $A^0 = \mathrm{diag}(1/\kappa \;\; \rho_1 \;\; \tilde{\varepsilon}_2/b \;\; \rho_2 \;\; \varepsilon_2/b)$,

$$A = -\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\varepsilon}_2 & 0 \\ 0 & 0 & \tilde{\varepsilon}_2 & 0 & \varepsilon_2 \\ 0 & 0 & 0 & \varepsilon_2 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_2\mu_2/b \end{pmatrix}.$$

Here, the corresponding characteristic equation is given by

$$\rho_1\rho_2(\lambda+\mu_2)\lambda^4 + \{\rho_1 b(\lambda+\mu_2\tilde{\varepsilon}_2) + \rho_2\kappa(\lambda+\mu_2)\}\xi^2\lambda^2 + \rho_1\kappa(\lambda+\mu_2)\lambda^2 + \kappa b(\lambda+\mu_2\tilde{\varepsilon}_2)\xi^4 = 0.$$

Using the same arguments as in the previous subsections, we obtain the expansion for the eigenvalues. For the sake of brevity, we omit the technical details below.

In the low frequency region ($|\xi| \to 0$), we have

$$\lambda_1(\xi) = -\mu_2 + O(|\xi|), \tag{3.17}$$

$$\lambda_j(\xi) = \pm\sqrt{\frac{\kappa}{\rho_2}}i - \frac{1}{2}\left\{\frac{b\mu_2\varepsilon_2}{\kappa+\rho_2\mu_2^2} \mp \frac{1}{\kappa}\left(\frac{\rho_1 b + \rho_2\kappa}{\rho_1} - \frac{\rho_2 b\mu_2^2\varepsilon_2}{\kappa+\rho_2\mu_2^2}\right)\sqrt{\frac{\kappa}{\rho_2}}i\right\}\xi^2 + O(|\xi|^3),$$

$$\lambda_{j+2}(\xi) = \pm\sqrt{\frac{b\tilde{\varepsilon}_2}{\rho_1}}i\xi^2 - \left(\frac{b\varepsilon_2}{2\rho_1\mu_2} \pm \frac{\rho_1 b\tilde{\varepsilon}_2 + \rho_2\kappa}{2\rho_1\kappa}\sqrt{\frac{b\tilde{\varepsilon}_2}{\rho_1}}i\right)\xi^4 + O(|\xi|^5)$$

for $j = 2, 3$. On the other hand, in the high frequency region ($|\xi| \to \infty$), we infer

$$\lambda_1(\xi) = -\mu_2\tilde{\varepsilon}_2 + O(|\xi|^{-1}), \tag{3.18}$$

and

$$\lambda_j(\xi) = \pm\sqrt{\frac{b}{\rho_2}}i\xi - \frac{\mu_2\varepsilon_2}{2} + O(|\xi|^{-1}), \tag{3.19}$$

$$\lambda_{j+2}(\xi) = \pm\sqrt{\frac{\kappa}{\rho_1}}i\xi \mp \frac{\rho_1\kappa}{2(\rho_1 b - \rho_2\kappa)}\sqrt{\frac{\kappa}{\rho_1}}i\xi^{-1} - \frac{\rho_1^2\kappa b\mu_2\varepsilon_2}{2(\rho_1 b - \rho_2\kappa)^2}\xi^{-2} + O(|\xi|^{-3}),$$

for $j = 2, 3$ if $\rho_1 b \neq \rho_2\kappa$,

$$\lambda_j(\xi) = \sqrt{\frac{\kappa}{\rho_1}} i\xi - \frac{1}{4}\left(\mu_2\varepsilon_2 \pm \sqrt{(\mu_2\varepsilon_2)^2 - \frac{4\kappa}{\rho_2}}\right) + O(|\xi|^{-1}), \tag{3.20}$$

$$\lambda_{j+2}(\xi) = -\sqrt{\frac{\kappa}{\rho_1}} i\xi - \frac{1}{4}\left(\mu_2\varepsilon_2 \pm \sqrt{(\mu_2\varepsilon_2)^2 - \frac{4\kappa}{\rho_2}}\right) + O(|\xi|^{-1}),$$

for $j = 2, 3$ if $\rho_1 b = \rho_2 \kappa$. At last, the asymptotic expansions (3.17), (3.18), (3.19) and (3.20) provides the optimality of the pointwise estimate in Theorem 2.4.

### 3.4. No dissipative structure

In the rest of this section, we consider the eigenvalues for (1.1) with $\kappa_0 = b_0 = 0$, simply to show that it has no dissipative structure under this undamped situation. Moreover, it does agree with the fact that the energy $E_0$ is conservative when $\kappa_0 = b_0 = 0$, see the energy identity (2.3).

Indeed, for this case, the initial problem (1.1) is rewritten as (3.1) with $u = (v, w, z, y)^\top$ and $A^0 = \mathrm{diag}(1/\kappa \ \ \rho_1 \ \ 1/b \ \ \rho_2)$,

$$A = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the corresponding characteristic equation is given by

$$\rho_1\rho_2\lambda^4 + (\rho_1 b + \rho_2\kappa)\xi^2\lambda^2 + \rho_1\kappa\lambda^2 + \kappa b\xi^4 = 0,$$

from where one sees that the eigenvalues satisfy

$$\lambda^2 = -\frac{1}{2\rho_1\rho_2}\{(\rho_1 b + \rho_2\kappa)\xi^2 + \rho_1\kappa \pm \sqrt{((\rho_1 b + \rho_2\kappa)\xi^2 + \rho_1\kappa)^2 - 4\rho_1\rho_2\kappa b\xi^4}\}.$$

A simple calculation shows that

$$\begin{aligned} &((\rho_1 b + \rho_2\kappa)\xi^2 + \rho_1\kappa)^2 - 4\rho_1\rho_2\kappa b\xi^4 \\ &\qquad = (\rho_1 b - \rho_2\kappa)^2\xi^4 + \rho_1^2\kappa^2 + 2\rho_1\kappa(\rho_1 b + \rho_2\kappa)\xi^2 > 0, \end{aligned}$$

from where we obtain $\lambda^2 \le 0$. Namely, we get $\lambda \in i\boldsymbol{R}$ for any $\xi \in \boldsymbol{R}$, which means (1.1) with $\kappa_0 = b_0 = 0$ has no dissipative structure.

## 4. Fundamental solution

In this section, we shall construct the fundamental solution of the system (1.5)–(1.6). Let $G(t,x)$ be a $4 \times 4$ matrix-valued function. Then $G(t,x)$ is called the fundamental solution of (1.5)–(1.6) if it satisfies the following problem:

$$(4.1) \qquad \begin{aligned} &A^0 G_t + AG_x + LG + M_1 g_1 * G_x + M_2 g_2 * G_x + N g_1 * G = 0, \\ &G(0,x) = \delta(x) I, \end{aligned}$$

where $I$ is the $4 \times 4$ unit matrix and $\delta(x)$ denotes the Dirac delta function. Applying the Fourier transform in (4.1), we obtain

$$A^0 \hat{G}_t + i\xi A\hat{G} + L\hat{G} + i\xi M_1 g_1 * \hat{G} + i\xi M_2 g_2 * \hat{G} + N g_1 * \hat{G} = 0, \qquad \hat{G}(0,\xi) = I.$$

Furthermore, applying the Laplace transform in the latter, we also obtain

$$(4.2) \qquad M(\lambda,\xi)\mathscr{L}[\hat{G}(\cdot,\xi)](\lambda) = I,$$

where the matrix coefficient $M(\lambda,\xi)$ is defined by

$$M(\lambda,\xi) := \lambda I + (A^0)^{-1}(i\xi A + L + i\xi M_1 \mathscr{L}[g_1](\lambda) + i\xi M_2 \mathscr{L}[g_2](\lambda) + N\mathscr{L}[g_1](\lambda)).$$

Then, we formally obtain $\hat{G}(t,\xi) = \mathscr{L}^{-1}[M(\cdot,\xi)^{-1}](t)$. Once this Laplace inverse transform exists, we get the fundamental solution $G(t,x)$ described as

$$G(t,x) = \mathscr{F}^{-1}[\hat{G}(t,\cdot)](x).$$

The next lemma guarantees that $\hat{G}(t,\xi)$ is well-defined.

**Lemma 4.1.** *For each $\xi \in \boldsymbol{R}$, the inverse matrix $M(\lambda,\xi)^{-1}$ exists as an analytic function of $\lambda$ in $\{\lambda \in \boldsymbol{C};\ \mathrm{Re}\,\lambda > 0\}$. Consequently, $\hat{G}(t,\xi)$ is well-defined and is given by the formula*

$$(4.3) \qquad \hat{G}(t,\xi) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} M(\lambda,\xi)^{-1}\, d\lambda,$$

*where $\gamma$ is a fixed positive number.*

*Proof.* For the case $\kappa_0 = 0$ and $b_0 > 0$, the proof of Lemma 4.1 is already knonwn by Liu-Kawashima [9]. Therefore, in the present proof we assume $\kappa_0 > 0$ and $b_0 \geq 0$.

Let us fix $\xi \in \boldsymbol{R}$. Let $\lambda = \gamma + i\nu$ and assume that $\gamma > -c_0$, where $c_0 := \min\{c_1, c_2\}$, and $c_1$ and $c_2$ are the positive constants defined in Assumption 1.1. Then, the Laplace transforms $\mathscr{L}[g_1](\lambda)$ and $\mathscr{L}[g_2](\lambda)$ are well-defined and give

$$\mathscr{L}[g_j](\lambda) = \int_0^\infty e^{-\gamma t} g_j(t) \cos(\nu t) dt - i \int_0^\infty e^{-\gamma t} g_j(t) \sin(\nu t) dt =: g_{j1}(\lambda) - i g_{j2}(\lambda),$$

for $j = 1, 2$. Since (1.3), we obtain

$$h_{11}(\lambda) := 1 - \kappa_0 g_{11}(\lambda) > 0, \qquad h_{21}(\lambda) := 1 - b_0 g_{21}(\lambda) > 0, \tag{4.4}$$

for $\gamma \geq 0$. On the other hand, we have

$$h_{j2}(\lambda) := \frac{g_{j2}(\lambda)}{\nu} = \int_0^\infty e^{-\gamma t} g_j(t) \frac{\sin(|\nu| t)}{|\nu|}\, dt > 0, \tag{4.5}$$

for $\gamma \geq 0$, $\nu \neq 0$, and $j = 1, 2$. The proof of (4.5) is similar to that presented in [8, 9].

By a straightforward computation, we have

$$\begin{aligned} \varDelta(\lambda, \xi) &:= \det M(\lambda, \xi) \\ &= \left\{ \lambda^2 + \frac{\kappa}{\rho_1}(1 - \kappa_0 \mathscr{L}[g_1](\lambda))\xi^2 \right\} \left\{ \lambda^2 + \frac{b}{\rho_2}(1 - b_0 \mathscr{L}[g_2](\lambda))\xi^2 \right\} \\ &\quad + \frac{\kappa}{\rho_2} \lambda^2 (1 - \kappa_0 \mathscr{L}[g_1](\lambda)), \end{aligned}$$

and this furnishes

$$\begin{aligned} \operatorname{Re} \varDelta(\lambda, \xi) &= \left( \gamma^2 - \nu^2 + \frac{\kappa}{\rho_1} h_{11}(\lambda) \xi^2 \right) \left( \gamma^2 - \nu^2 + \frac{b}{\rho_2} h_{21}(\lambda) \xi^2 \right) \\ &\quad + \frac{\kappa}{\rho_2} (\gamma^2 - \nu^2) h_{11}(\lambda) \\ &\quad - \left( 2\gamma\nu + \frac{\kappa\kappa_0}{\rho_1} g_{12}(\lambda) \xi^2 \right) \left( 2\gamma\nu + \frac{b b_0}{\rho_2} g_{22}(\lambda) \xi^2 \right) - \frac{2\kappa\kappa_0}{\rho_2} \gamma\nu g_{12}(\lambda), \end{aligned} \tag{4.6}$$

$$\begin{aligned} \operatorname{Im} \varDelta(\lambda, \xi) &= \left( \gamma^2 - \nu^2 + \frac{\kappa}{\rho_1} h_{11}(\lambda) \xi^2 \right) \left( 2\gamma\nu + \frac{b b_0}{\rho_2} g_{22}(\lambda) \xi^2 \right) \\ &\quad + \frac{\kappa\kappa_0}{\rho_2} (\gamma^2 - \nu^2) g_{12}(\lambda) \\ &\quad + \left( \gamma^2 - \nu^2 + \frac{b}{\rho_2} h_{21}(\lambda) \xi^2 \right) \left( 2\gamma\nu + \frac{\kappa\kappa_0}{\rho_1} g_{12}(\lambda) \xi^2 \right) + \frac{2\kappa}{\rho_2} \gamma\nu h_{11}(\lambda). \end{aligned} \tag{4.7}$$

To prove the analyticity for $M(\lambda, \xi)^{-1}$, we will study roots of

$$\varDelta(\lambda, \xi) = 0. \tag{4.8}$$

We suppose that $\lambda = \gamma + i\nu$ with $\gamma > -c_0$ is a root of $\Delta(\lambda, \xi) = 0$. Then we consider the two situations as follows.

Firstly, we consider in the case $\xi \neq 0$. Then, if $\nu = 0$, we have $\lambda = \gamma$ and (4.6) gives

$$\mathrm{Re}\,\Delta(\lambda,\xi) = \left(\gamma^2 + \frac{\kappa}{\rho_1} h_{11}(\lambda)\xi^2\right)\left(\gamma^2 + \frac{b}{\rho_2} h_{21}(\lambda)\xi^2\right) + \frac{\kappa}{\rho_2}\gamma^2 h_{11}(\lambda) > 0,$$

for $\gamma \geq 0$, which estimate comes from (4.4). Namely, there is no root for (4.8) with $\lambda = \gamma \geq 0$. On the other hand, if $\nu \neq 0$, we assume $\mathrm{Im}\,\Delta(\lambda, \xi) = 0$. Then (4.6) and (4.7) lead to

$$\begin{aligned}
&\left(2\gamma + \frac{\kappa\kappa_0}{\rho_1} h_{12}(\lambda)\xi^2\right) \mathrm{Re}\,\Delta(\lambda,\xi) \\
&\quad = -\left\{\left(\gamma^2 - \nu^2 + \frac{\kappa}{\rho_1} h_{11}(\lambda)\xi^2\right)^2 + \nu^2\left(2\gamma + \frac{\kappa\kappa_0}{\rho_1} h_{12}(\lambda)\xi^2\right)^2\right\} \\
&\qquad \times \left(2\gamma + \frac{bb_0}{\rho_2} h_{22}(\lambda)\xi^2\right) - \frac{\kappa\kappa_0}{\rho_2}(\gamma^2 + \nu^2)^2 h_{12}(\lambda) \\
&\qquad - \frac{2\kappa^2}{\rho_1\rho_2}(h_{11}(\lambda)^2 + \kappa_0^2\nu^2 h_{12}^2(\lambda))\gamma\xi^2.
\end{aligned}$$

Thus, we obtain $\mathrm{Re}\,\Delta(\lambda, \xi) < 0$ for $\gamma \geq 0$. Namely, there is no root for (4.8) with $\lambda = \gamma + i\nu$, $\gamma \geq 0$ and $\nu \neq 0$.

Secondly, we consider in the case $\xi = 0$. Then (4.8) yields

$$\lambda^2\left(\lambda^2 + \frac{\kappa}{\rho_2}(h_{11}(\lambda) + i\kappa_0 g_{12}(\lambda))\right) = 0,$$

from where $\lambda = 0$ a root, and the other roots satisfy $\tilde{\Delta}(\lambda) = 0$, where

$$\tilde{\Delta}(\lambda) := \lambda^2 + \frac{\kappa}{\rho_2}(h_{11}(\lambda) + i\kappa_0 g_{12}(\lambda)).$$

Here, we have

$$\mathrm{Re}\,\tilde{\Delta}(\lambda) = \gamma^2 - \nu^2 + \frac{\kappa}{\rho_2} h_{11}(\lambda), \qquad \mathrm{Im}\,\tilde{\Delta}(\lambda) = 2\gamma\nu + \frac{\kappa\kappa_0}{\rho_2} g_{12}(\lambda).$$

If $\nu = 0$, we have $\mathrm{Re}\,\tilde{\Delta}(\lambda) > 0$ for $\gamma \geq 0$. On the other hand, if $\nu \neq 0$, we obtain $\mathrm{Im}\,\tilde{\Delta}(\lambda) > 0$ for $\gamma \geq 0$. Namely, there is no root for (4.8) with $\lambda = \gamma + i\nu$ and $\gamma > 0$.

We conclude that $M(\lambda, \xi)^{-1}$ is analytic in $\{\lambda \in \boldsymbol{C};\ \mathrm{Re}\,\lambda > 0\}$ for $\xi = 0$ and in $\{\lambda \in \boldsymbol{C};\ \mathrm{Re}\,\lambda \geq 0\}$ for $\xi \neq 0$. Therefore we obtain $\mathscr{L}[\hat{G}(\cdot, \xi)](\lambda) = M(\lambda, \xi)^{-1}$ for $\mathrm{Re}\,\lambda > 0$.

Consequently, for any fixed $\gamma > 0$, we can formally express its Laplace inverse transform as

$$(4.9)\qquad \hat{G}(t,\xi) = \mathscr{L}^{-1}[M(\cdot,\xi)^{-1}](t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}M(\lambda,\xi)^{-1}d\lambda$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t}M(\lambda,\xi)^{-1}dv,$$

where $\lambda = \gamma + iv$ in the last equation. In the rest of this proof, we show the convergence for the last integral in (4.9). To this end, because $e^{\lambda t}M(\lambda,\xi)^{-1}$ is integrable over $|v| \le R$ for any $R > 0$, it is enough to prove that $\int_{|v|\ge R} e^{\lambda t}M(\lambda,\xi)^{-1}dv$ converges. Since

$$\int_{|v|\ge R} e^{\lambda t}M(\lambda,\xi)^{-1}dv$$
$$= \int_{|v|\ge R} e^{\lambda t}\lambda^{-1}I\,dv + \int_{|v|\ge R} e^{\lambda t}\lambda^{-1}M(\lambda,\xi)^{-1}(\lambda I - M(\lambda,\xi))dv,$$

and

$$|\mathscr{L}[g_1](\lambda)| \le \int_0^\infty |g_1(t)|dt < \frac{1}{\kappa_0}, \qquad |\mathscr{L}[g_2](\lambda)| \le \int_0^\infty |g_2(t)|dt < \frac{1}{b_0},$$

it is not hard to prove $\int_{|v|\ge R} e^{\lambda t}M(\lambda,\xi)^{-1}dv$ converges. Therefore, we can conclude that the last integral in (4.9) converges. Hence, the existence of $\hat{G}(t,\xi)$ is proved as well as it is given by the formula (4.3). This completes the proof of Lemma 4.1. □

### 4.1. Pointwise estimates in the Fourier space

Using the fundamental solution $G(t,x)$ of (1.5)–(1.6), the solution of problem (2.1) is given by

$$\hat{u}(t,\xi) = \hat{G}(t,\xi)\hat{u}_0(\xi),$$

where $\hat{G}(t,\xi)$ is set in (4.3). Therefore, as a prompt consequence of Theorems 2.2, 2.3, and 2.4, we have the following pointwise estimates for $\hat{G}(t,\xi)$ for all cases with respect to $\kappa_0$ and $b_0$.

**Corollary 4.2.** *Let us consider $\kappa_0 > 0$ and $b_0 > 0$, and let $\hat{G}$ be given by* (4.3). *Then, $\hat{G}$ satisfies the pointwise estimate in the Fourier space*: $|\hat{G}(t,\xi)| \le Ce^{-c\rho(\xi)t}$ *with $\rho(\xi)$ set in* (2.4).

**Corollary 4.3.** *Let us consider $\kappa_0 > 0$ and $b_0 = 0$, and let $\hat{G}$ be given by* (4.3). *Then, $\hat{G}$ satisfies the pointwise estimate in the Fourier space*: $|\hat{G}(t,\xi)| \leq Ce^{-c\eta(\xi)t}$ *with $\eta(\xi)$ set in* (2.24).

**Corollary 4.4.** *Let us consider $\kappa_0 = 0$ and $b_0 > 0$, and let $\hat{G}$ be given by* (4.3). *Then, $\hat{G}$ satisfies the pointwise estimate in the Fourier space*: $|\hat{G}(t,\xi)| \leq Ce^{-c\zeta(\xi)t}$ *with $\zeta(\xi)$ set in* (2.42).

### 4.2. $L^2$-estimates via the fundamental solution

By means of the fundamental solution, we know that the solution of problem (1.5)–(1.6) is given by the formula

$$u(t,x) = (G(t,\cdot) * u_0)(x), \tag{4.10}$$

where $*$ denotes the standard convolution with respect to $x \in \boldsymbol{R}$.

Therefore, through Corollaries 4.2, 4.3, and 4.4, we are able to express the $L^2$-estimates for the solution operator $G(t)*$ (and its derivatives) set by the solution formula (4.10). More precisely, we have:

**Proposition 4.5.** *Let us consider $\kappa_0 > 0$ and $b_0 > 0$. Let us also take $k \geq 0$ and $1 \leq p \leq 2$. Then the solution* (4.10) *satisfies the following decay estimates*:

$$\|\partial_x^k G(t) * u_0\|_{L^2} \leq C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|u_0\|_{L^p} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2} \tag{4.11}$$

*for some constants $C, c > 0$.*

**Proposition 4.6.** *Let us consider $\kappa_0 > 0$ and $b_0 = 0$, and also take $k \geq 0$, $\ell \geq 0$, $1 \leq p \leq 2$. Then the solution* (4.10) *satisfies the following decay estimates*:

$$\|\partial_x^k G(t) * u_0\|_{L^2} \leq C(1+t)^{-\frac{1}{6}(\frac{1}{p}-\frac{1}{2})-\frac{k}{6}}\|u_0\|_{L^p} + C(1+t)^{-\frac{\ell}{2}}\|\partial_x^{k+\ell} u_0\|_{L^2} \quad \text{if } \frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2} \tag{4.12}$$

*and*

$$\|\partial_x^k G(t) * u_0\|_{L^2} \leq C(1+t)^{-\frac{1}{6}(\frac{1}{p}-\frac{1}{2})-\frac{k}{6}}\|u_0\|_{L^p} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2} \qquad \text{if } \frac{\kappa}{\rho_1} = \frac{b}{\rho_2} \tag{4.13}$$

*for some constants $C, c > 0$.*

**Proposition 4.7.** *Let us consider $\kappa_0 = 0$ and $b_0 > 0$, and also take $k \geq 0$, $\ell \geq 0$, $1 \leq p \leq 2$. Then the solution* (4.10) *satisfies the following decay estimates*:

$$(4.14)\qquad \|\partial_x^k G(t) * u_0\|_{L^2} \le C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|u_0\|_{L^p} + C(1+t)^{-\frac{\ell}{2}}\|\partial_x^{k+\ell} u_0\|_{L^2} \quad \text{if } \frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2}$$

*and*

$$(4.15)\qquad \|\partial_x^k G(t) * u_0\|_{L^2} \le C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|u_0\|_{L^p} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2} \qquad \text{if } \frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$$

*for some constants* $C, c > 0$.

*Remark* 1. In the case of different wave speeds, one sees from the decay estimates (4.12) and (4.14) that the decay estimates are of the regularity-loss type because the decay rate $(1+t)^{-\ell/2}$ is only achieved by assuming the additional $\ell$th-order regularity for the initial data. On the other hand, in the case of equal wave speeds, the decay estimates (4.13) and (4.15) are not of regularity-loss type, by having similar features to the decay rate (4.11).

The proofs of Propositions 4.5, 4.6 and 4.7 are derived by the standard argument based on Theorems 2.2, 2.3 and 2.4, respectively. Indeed, to derive such desired results we apply the Plancherel theorem and then analyze the low and high frequency region. We omit the detailed proofs here since they will be encompassed by the proofs of Theorems 5.4, 5.5, and 5.6 in the subsequent Section 5. See also [9, 16] for similar approach to other models.

## 5. Main results on stability

In this section, we show the energy estimate and the decay estimate of solutions to the problem (1.5)–(1.6). The key argument is the energy method in the Fourier space, which was implicitly deduced in the proofs presented in Section 2.

### 5.1. Energy estimates

**Theorem 5.1.** *Let us suppose that* $\kappa_0 > 0$ *and* $b_0 > 0$, *and also take on* $u_0 \in H^s$ *for* $s \ge 0$. *Then the solution* $u$ *of problem* (1.5)–(1.6), *which is given by the formula* (4.10), *belongs to the class* $u \in C^0([0,\infty); H^s)$ *and satisfies the energy estimate*:

$$(5.1)\qquad \|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x^2 u(\tau)\|_{H^{s-2}}^2 d\tau \le C\|u_0\|_{H^s}^2$$

*for some constant* $C > 0$.

**Theorem 5.2.** *Let us suppose that* $\kappa_0 > 0$ *and* $b_0 = 0$, *and also take on* $u_0 \in H^s$ *for* $s \geq 0$. *Then the solution* $u$ *of problem* (1.5)–(1.6), *which is given by the formula* (4.10), *lies in the class* $u \in C^0([0, \infty); H^s)$ *and satisfies the energy estimates*:

$$\|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x^3 u(\tau)\|_{H^{s-4}}^2 d\tau \leq C\|u_0\|_{H^s}^2 \qquad \text{if } \frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2} \tag{5.2}$$

*and*

$$\|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x^3 u(\tau)\|_{H^{s-3}}^2 d\tau \leq C\|u_0\|_{H^s}^2 \qquad \text{if } \frac{\kappa}{\rho_1} = \frac{b}{\rho_2} \tag{5.3}$$

*for some constant* $C > 0$.

**Theorem 5.3.** *Let us suppose* $\kappa_0 = 0$ *and* $b_0 > 0$, *and also take on* $u_0 \in H^s$ *for* $s \geq 0$. *Then the solution* $u$ *of problem* (1.5)–(1.6), *which is given by the formula* (4.10), *is in the class* $u \in C^0([0, \infty); H^s)$ *and satisfies the energy estimates*:

$$\|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x^2 u(\tau)\|_{H^{s-3}}^2 d\tau \leq C\|u_0\|_{H^s}^2 \qquad \text{if } \frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2} \tag{5.4}$$

*and*

$$\|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x^2 u(\tau)\|_{H^{s-2}}^2 d\tau \leq C\|u_0\|_{H^s}^2 \qquad \text{if } \frac{\kappa}{\rho_1} = \frac{b}{\rho_2} \tag{5.5}$$

*for some constant* $C > 0$.

*Proof of Theorems* 5.1, 5.2, *and* 5.3. In the previous section, we have just proved that the solution $u$ is given by the formula (4.10) for all cases. Furthermore, in Section 2, we have shown that this solution satisfies, in each specific case, the energy estimates (2.22), (2.34) and (2.40), (2.49) and (2.55) in the Fourier space. Therefore, the completion of the proof is done as follows.

*Case* $\kappa_0 > 0$ *and* $b_0 > 0$. Multiplying (2.22) by $(\xi^2 + 1)^s$ and integrating the resultant inequality with respect to $t$ and $\xi$, we have

$$\|u(t)\|_{H^s}^2 + \int_0^t (\|\partial_x(v, y)(\tau)\|_{H^{s-1}}^2 + \|\partial_x^2(w, z)(\tau)\|_{H^{s-2}}^2) d\tau \leq C\|u_0\|_{H^s}^2,$$

which gives (5.1).

*Case* $\kappa_0 > 0$ *and* $b_0 = 0$. Here, we apply the same argument to (2.34) and (2.40). Then we arrive at

$$\|u(t)\|_{H^s}^2 + \int_0^t (\|\partial_x^2 v(\tau)\|_{H^{s-2}}^2 + \|\partial_x^3 w(\tau)\|_{H^{s-3}}^2 + \|\partial_x^3 z(\tau)\|_{H^{s-4}}^2 + \|\partial_x^2 y(\tau)\|_{H^{s-3}}^2) d\tau$$
$$\leq C\|u_0\|_{H^s}^2$$

for $\rho_1 b \neq \rho_2 \kappa$, and

$$\|u(t)\|_{H^s}^2 + \int_0^t (\|\partial_x^2 (v, y)(\tau)\|_{H^{s-2}}^2 + \|\partial_x^3 (w, z)(\tau)\|_{H^{s-3}}^2) d\tau \leq C\|u_0\|_{H^s}^2$$

for $\rho_1 b = \rho_2 \kappa$. These estimates imply that (5.2) and (5.3) hold true.

*Case $\kappa_0 = 0$ and $b_0 > 0$.* Now, we note that the estimates (2.49) and (2.55) lead to

$$\|u(t)\|_{H^s}^2 + \int_0^t (\|\partial_x v(\tau)\|_{H^{s-2}}^2 + \|\partial_x^2 w(\tau)\|_{H^{s-3}}^2 + \|\partial_x^2 z(\tau)\|_{H^{s-2}}^2 + \|\partial_x y(\tau)\|_{H^{s-1}}^2) d\tau$$
$$\leq C\|u_0\|_{H^s}^2$$

for $\rho_1 b \neq \rho_2 \kappa$, and

$$\|u(t)\|_{H^s}^2 + \int_0^t (\|\partial_x (v, y)(\tau)\|_{H^{s-1}}^2 + \|\partial_x^2 (w, z)(\tau)\|_{H^{s-2}}^2) d\tau \leq C\|u_0\|_{H^s}^2$$

for $\rho_1 b = \rho_2 \kappa$. These estimates also provide (5.4) and (5.5). □

### 5.2. Decay rate estimates

**Theorem 5.4.** *Under the same assumptions of Theorem* 5.1, *let us additionally consider $u_0 \in L^p$ for $1 \leq p \leq 2$. Then the solution satisfies the following decay estimate*:

$$\|\partial_x^k u(t)\|_{H^{s-k}} \leq C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}(\|u_0\|_{L^p} + \|u_0\|_{H^s}), \qquad 0 \leq k \leq s$$

*for some constant $C > 0$.*

**Theorem 5.5.** *Under the same assumptions of Theorem* 5.2, *let us additionally consider $u_0 \in L^p$ for $1 \leq p \leq 2$ and $s \geq (1/p - 1/2)/3$, and set the number*

$$q_{s,p} := \frac{1}{4}\left\{3s - \left(\frac{1}{p} - \frac{1}{2}\right)\right\} \geq 0.$$

*Then the solution satisfies the decay estimates*:

$$\left.\begin{aligned} \|\partial_x^k u(t)\|_{H^{s-(k+\ell)}} &\le C(1+t)^{-\frac{1}{6}(\frac{1}{p}-\frac{1}{2})-\frac{k}{6}}(\|u_0\|_{L^p}+\|u_0\|_{H^s}), \quad 0 \le k \le q_{s,p} \\ \|\partial_x^k u(t)\|_{L^2} &\le C(1+t)^{-\frac{1}{2}(s-k)}(\|u_0\|_{L^p}+\|u_0\|_{H^s}), \qquad q_{s,p} \le k \le s \end{aligned}\right\}$$

$$\text{if } \frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2},$$

*where* $\ell = (1/p - 1/2)/3 + k/3$; *and*

$$\|\partial_x^k u(t)\|_{H^{s-k}} \le C(1+t)^{-\frac{1}{6}(\frac{1}{p}-\frac{1}{2})-\frac{k}{6}}(\|u_0\|_{L^p}+\|u_0\|_{H^s}), \qquad 0 \le k \le s \text{ if } \frac{\kappa}{\rho_1} = \frac{b}{\rho_2},$$

*for some constant* $C > 0$ *in both cases.*

**Theorem 5.6.** *Under the same assumptions of Theorem* 5.3, *let us additionally consider* $u_0 \in L^p$ *for* $1 \le p \le 2$ *and* $s \ge (1/p - 1/2)/2$, *and set the number*

$$r_{s,p} := \frac{1}{3}\left\{2s - \left(\frac{1}{p} - \frac{1}{2}\right)\right\} \ge 0.$$

*Then the solution satisfies the decay estimates*:

$$\left.\begin{aligned} \|\partial_x^k u(t)\|_{H^{s-(k+\ell)}} &\le C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}(\|u_0\|_{L^p}+\|u_0\|_{H^s}), \quad 0 \le k \le r_{s,p} \\ \|\partial_x^k u(t)\|_{L^2} &\le C(1+t)^{-\frac{1}{2}(s-k)}(\|u_0\|_{L^p}+\|u_0\|_{H^s}), \qquad r_{s,p} \le k \le s \end{aligned}\right\}$$

$$\text{if } \frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2},$$

*where* $\ell = (1/p - 1/2)/2 + k/2$; *and*

$$\|\partial_x^k u(t)\|_{H^{s-k}} \le C(1+t)^{-\frac{1}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}(\|u_0\|_{L^p}+\|u_0\|_{H^s}), \qquad 0 \le k \le s, \text{ if } \frac{\kappa}{\rho_1} = \frac{b}{\rho_2},$$

*for some constant* $C > 0$ *in both cases.*

*Proof of Theorems* 5.4, 5.5, *and* 5.6. Since the proof of Theorem 5.4 is the same (not to say simpler) as the other proofs, we only detail the proof of Theorems 5.5 and 5.6. Furthermore, because of the similarity of the arguments (again not to say simpler) in case $\kappa/\rho_1 = b/\rho_2$, we only give the proof in case $\kappa/\rho_1 \neq b/\rho_2$.

Firstly, we prove Theorem 5.5 with $\kappa/\rho_1 \neq b/\rho_2$. Since $\eta(\xi) \ge c\xi^6$ for $|\xi| \le 1$ and $\eta(\xi) \ge c\xi^{-2}$ for $|\xi| \ge 1$, we have

$$
\begin{aligned}
\|\partial_x^k u(t)\|_{H^{s-(k+\ell)}}^2 &= \int_{\boldsymbol{R}} (\xi^2+1)^{s-(k+\ell)} \xi^{2k} |\hat{u}(t,\xi)|^2 d\xi \\
&\le C \int_{\boldsymbol{R}} (\xi^2+1)^{s-(k+\ell)} \xi^{2k} e^{-c\eta(\xi)t} |\hat{u}_0(\xi)|^2 d\xi \\
&\le C \int_{|\xi|\le 1} (\xi^2+1)^{s-(k+\ell)} \xi^{2k} e^{-c\xi^6 t} |\hat{u}_0(\xi)|^2 d\xi \\
&\quad + C \int_{|\xi|\ge 1} (\xi^2+1)^{s-(k+\ell)} \xi^{2k} e^{-c\xi^{-2} t} |\hat{u}_0(\xi)|^2 d\xi \\
&:= J_L + J_H.
\end{aligned}
$$

In the low frequency region, employing the Hölder inequality, we estimate

$$
\begin{aligned}
J_L &\le C \int_{|\xi|\le 1} \xi^{2k} e^{-c\xi^6 t} |\hat{u}_0(\xi)|^2 d\xi \le C \|\xi^{2k} e^{-c\xi^6 t}\|_{L^r(|\xi|\le 1)} \|u_0\|_{L^{p'}}^2 \\
&\le C(1+t)^{-\frac{1}{3}(\frac{1}{p}-\frac{1}{2})-\frac{1}{3}k} \|u_0\|_{L^p}^2
\end{aligned}
$$

for $k \ge 0$, where $p'$ and $r$ are satisfied $1/p + 1/p' = 1$ and $1/r + 2/p' = 1$ for $1 \le p \le 2$. On the other hand, in the high frequency region, we also estimate

$$
J_H \le C \sup_{|\xi|\ge 1} (\xi^{-2\ell} e^{-c\xi^{-2} t}) \int_{|\xi|\ge 1} (\xi^2+1)^s |\hat{u}_0(\xi)|^2 d\xi \le C(1+t)^{-\ell} \|u_0\|_{H^s}^2
$$

for $k \ge 0$ and $\ell \ge 0$. Therefore, combining these estimates, we obtain

$$
\|\partial_x^k u(t)\|_{H^{s-(k+\ell)}}^2 \le C(1+t)^{-\frac{1}{3}(\frac{1}{p}-\frac{1}{2})-\frac{1}{3}k} \|u_0\|_{L^p}^2 + C(1+t)^{-\ell} \|u_0\|_{H^s}^2. \tag{5.6}
$$

For

$$
0 \le k \le \frac{1}{4}\left\{3s - \left(\frac{1}{p} - \frac{1}{2}\right)\right\},
$$

substituting $\ell = (1/p - 1/2)/3 + k/3$ in (5.6), we arrive at

$$
\|\partial_x^k u(t)\|_{H^{s-(k+\ell)}}^2 \le C(1+t)^{-\frac{1}{3}(\frac{1}{p}-\frac{1}{2})-\frac{1}{3}k} (\|u_0\|_{L^p}^2 + \|u_0\|_{H^s}^2).
$$

On the other hand, for

$$
\frac{1}{4}\left\{3s - \left(\frac{1}{p} - \frac{1}{2}\right)\right\} \le k \le s,
$$

substituting $\ell = s - k$ in (5.6), we get

$$
\|\partial_x^k u(t)\|_{L^2}^2 \le C(1+t)^{-(s-k)} (\|u_0\|_{L^p}^2 + \|u_0\|_{H^s}^2).
$$

These estimates lead to the desired estimates in Theorem 5.5.

Secondly, we prove Theorem 5.6 with $\kappa/\rho_1 \neq b/\rho_2$. Because of $\eta(\xi) \geq c\xi^4$ for $|\xi| \leq 1$ and $\eta(\xi) \geq c\xi^{-2}$ for $|\xi| \geq 1$, we also have

$$\|\partial_x^k u(t)\|^2_{H^{s-(k+\ell)}} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}k}\|u_0\|^2_{L^p} + C(1+t)^{-\ell}\|u_0\|^2_{H^s}. \tag{5.7}$$

For

$$0 \leq k \leq \frac{1}{3}\left\{2s - \left(\frac{1}{p} - \frac{1}{2}\right)\right\},$$

replacing $\ell = (1/p - 1/2)/2 + k/2$ in (5.7), we arrive at

$$\|\partial_x^k u(t)\|^2_{H^{s-(k+\ell)}} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}k}(\|u_0\|^2_{L^p} + \|u_0\|^2_{H^s}).$$

On the other hand, for

$$\frac{1}{3}\left\{2s - \left(\frac{1}{p} - \frac{1}{2}\right)\right\} \leq k \leq s,$$

replacing $\ell = s - k$ in (5.7), we get

$$\|\partial_x^k u(t)\|^2_{L^2} \leq C(1+t)^{-(s-k)}(\|u_0\|^2_{L^p} + \|u_0\|^2_{H^s}).$$

Thus, these estimates also lead to the desired estimates in Theorem 5.6. Therefore, the proofs are over by noting that the remaining estimates follow similarly. □

*Remark* 2. We finally stress the novelties and improvements of this section as follows.

( i ) In case $\kappa_0 > 0$ and $b_0 > 0$, Theorems 5.1 and 5.4 provide, for the first time, new decay rate estimates for the system (1.5)–(1.6), independently of any relation among the coefficients and also without regularity-loss of decaying.

(ii) In case $\kappa_0 > 0$ and $b_0 = 0$, Theorems 5.2, and 5.5, also prove for the first time new decay estimates as expressed therein. They are optimal in sense of Section 3 but in case of different wave speeds we note that the results are of regularity-loss type.

(iii) In case $\kappa_0 = 0$ and $b_0 > 0$, as already mentioned before, Liu-Kawashima [9] and Mori [11] derived the energy estimate and the decay estimate previously. The estimates achieved in Theorems 5.3 and 5.6 are optimal and revealed the relationship between the decay estimates and the physical parameters.

## A. Physical modeling and mathematical aspects

For the sake of completeness, and also to clarify the physical modeling behind the mathematical system (1.1), we bring to our context the main ideas developed recently by Alves et al. [1]. Then, as a mathematical curiosity, we employ different geometric aspects on distance in $\boldsymbol{R}$ to consider the model along the whole real line.

Our starting point is the Boltzmann theory for viscoelastic materials where the stress $\sigma$ is assumed to rely on both the instantaneous strain $\varepsilon$ and the strain history $\{\varepsilon(s);\ 0 \le s \le t\}$. Thus, according to Boltzmann [3, 4], the following stress-strain constitutive law is in place

$$\sigma(t) = E\left\{\varepsilon(t) - \int_0^t g(t-s)\varepsilon(s)ds\right\} := E\{\varepsilon(t) - (g * \varepsilon)(t)\}, \tag{A.1}$$

where the constant $E$ stands for the Young modulus of elasticity, and the function $g$ is known as relaxation measure of the material or simply *memory kernel*.

On the other hand, in what concerns a thin 3D-beam

$$[-L, L] \times \Omega = \{(x, y, z);\ x \in [-L, L] \text{ and } (y, z) \in \Omega\}$$

of length $L > 0$ and uniform cross section $\Omega \subset \boldsymbol{R}^2$ made of homogeneous isotropic viscoelastic material, the following Timoshenko Hypotheses (**H**) are assumed (cf. Prüss [12, Chapter 9] and Drozdov-Kolmanovskii [5, Chapter 5]):

($\mathbf{H_1}$) $(0, 0)$ is the center of $\Omega$ so that $\int_\Omega z\, dydz = \int_\Omega y\, dydz = 0$;

($\mathbf{H_2}$) $\operatorname{diam} \Omega \ll L$ so that the thickness of the beam is very thin when compared to length;

($\mathbf{H_3}$) the bending takes place only on the $(x, z)$-plane, that is, normal stresses in the $y$-axis are negligible in general;

($\mathbf{H_4}$) the matrix of stress tensor $\sigma = (\sigma_{ij})_{1 \le i,j \le 3}$ is considered with only two effective stresses, namely, $\sigma_{11}$ and $\sigma_{13}$, and the remaining stresses are neglected ($\sigma_{ij} \approx 0$).

Additionally, the displacements and the rotation angle in the $(x, z)$-plane are expressed by means of the following Notations (**N**):

($\mathbf{N_1}$) $u = u(t, x)$: the longitudinal displacement of points lying on the horizontal $x$-axis;

($\mathbf{N_2}$) $\psi = \psi(t, x)$: the angle of rotation for the normal to the $x$-axis;

($\mathbf{N_3}$) $w_1(t, x, z) = u(t, x) + z\psi(t, x)$: the longitudinal displacement;

($\mathbf{N_4}$) $w_2(t, x, z) = \phi(t, x)$: the vertical beam displacement.

Under the above structural conditions, we can derive the linear model (1.1) which refers to viscoelastic Timoshenko beams with a viscoelastic coupling on both the shear force and the bending moment. To clarify this statement, we proceed in some steps as designed below.

*Stress-Strain relations.* Taking into account the relevant stresses $\sigma_{11}$ and $\sigma_{13}$ in the Boltzmann context (A.1), then the stress-strain relations for viscoelastic

Timoshenko beams (cf. [5, 12]) are given by

$$\sigma_{13}(t,\cdot) = 2kG\{\varepsilon_{13}(t,\cdot) - \kappa_0(g_1 * \varepsilon_{13})(t,\cdot)\}, \tag{A.2}$$

$$\sigma_{11}(t,\cdot) = E\{\varepsilon_{11}(t,\cdot) - b_0(g_2 * \varepsilon_{11})(t,\cdot)\}, \tag{A.3}$$

where $G$ is the constant shear modulus, $k$ is a shear correction coefficient, $g := \kappa_0 g_1$ and $g := b_0 g_2$ are relaxation memory kernels with non-negative weighted coefficients $\kappa_0$ and $b_0$ that might cancel the viscoelastic effect on bending and shear deformations, and the index "$\cdot$" denotes points lying in the $(x,z)$-plane.

*Elastic strains.* Now, following again [5] (see (2.4) on page 339 therein), the standard formulas for the components of the infinitesimal elastic strain tensor can be exhibited by

$$\varepsilon_{13}(t,\cdot) := \frac{1}{2}\left(\frac{\partial w_1}{\partial z} + \frac{\partial w_2}{\partial x}\right)(t,\cdot) = \frac{1}{2}(\psi(t,x) + \phi_x(t,x)), \tag{A.4}$$

$$\varepsilon_{11}(t,\cdot) := \frac{\partial w_1}{\partial x}(t,\cdot) = u_x(t,x) + z\psi_x(t,x). \tag{A.5}$$

*Shear and Bending relations.* Going back to postulations $(\mathbf{H_1})$–$(\mathbf{H_4})$ and following once again [12] (see the identities (9.10)–(9.11) therein) the standard formulas for the bending moment and the shear force are given by

$$S(t,x) = \int_\Omega \sigma_{13}(t,\cdot)dydz \qquad \text{and} \qquad M(t,x) = \int_\Omega z\sigma_{11}(t,\cdot)dydz, \tag{A.6}$$

respectively, where we have normalized the equations in (A.6) by the area $A := \int_\Omega dydz$ and inertial moment $I := \int_\Omega z^2\, dydz$ of the cross section $\Omega$.

*Viscoelastic coupling on the shear force.* Regarding (A.2), (A.4), and the first identity in (A.6), the following (not so classical) viscoelastic law for the shear force comes up

$$S(t,x) = kGA\{(\phi_x + \psi)(t,x) - \kappa_0(g_1 * (\phi_x + \psi))(t,x)\}, \tag{A.7}$$

which has been firstly derived in [1, Section 2].

*Viscoelastic coupling on the bending moment.* Now, using (A.3), (A.5), the second identity in (A.6), and also $(\mathbf{H_1})$, one gets the (classical) viscoelastic law for the bending moment

$$M(t,x) = EI\{\psi_x(t,x) - b_0(g_2 * \psi_x)(t,x)\}, \tag{A.8}$$

which eliminates the variable $u = u(t,x)$ corresponding to the longitudinal displacement on the $x$-axis. In other words, it can be interpreted as a too small horizontal displacement ($u \approx 0$) when compared to the vertical displacement $\varphi$ and the rotation angle $\psi$ in the beam deformation.

*Motion equations for beams of Timoshenko-Ehrenfest type.* In order to derive the desired viscoelastic system (1.1), we are going to consider a well accepted model in differential equations encompassing bending and shear deformations. To do so, we follow the model for vibrations of prismatic beams developed by Timoshenko-Ehrenfest, cf. [6, 14, 15], namely,

$$\text{(A.9)} \qquad \begin{cases} \rho A\phi_{tt}(t,x) - S_x(t,x) = 0, \\ \rho I\psi_{tt}(t,x) - M_x(t,x) + S(t,x) = 0, \end{cases}$$

for $(t,x) \in (0,\infty) \times (-L,L)$, where $\rho$ represents the mass density per area unit. The remaining variables are already set above.

*The viscoelastic model on bounded intervals.* Under the above structured steps, we can deduct the viscoelastic Timoshenko system related to (1.1) but on bounded domains. Indeed, replacing (A.7)–(A.8) in (A.9), we arrive at the next viscoelastic beam system:

$$\text{(A.10)} \qquad \begin{cases} \rho A\phi_{tt} - kGA((\phi_x + \psi)_x - \kappa_0(g_1 * (\phi_x + \psi)_x)) = 0, \\ \rho I\psi_{tt} - EI(\psi_{xx} - b_0(g_2 * \psi_{xx})) \\ \qquad + kGA((\phi_x + \psi) - \kappa_0(g_1 * (\phi_x + \psi))) = 0, \end{cases}$$

for $(t,x) \in (0,\infty) \times (-L,L)$. Therefore, by using the notation

$$\text{(A.11)} \qquad \rho_1 = \rho A, \qquad \rho_2 = \rho I, \qquad \kappa = kGA, \qquad b = EI,$$

one can see that (A.10) corresponds to (1.1) but for spatial $x$-variable belonging to the bounded domain $(-L,L)$, $L > 0$. For $\kappa_0 > 0$ and $b_0 = 0$, (A.10) means the Timoshenko system with viscoelastic coupling on shear force, being treated for $x \in [0,L]$ only recently in [1]. On the other hand, when $\kappa_0 = 0$ and $b_0 > 0$, then (A.10) reduces to the classical viscoelastic Timoshenko system firstly introduced by [2], and subsequently studied by several authors up to nowadays, still in bounded intervals like $[0,L]$. Finally, for $\kappa_0 > 0$ and $b_0 > 0$, (A.10) represents a fully damped viscoelastic Timoshenko-Ehrenfest system. A slightly modified version of (A.10) was considered by Grasselli et al. [7], where the authors present the system with history, nonlinear source terms and external forces.

*The model posed on 1D-spaces: a mathematical curiosity.* Under the physical meanings aforementioned, a simple way to reach (1.1) by means of (A.10)–(A.11) it is, mathematically speaking, to take the limit $L \to \infty$ over the interval $(-L,L)$, that is, to consider the beam length $L > 0$ large enough to understand the problem over $\boldsymbol{R} = \lim_{L\to\infty}(-L, L)$. This procedure may lead to the issue of *infinite beam length*. On the other hand, according to the stereographic projection $\pi$ of $\boldsymbol{R}$ onto the unit sphere $\boldsymbol{S}^1 - \{(0,1)\} \subset \boldsymbol{R}^2$, one can see $\boldsymbol{R}$ as a subset of the extended real line $\mathscr{R} := \{+\infty\} \cup \boldsymbol{R}$ which in turn is isometric to $\boldsymbol{S}^1$ with some proper notion of distance, called *chordal metric*, leading to the notion of *finite length* in $\boldsymbol{S}^1 - \{(0,1)\}$. For all details of these statements we refer to [13, Chapter 4] (see §4.2 therein). Below we present the main ideas adapted to our case. Indeed, one knows that

$$\begin{aligned}\pi : \boldsymbol{R} &\to \boldsymbol{S}^1 - \{(0,1)\}\\ x &\mapsto \pi(x) = \left(\frac{2x}{1+x^2}, \frac{x^2-1}{x^2+1}\right)\end{aligned}$$

is a bijection with explicit inverse

$$\begin{aligned}\pi^{-1} : \boldsymbol{S}^1 - \{(0,1)\} &\to \boldsymbol{R}\\ \mathbf{x} = (x_1, x_2) &\mapsto \pi^{-1}(\mathbf{x}) = \frac{x_1}{1-x_2}.\end{aligned}$$

Thus, we can extend $\pi$ to a bijection $\Pi : \mathscr{R} \to \boldsymbol{S}^1$ by setting $\Pi(+\infty) = (0,1)$. Moreover, we can define a metric $d$ on $\mathscr{R}$ by the formula

$$d(x, y) = |\Pi(x) - \Pi(y)|.$$

According to [13, Theorem 4.2.1], we have

$$d(x, y) = \begin{cases} \dfrac{2}{(1+|x|^2)^{1/2}} & \text{if } x \in \boldsymbol{R},\ y = +\infty,\\[2ex] \dfrac{2|x-y|}{(1+|x|^2)^{1/2}(1+|y|^2)^{1/2}} & \text{if } x, y \in \boldsymbol{R}.\end{cases}$$

The metric $d$ is called the *chordal metric* on $\mathscr{R}$, and by definition one can see that the map $\Pi$ is an isometry from $(\mathscr{R}, d)$ to $\boldsymbol{S}^1$ with the Euclidean metric induced by $\boldsymbol{R}^2$, which proves the desired. Moreover, we observe that: (i) the metric space $\mathscr{R} = \Pi^{-1}(\boldsymbol{S}^1)$ is compact, being known as *one-point compactification* of $\boldsymbol{R}$; (ii) the metric topology on $\boldsymbol{R}$ induced by the chordal metric is the same as the Euclidean topology, once $\pi$ maps $\boldsymbol{R}$ homeomorphically onto the open subset $\boldsymbol{S}^1 - \{(0,1)\}$ of $\boldsymbol{S}^1$. Therefore, the metric space $\boldsymbol{R}$, seen as a subset of $\mathscr{R}$ and with the notion of chordal distance therein, is bounded. More specifically, we have

$$\boldsymbol{R} = \pi^{-1}(\boldsymbol{S}^1 - \{(0,1)\}) = \Pi^{-1}(\boldsymbol{S}^1 - \{(0,1)\}) \subset \Pi^{-1}(\boldsymbol{S}^1) = \mathscr{R}.$$

Under the above mathematical concerns, one can interpret the issue of infinite beam length as a beam given by the intrinsic measure of $\boldsymbol{S}^1 - \{(0,1)\}$ in $\boldsymbol{S}^1$, which is for sure finite. In conclusion, this fact provides mathematical aspects, just for the reader's curiosity, to consider the viscoelastic beam system (A.10) in $(0,\infty)\times\boldsymbol{R}$, by leading to the beginning problem (1.1).

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