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DYNAMICS OF A CLASS OF EXTENSIBLE BEAMS WITH DEGENERATE AND NON-DEGENERATE NONLOCAL DAMPING

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Abstract. This work is concerned with new results on long-time dynamics of a class of hyperbolic evolution equations related to extensible beams with three distinguished nonlocal nonlinear damping terms. In the first possibly degenerate case, the results feature the existence of a family of compact global attractors and a thickness estimate for their Kolmogorov's ε-entropy. Then, in the non-degenerate context, the structure of the helpful nonlocal damping leads to the existence of finite-dimensional compact global and exponential attractors. Lastly, in a degenerate and critical framework, it is proved the existence of a bounded closed global attractor but not compact. To the proofs, we provide several new technical results by means of refined estimates that open up perspectives for a new branch of nonlinearly damped problems.

1. Introduction

In the present article, we address the following evolution problem of hyperbolic type with nonlocal nonlinear damping term

$$\begin{cases} u_{tt} + \kappa A u + A_1 u + f(u) + k\big(\mathcal{E}_\alpha(u,u_t)\big)u_t = h_\lambda, \quad t>0, \\ u(0)=u_0, \quad u_t(0)=u_1, \end{cases} \tag{1.1}$$

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where κ is non-negative parameter, A and A_1 are linear self-adjoint positive definite operators related to Laplacian and bi-harmonic differential operators, respectively, $f(u)$ corresponds to a nonlinear source of lower order and growth exponent p, $h_\lambda = \lambda h$ represents an external force with $\lambda \in [0,1]$ and h lying in a Hilbert space $(H, \|\cdot\|)$, and

$$\mathcal{E}_\alpha(u, u_t) = \|A^\alpha u\|^2 + \|u_t\|^2, \quad \alpha \in [0,1], \tag{1.2}$$

with fractional powers A^α to be well defined later. Our study encompasses the abstract model (1.1) including three possibilities concerning the scalar function $k(\cdot)$ as follows:

(**k.1**) $k(\cdot)$ is a monomial-like function on $[0,\infty)$ with exponent $q \geq 1/2$, namely,

$$k(s) = \gamma s^q, \; s \geq 0, \quad \text{with} \quad \gamma > 0;$$

(**k.2**) $k(\cdot)$ is any C^1-function on $[0,\infty)$ such that $k(s) > 0$, $s \geq 0$;

(**k.3**) $k(\cdot)$ is a bounded Lipschitz function on $[0,\infty)$ such that $k \equiv 0$ on $[0,1]$ and $k(s)$ is strictly increasing for $s > 1$.

Through the coming statements, we try to be more transparent as possible in clarifying that problem (1.1)-(1.2) with damping coefficient obeying the three possibilities (**k.1**)-(**k.3**) has not been studied in the literature, by reaching in each specific case a level of results concerning its long-time behavior. In what follows, we state our main achievements.

1.1. **The possibly degenerate case (k.1).** In such a case the "polynomial" structure of the function $k(\cdot)$ says that the nonlocal damping coefficient can degenerate whenever the argument $\mathcal{E}_\alpha(u, u_t)$ vanishes, and the rate of degeneracy (of such unknown degenerate points) is determined by the exponent q. Additionally, the long-time dynamics of the system (1.1) is characterized by a "polynomial behavior" of type $1/q$, which can be very slow when q is very large. Within this scenario, our main achievements are stated in Sections 4 and 5 whose highlights are presented as follows:

(1) We prove a new key inequality (see Proposition 4.9) that gives a suitable estimate for the difference of two trajectory solutions of (1.1). Moreover, it will be very useful in the proof of attractors (smooth properties) for critical aspects in terms of the exponent p. To its proof, we launch a generalized Nakao's inequality (see Proposition 4.8) whose proof is only given in Appendix A.2. We point out that such a generalized Nakao's lemma will be also applicable to a wide class of autonomous and nonautonomous dynamical systems in the near future.

(2) Theorem 5.1 brings out the existence of a family of attractors for the dynamical system associated with problem (1.1) as well as their geometric and continuity properties. All statements in this result are reached for critical parameters p and subcritical power $\alpha \in [0,1)$ due to lack of compactness when $\alpha = 1$ (which is the subject of the case (**k.3**)). To the proof of asymptotic smoothness/compactness properties, we have combined our new generalized Nakao's inequality with an extended version of Khanmamedov's limit (see Proposition 4.11). For the continuity properties with respect to parameter $\lambda \in [0,1]$, we still prove a new Lipschitz continuous result (Proposition 4.12)

(3) In the subcritical framework with respect to both parameters α and p, we can compute an estimate of Kolmogorov's ε-entropy of the family of attractors, see Theorem 5.2, which roughly speaking measures the "thickness" of the compact attractors. For such an achievement, we have regarded the stabilizability estimate provided by Corollary 4.10 instead of the above mentioned key inequality and this reveals why we must go down to the subcritical case concerning the growth exponent p.

As far as we know, the model (1.1)-(1.2) in case (**k.1**), as well as the above mentioned results, have not been addressed in the literature in the sense of dynamical systems. In what concerns the stability to equilibrium, closer models we have found are [9, 31]. In [9] an interesting local stability result is presented with degenerate coefficient $\gamma\big[\|Au\|^2\big]^q u_t$, but only when initial data is taken regular and bounded, cf. [9, Theorem 3.1], which is not viable in the study of dynamical systems in the pattern weak phase space. In [31] the authors address the problem with nonlocal structural (strong) damping $\gamma\big[\|Au\|^2+\|u_t\|^2\big]^q Au_t$ instead of the nonlocal weak one $\gamma\big[\|A^\alpha u\|^2+\|u_t\|^2\big]^q u_t$. However, due to technical difficulties, in [31, Theorems 2.1 and 3.1] the authors only deal with existence and stability of regular solution when $q \geq 1$. Here, we have surpassed such difficulties and all results are provided for $q \geq \frac{1}{2}$. Moreover, it is worth pointing out that our particular Corollary 4.5 (see also Remark 4.2) gives the precise answer on polynomial stability to the related homogeneous problem, which finally clarifies the prediction estimate stated in [31, Theorem 4.1]. We still stress that, in this case, all results encompass the particular one with nonlocal averaged damping $\gamma\|u_t\|^{2q}u_t$, that is, by neglecting the potential energy $\|A^\alpha u\|^2$ in (1.2).

Finally, it is worth notifying that we also put some strength in the attempts of proving finite dimensionality for the family of compact global attractors obtained in Theorem 5.1. Indeed, throughout the whole Subsection 5.3, we clarified that it seems a delicate task due to the nature of such a damping term in case (**k.1**). Our conclusion is that we could not reach the assumptions of the current abstract results in dynamical systems to prove the finiteness of dimension in this case of nonlocal nonlinear possibly degenerate damping and a keener theory must be done for this purpose. All technical details concerning this issue are elucidated in Subsection 5.3.

1.2. **The non-degenerate case (k.2).** It is for sure a more touchable case where we can thrive up to the existence of exponential attractors. As a matter of fact, such an assumption in (**k.2**) is already employed by the authors in [28, 29, 30] for related extensible beam models with nonlocal averaged damping coefficient $k\big(\|A^{1/4}u\|^2\big)$ instead of the energy damping coefficient $k\big(\|A^{\alpha}u\|^2 + \|u_t\|^2\big)$. Nevertheless, our results here improve and generalize those provided in these references. Indeed, this case is treated in Section 6 and the highlights are:

(1) In Theorem 6.3, we catch up the same results as in Theorem 5.1 by aggregating finite fractal dimension and regularity of the existing global attractors, and also the existence of generalized fractal exponential attractors, that is, exponential attractors whose fractal dimensional is finite only in an extended space. Moreover, thanks to a new stabilizability estimate in this case (see Proposition 6.2), all these properties in Theorem 6.3 are achieved for the critical source exponent p but still in the subcritical case with respect to fractional power α.

(2) We also draw attention to the fact that positive constant functions are also incorporated in this case, and for such a very specific situation we can go further and prove the existence of time-dependent exponential attractors (with finite fractal dimensional in the standard phase space). This is the subject of Theorem 6.6 whose proof is mainly achieved by means of a new smoothing property (see Proposition 6.5) in the subcritical aspect with respect to p and assuming the commutative case $A = A_1^{1/2}$.

Therefore, the above facts furnish a considerable extension of the previous results achieved by authors in [28, 29, 30] concerning the criticality of the exponent p and the existence of exponential attractors in the linear damping case. Additionally, we note that standard examples of functions $k(\cdot)$ are:

$$k(s) = \gamma e^{\pm s}, \quad k(s) = \frac{\gamma}{1+s}, \quad k(s) = \gamma, \quad s \geq 0, \ \gamma > 0.$$

1.3. **The degenerate case (k.3).** This is exactly the case where we address the critical parameter $\alpha = 1$ and it justifies why in the previous ones we only get compact global attractors for subcritical powers $\alpha \in [0, 1)$. In fact, due to lack of compactness of the damping coefficient in the standard phase space and since $k(\cdot)$ vanishes on $[0,1]$, then we shall prove in Section 7 the existence of a noncompact global attractor even by assuming $f = h_\lambda = 0$ in the model (1.1). Here, our main result is Theorem 7.1 that proves the existence of a closed bounded (but not compact) global attractor for $\kappa > 0$ small enough, and due to the uniqueness of a global attractor (when it there exists), this precludes the existence of a compact global attractor under the assumptions considered in the present third case. We also point out that our main result is an extension of the one stated in [12, Proposition 5.3.9] where the particular case $\kappa = 0$ is considered. To our knowledge, the case $\kappa > 0$ has never been approached and, although similar, it requires different computations.

Examples of functions $k(\cdot)$ in this case are given as follows.

$$k(s) = \begin{cases} 0, & 0 \le s \le 1, \\ \gamma(1 - s^{-1}), & s > 1, \end{cases} \qquad k(s) = \begin{cases} 0, & 0 \le s \le 1, \\ \gamma(1 - e^{-s}), & s > 1, \end{cases} \qquad \gamma > 0.$$

We finish the introduction with the organization of the remaining paper:

In Section 2, we provide a physical motivation for studying the problem (1.1) and its abstract formulation through concrete problems in mathematical physics;

In Section 3, we state the well-posedness of the model (1.1)-(1.2) under the three conditions (**k.1**)-(**k.3**) and set the corresponding dynamical system.

From Section 4 to Section 7, we state and prove our main results concerning the long-time dynamics of problem (1.1) as well as it is exhibited additional remarks on the novelties introduced in the present article comparing to previous literature;

In Appendix A, we present some supplementary technical proofs.

Last, aiming reader's convenience and also for the sake of selfcontainment, in Appendix B, we remind several concepts and results coming from the meaningful literature in dynamical systems.

2. Physical and mathematical patterns

In this section, our main goal is to clarify that the abstract problem (1.1)-(1.2) is motivated by concrete problems in mathematical physics. More precisely, we are going to set it up as an abstract version of the following generalized n-dimensional extensible beam equation with nonlinear source and nonlocal damping terms

$$u_{tt} - \kappa\Delta u + \Delta^2 u + f(u) + k\big(\|(-\Delta)^\alpha u\|^2 + \|u_t\|^2\big)u_t = h_\lambda \ \text{ in } \ \Omega\times(0,\infty), \quad (2.1)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, κ is a non-negative constant, $f(u)$ is a nonlinear source whose assumption will be given in Section 3 (see Assumption 3.1), $\alpha \in [0,1]$, $h_\lambda = \lambda h$ with $\lambda \in [0,1]$ and $h \in L^2(\Omega)$, the notation $\|\cdot\|$ stands for the usual norm in $L^2(\Omega)$, and the scalar function $k(\cdot)$ is given in some class of functions encompassed by (**k.1**)-(**k.3**). Additionally, problem (2.1) is considered with initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{2.2}$$

and either physical boundary conditions: clamped

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{2.3}$$

where ν is the outward normal to $\partial\Omega$, or hinged (simply supported)

$$u = \Delta u = 0 \quad \text{on} \quad \partial\Omega. \tag{2.4}$$

2.1. **Physical motivation.** In Balakrishnan [4] it is presented the damping phenomena in flight structures with a free response. Accordingly, the one-dimensional model is described in terms of the basic second-order dynamics for the displacement variable $y(t)$ as follows

$$\ddot{y}(t) + \omega^2 y(t) + \gamma D(y(t), \dot{y}(t)) = 0, \tag{2.5}$$

where ω is the mode frequency and $\gamma > 0$ corresponds to a (small) damping coefficient. Here, $\dot{y} = \frac{dy}{dt}$ stands for the time derivative. Since $D(y(t), \dot{y}(t))$ is responsible by damping effects, it has been firstly considered as function depending on $\dot{y}(t)$ only, namely,

$$\text{sign}\dot{y}(t), \quad |\dot{y}(t)|\dot{y}(t), \quad |\dot{y}(t)|^\alpha \dot{y}(t), \quad \alpha \in (0,1).$$

After that, employing Krylov-Bogoliubov's approximation, Balakrishnan and Taylor [5] suggested a new class of damping models, called by *energy damping,* that are based on the instantaneous total energy of the system. More specifically, denoting the instantaneous energy associated with the system (2.5) by

$$\mathcal{E}(t) = \frac{w^2}{2}[y(t)]^2 + \frac{1}{2}[\dot{y}(t)]^2,$$

then $D(y(t), \dot{y}(t))$ is represented as

$$D(y(t), \dot{y}(t)) = [\mathcal{E}(t)]^q \dot{y}(t), \quad q > 0. \tag{2.6}$$

Thus, a straightforward computation with (2.5)-(2.6) reveals

$$\frac{d}{dt}\mathcal{E}(t) \ = \ -\gamma [\mathcal{E}(t)]^q [\dot{y}(t)]^2,$$

from where one sees that the stability of the system is driven by a possibly degenerate nonlocal nonlinear dissipative term involving the energy as a damping coefficient.

The same conclusion can be done with Krasovskii's system presented in [34], namely,

$$\ddot{y}(t) + y(t) + k\big([y(t)]^2 + [\dot{y(t)}]^2\big)\dot{y} = 0, \tag{2.7}$$

where the damping coefficient function $k(\cdot)$ is assumed to satisfy suitable properties for the generation of a dissipative dynamical system in $\mathbb{R}^2$. See also [12, Subsection 5.3.3] for more details on the infinite-dimensional version of (2.7).

By following the same spirit of [5, Section 4], where the authors derive some prototypes of models for uniform Bernoulli beams, we consider the following one-dimensional beam bending for a beam of length $2L$ and nonlocal damping coefficient in terms of the energy

$$u_{tt} - 2\zeta\sqrt{\lambda}u_{xx} + \lambda u_{xxxx} - \gamma\Big[\int_{-L}^{L}\big(\lambda|u_{xx}|^2 + |u_t|^2\big)dx\Big]^q u_{xxt} = 0, \tag{2.8}$$

where $u = u(x,t)$ represents the transversal deflection of the beam, $\gamma > 0$ is a damping coefficient, ζ is the constant coming from the Krylov-Bogoliubov approximation and $\lambda = \frac{2\zeta w}{\sigma^2}$ with w being the mode frequency and σ^2 the spectral density of a Gaussian external force. Moreover, for materials whose viscosity can be essentially seen as friction between moving solids, one may interpret beam modes like (2.8) with nonlocal frictional damping term instead of the viscous one. In this way, and also in accordance with the energy damping (2.6), the following equation emerges

$$u_{tt} - \kappa u_{xx} + u_{xxxx} + \gamma\Big[\int_{-L}^{L}\big(|u_{xx}|^2 + |u_t|^2\big)dx\Big]^q u_t = 0, \tag{2.9}$$

where we have normalized the equation with respect to an appropriate structural constant ($\lambda = 1$) and also denoted $\kappa = 2\zeta$ for the sake of notation.

Furthermore, in order to see (2.9) in the n-dimensional scenario, we consider the representative mathematical model

$$u_{tt} - \kappa\Delta u + \Delta^2 u + \gamma\Big[\int_{\Omega}\Big(|\Delta u|^2 + |u_t|^2\Big)dx\Big]^q u_t = 0, \tag{2.10}$$

where Ω is a bounded domain of the Euclidian space $\mathbb{R}^n$, $n \geq 1$.

Now, we note that (2.10) represents (2.1) with

$$\begin{aligned} &f = h_\lambda = 0, \quad k(s) = \gamma s^q, \quad \alpha = 1, \\ &\mathcal{E}_1(u,u_t) = \int_\Omega \big(|\Delta u|^2 + |u_t|^2\big)dx := \|\Delta u\|^2 + \|u_t\|^2, \end{aligned}$$

and, roughly speaking, problem (2.1) is a particular case of (1.1)-(1.2) with

$$A = -\Delta \quad \text{and} \quad A_1 = \Delta^2. \tag{2.11}$$

We still remark (1.1) can be seen as an infinite-dimensional generalization of Balakrishnan-Taylor's model (2.5)-(2.6) and Krasovskii's system (2.7).

Finally, since the long-time dynamics of problem (2.1)-(2.4) seems to be not studied in the literature so far, we feel motivated to investigate this issue by means of the abstract version (1.1) whose precise details on its abstract configuration are presented thereupon.

2.2. **Abstract formulation of the problem.** Below, we provide the precise details to set problem (2.1)-(2.4) up in an abstract formulation as given in (1.1), not only formally taking operators as in (2.11). We notice that the theory on functional analysis used below can be found e.g. in [6, 12, 18, 37, 38, 47, 48].

Throughout this work, the notation $(\cdot,\cdot)$ stands for the L^2-inner product, and $\|\cdot\|_p$ denotes the L^p-norm, $p \geq 1$. For commodity, when $p = 2$, we design $\|\cdot\|_2 = \|\cdot\|$. As usual, we denote by $H^s(\Omega)$ the L^2-based Sobolev space of the order $s > 0$ and by $H_0^s(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$. Let us also set $H = L^2(\Omega)$.

We first introduce the second-order (Laplacian) unbounded linear operator A by the formula

$$Au = -\Delta u, \quad u \in \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

It is well-known that A is a positively self-adjoint operator in H and, consequently, we can define the fractional powers A^s of A, $s \in \mathbb{R}$, with domains $\mathcal{D}(A^s)$. The spaces $\mathcal{D}(A^s)$ are Hilbert spaces with inner product and norm

$$(u, v)_{\mathcal{D}(A^s)} = (A^s u, A^s v), \quad \|u\|_{\mathcal{D}(A^s)} = \|A^s u\|.$$

Now, we consider the fourth-order (Biharmonic) unbounded linear operator $A_1 u = \Delta^2 u$ with domain

$$u \in \mathcal{D}(A_1) = \begin{cases} H^4(\Omega) \cap H_0^2(\Omega) & \text{for } (2.3), \\ \{u \in H^4(\Omega);\ u = \Delta u = 0 \text{ on } \partial\Omega\} & \text{for } (2.4). \end{cases}$$

From the spectral theory, we know that there exists a complete orthonormal basis $\{w_j\}_{j\in\mathbb{N}}$ in H consisting of eigenvectors of A_1, say with eigenvalues $\{\sigma_j\}_{j\in\mathbb{N}}$, such that

$$\begin{aligned} &w_j \in \mathcal{D}(A_1), \quad A_1 w_j = \sigma_j w_j, \quad j \geq 1, \\ &0 < \sigma_1 \leq \sigma_2 \leq \cdots \quad \text{with} \quad \sigma_j \to +\infty \ \text{ as } \ j \to +\infty. \end{aligned}$$

Likewise, we can also define the fractional powers A_1^s of A_1, $s \in \mathbb{R}$, with domains $\mathcal{D}(A_1^s)$ which are Hilbert spaces with inner product and norm

$$(u,v)_{\mathcal{D}(A_1^s)} = (A_1^s u, A_1^s v), \quad \|u\|_{\mathcal{D}(A_1^s)} = \|A_1^s u\|.$$

In both cases, we have densely inclusions

$$\mathcal{D}(A^{s_1}) \subset \mathcal{D}(A^{s_2}) \quad \text{and} \quad \mathcal{D}(A_1^{s_1}) \subset \mathcal{D}(A_1^{s_2}), \quad s_1, s_2 \in \mathbb{R},\ s_1 \geq s_2,$$

with continuous embedding when $s_1 \geq s_2$ and compact embedding in case $s_1 > s_2$.

Let us still stress some particular properties of the above operators and spaces. In fact, we first note that $\mathcal{D}(A_1^0) = \mathcal{D}(A^0) = H$, and

$$\mathcal{D}(A_1^{\frac{1}{2}}) = \begin{cases} H_0^2(\Omega) & \text{for } \ (2.3), \\ H^2(\Omega) \cap H_0^1(\Omega) & \text{for } \ (2.4). \end{cases}$$

Hence, in any case, concerning the boundary conditions, we have

$$\mathcal{D}(A_1^{\frac{1}{2}}) \subseteq \mathcal{D}(A) \quad \text{and} \quad \|A_1^{\frac{1}{2}} u\| = \|Au\|, \quad \forall\, u \in \mathcal{D}(A_1^{\frac{1}{2}}). \tag{2.12}$$

More particularly, in the specific case of hinged boundary condition (2.4), one knows that

$$A^s u = A_1^{s/2} u, \quad \forall\, u \in \mathcal{D}(A^s) = \mathcal{D}(A_1^{s/2}), \tag{2.13}$$

providing a good symmetry (say commutativity) to the extensible beam model (2.1), and consequently to problem (1.1). Moreover, the next particular embedding inequalities will be useful throughout this text

$$\sigma_1 \|u\|^2 \leq \|A_1^{\frac{1}{2}} u\|^2, \quad \sigma_1^{\frac{1}{2}} \|A_1^{1/4} u\|^2 \leq \|A_1^{\frac{1}{2}} u\|^2, \quad \forall\ u \in \mathcal{D}(A_1^{\frac{1}{2}}),$$

which can be taken for both boundary conditions.

Under the above construction, we are in the position to rewrite the concrete problem (2.1)-(2.4) in the following abstract class of second-order evolution problems

$$\begin{cases} u_{tt} + \kappa A u + A_1 u + f(u) + k\big(\|A^\alpha u\|^2 + \|u_t\|^2\big) u_t = h_\lambda, \quad t > 0, \\ u(0) = u_0, \quad u_t(0) = u_1, \end{cases}$$

which fairly corresponds to the beginning problem (1.1)-(1.2), where the damping coefficient is assumed to satisfy (**k.1**)-(**k.3**).

3. Well-posedness and associated dynamical system

The asymptotic behavior of the solutions of problem (1.1) will be considered on the Hilbert phase space

$$\mathcal{H} = \mathcal{D}(A_1^{\frac{1}{2}}) \times H, \quad ||(u,v)||^2_{\mathcal{H}} = \|A_1^{\frac{1}{2}} u\|^2 + \|v\|^2, \quad (u,v) \in \mathcal{H}.$$

As we shall see later, $\mathcal{H}$ is natural finite energy space for (1.1), and if we define the Sobolev phase space

$$\mathcal{H}_\alpha = \mathcal{D}(A^\alpha) \times H, \quad ||(u,v)||^2_{\mathcal{H}_\alpha} = \|A^\alpha u\|^2 + \|v\|^2, \quad \alpha \in [0,1],$$

then

$$\mathcal{H} = \mathcal{D}(A_1^{\frac{1}{2}}) \times H \subseteq \mathcal{D}(A) \times H = \mathcal{H}^1, \quad ||(\cdot,\cdot)||_{\mathcal{H}} = ||(\cdot,\cdot)||_{\mathcal{H}_1}.$$

The energy functional $E(u(t), u_t(t)) := E(t)$, $t \geq 0$, corresponding to problem (1.1) is expressed by

$$E(t) = \frac{1}{2}\left[\, \|u_t(t)\|^2 + \|A_1^{\frac{1}{2}} u(t)\|^2 + \kappa \|A^{\frac{1}{2}} u(t)\|^2 \,\right] + \big(\widehat{f}(u(t)), 1\big) - \big(h_\lambda, u\big), \tag{3.1}$$

where we set hereafter $\widehat{f}(u) = \int_0^u f(\tau)d\tau$ as the primitive of the function f, whose assumptions are given below in order to address the Hadamard well-posedness of problem (1.1) as well as its long-time dynamics results.

Assumption 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function with $f(0) = 0$ and satisfying*

$$|f'(u)| \leq C_{f'}(1 + |u|^p), \quad u \in \mathbb{R}, \tag{3.2}$$

$$-C_f - \frac{c_f}{2}|u|^2 \leq \widehat{f}(u) \leq f(u)u + \frac{c_f}{2}|u|^2, \quad u \in \mathbb{R}, \tag{3.3}$$

for some constants $C_f, C_{f'} > 0$, $c_f \in [0, \sigma_1)$ and growth exponent $p \leq \frac{4}{n-4}$ for $n \geq 5$.

Remark 3.1. We observe that $p^* := 2(p+1) \leq \frac{2n}{n-4}$ can be the critical (Sobolev) exponent for the continuous embedding $\mathcal{D}(A_1^{\frac{1}{2}}) \hookrightarrow L^{p^*}(\Omega)$, which is not compact for the critical case $p = \frac{4}{n-4}$. From Section 4 to Section 6, we present several results on long-time behavior in critical and subcritical frameworks. We do not work in lower dimensions $1 \leq n \leq 4$ since they do not require compactness issues with respect to p, by holding the same results with unchanged computations.

3.1. **Well-posedness.**

Theorem 3.1 (Hadamard Well-Posedness). *Let us assume that* $\alpha, \lambda \in [0,1]$, $h_\lambda := \lambda h \in H$, *and* f *satisfies Assumption* 3.1. *Then, problem* (1.1) *is Hadamard well-posed for* $k(\cdot)$ *given in any case* (**k.1**)-(**k.3**).

The proof of Theorem 3.1 is given below separately concerning the cases (**k.1**)-(**k.3**). We start with the possibly degenerate case (**k.1**) in what concerns the function $k(\cdot)$. The other cases (**k.2**)-(**k.3**) shall be treated at the end of this section.

Well-posedness: case (k.1). In this case, problem (1.1) can be written explicitly as

$$\begin{cases} u_{tt} + \kappa A u + A_1 u + \gamma\big[\, \|A^\alpha u\|^2 + \|u_t\|^2 \,\big]^q u_t + f(u) = h_\lambda, \quad t > 0, \\ u(0) = u_0, \quad u_t(0) = u_1. \end{cases} \tag{3.4}$$

Theorem 3.2 (Existence and Uniqueness). *Let* $\gamma > 0$, $q \geq \frac{1}{2}$ *and* $\kappa \geq 0$ *be given constants. Additionally, let us take on* $\alpha, \lambda \in [0,1]$, $h_\lambda = \lambda h \in H$, *and Assumption* 3.1.

(*i*) *If* $(u_0, u_1) \in \mathcal{H}$, *then there exists* $T_{\max} > 0$ *such that problem* (3.4) *has a unique mild (weak) solution* $(u^\lambda, u_t^\lambda) := (u, u_t)$ *in the class*

$$(u, u_t) \in C([0, T_{\max}), \mathcal{H}).$$

(*ii*) *If* $(u_0, u_1) \in \mathcal{D}(A_1) \times \mathcal{D}(A_1^{\frac{1}{2}})$, *then the solution* U *is more regular (strong).*

In both cases, we have $T_{\max} = +\infty$.

Proof. We first define vector-valued function $U(t) := (u(t), v(t))$, $t \geq 0$, with $v = u_t$. Then we can rewrite system (3.4) as the following first order abstract problem

$$\begin{cases} U_t = \mathcal{A}U + \mathcal{M}(U), \quad t > 0, \\ U(0) = (u_0, u_1) := U_0, \end{cases} \tag{3.5}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator given by

$$\mathcal{A}U = (v, -A_1 u), \quad U \in \mathcal{D}(\mathcal{A}) := \mathcal{D}(A_1) \times \mathcal{D}(A_1^{\frac{1}{2}}), \tag{3.6}$$

and $\mathcal{M} : \mathcal{H} \to \mathcal{H}$ is the nonlinear operator

$$\mathcal{M}(U) = (0, -\kappa A u + \gamma ||U||_{\mathcal{H}^\alpha}^{2q} v - f(u) + h_\lambda), \quad U \in \mathcal{H}. \tag{3.7}$$

Therefore, the existence and uniqueness of solution to the system (3.4) rely on the study of problem (3.5). To this purpose, and according to Pazy [41, Chapter 6], it is enough to prove that $\mathcal{A}$ given in (3.6) is the infinitesimal generator of a C_0-semigroup of contractions $e^{\mathcal{A}t}$ (which is very standard) and $\mathcal{M}$ set in (3.7) is locally Lipschitz on $\mathcal{H}$ (which will be done in Appendix A.1 for the sake of completeness – here is the precise moment where, for a technical reason, we consider $q \geq 1/2$ as clarified later). From this, one concludes the proof of items (i) and (ii).

It remains to check that $T_{\max} = +\infty$, that is, both mild and regular solutions are globally defined in time. Indeed, taking the scalar product in H of (3.4) with u_t, we obtain

$$\frac{d}{dt}E(t) + \gamma||(u(t), u_t(t))||_{\mathcal{H}^\alpha}^{2q} \, \|u_t(t)\|^2 = 0, \quad t > 0. \tag{3.8}$$

Integrating (3.8) over $(0, t)$, $t > 0$, we get

$$E(t) + \gamma \int_0^t ||(u(\tau), u_t(\tau))||_{\mathcal{H}^\alpha}^{2q} \, \|u_t(\tau)\|^2 \, d\tau = E(0), \quad t > 0. \tag{3.9}$$

Now, using (3.3) and Young's inequality with $\sigma_1\omega := \sigma_1 - c_f > 0$, we have

$$\begin{aligned} &\left[\big(\widehat{f}(u(t)), 1\big) - \big(h_\lambda, u(t)\big) \right] \\ &\geq -\frac{c_f}{2\sigma_1}\|A_1^{\frac{1}{2}}u(t)\|^2 - C_f|\Omega| - \frac{1}{\omega\sigma_1}\|h_\lambda\|^2 - \frac{\omega}{4}\|A_1^{\frac{1}{2}}u(t)\|^2. \end{aligned}$$

From the energy defined (3.1), we infer

$$\begin{aligned} E(t) &\geq \frac{\omega}{4}\|A_1^{\frac{1}{2}}u(t)\|^2 + \frac{1}{2}\|u_t(t)\|^2 - C_f|\Omega| - \frac{1}{\omega\sigma_1}\|h\|^2 \\ &\geq \frac{\omega}{4}||(u(t), u_t(t))||_{\mathcal{H}}^2 - K_\lambda, \end{aligned}$$

where we denote $K_\lambda = \left[C_f|\Omega| + \frac{1}{\omega\sigma_1}\|h_\lambda\|^2\right] > 0$. From this and (3.9), we arrive at

$$\frac{\omega}{4}\|(u(t), u_t(t))\|_{\mathcal{H}}^2 \leq E(t) + K_\lambda \leq E(0) + K_\lambda, \quad \forall\, t \in [0, T_{\max}). \tag{3.10}$$

If $T_{\max} < +\infty$, we have that $\|(u(t), u_t(t))\|_{\mathcal{H}}$ blows up in finite time (cf. [41, Theorem 1.4]), which is contraction with (3.10). Therefore, $T_{\max} = +\infty$. □

Hereafter, for the sake of notation, we still omit the parameter $\lambda \in [0, 1]$ indexed to the solution of (3.4), by simply writing down $u^\lambda := u$.

To the continuous dependence result, we invoke the following technical lemma on the power function s^r for $r \geq 1$ (cf. [2]).

Lemma 3.3 ([2, Lemma 2.2]). *Let X be a normed space with norm $\|\cdot\|_X$. Then, for any $r \geq 1$, we have*

$$\big|\|u\|_X^r - \|v\|_X^r\big| \leq r\max\{\|u\|_X, \|v\|_X\}^{r-1}\|u-v\|_X, \quad \forall\, u, v \in X. \tag{3.11}$$

Theorem 3.4 (Continuous Dependence). *Under the assumptions of Theorem 3.2, let $U^1 = (u^1, u_t^1)$ and $U^2 = (u^2, u_t^2)$ be two (strong or weak) solutions of problem (3.4) corresponding to initial data $U_0^1 = (u_0^1, u_1^1)$ and $U_0^2 = (u_0^2, u_1^2)$, respectively, and let $T > 0$ be any positive time. Then, there exists a positive non-decreasing function $\mathcal{Q}(t) = \mathcal{Q}\big(||U_0^1||_{\mathcal{H}}, ||U_0^2||_{\mathcal{H}}, t\big)$ such that*

$$||U^1(t) - U^2(t)||_{\mathcal{H}} \leq \mathcal{Q}(t)||U_0^1 - U_0^2||_{\mathcal{H}}, \quad t \in [0, T]. \tag{3.12}$$

Proof. Setting $w = u^1 - u^2$ and $F(w) = f(u^1) - f(u^2)$, the difference $U^1 - U^2 = (w, w_t)$ is a solution (in the strong and weak sense) of the following problem

$$\begin{cases} w_{tt} + \kappa A w + A_1 w + \dfrac{\gamma}{2}\Pi_1 w_t + \dfrac{\gamma}{2}\Pi_2\big[\, u_t^1 + u_t^2 \,\big] + F(w) = 0, \\ w(0) = u_0^1 - u_0^2 := w_0, \quad w_t(0) = u_1^1 - u_1^2 := w_1, \end{cases} \tag{3.13}$$

where

$$\Pi_i(t) = ||U^1(t)||_{\mathcal{H}^\alpha}^{2q} + (-1)^{1-i}||U^2(t)||_{\mathcal{H}^\alpha}^{2q}, \quad i = 1, 2.$$

The next estimates are done for strong solutions, and the same result can be achieved for weak solution through density arguments.

We still denote

$$\mathcal{E}_w(t) = \|w_t(t)\|^2 + \kappa\|A^{\frac{1}{2}}w(t)\|^2 + \|A_1^{\frac{1}{2}}w(t)\|^2, \quad t \geq 0. \tag{3.14}$$

Taking the inner product in H of (3.13) with w_t, we obtain

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}_w(t) + \frac{\gamma}{2}\Pi_1(t)\|w_t(t)\|^2 = \mathcal{J}_1 + \mathcal{J}_2, \tag{3.15}$$

where

$$\mathcal{J}_1 = -\big(\, F(w), w_t \,\big), \quad \mathcal{J}_2 = -\frac{\gamma}{2}\Pi_2(t)\big(\,(u_t^1 + u_t^2), w_t\,\big).$$

Using $\mathcal{D}(A_1^{\frac{1}{2}}) \hookrightarrow \mathcal{D}(A) \hookrightarrow \mathcal{D}(A^{\frac{1}{2}})$, it is easy to see that

$$||U^1(t) - U^2(t)||_{\mathcal{H}}^2 \leq \mathcal{E}_w(t) \leq (1 + \kappa\mu_0)||U^1(t) - U^2(t)||_{\mathcal{H}}^2, \tag{3.16}$$

where $\mu_0 >$ is a constant independent of initial data. In what follows, we will denote by C several constants that depend on the initial data. From assumption (3.2), Hölder's inequality with $\frac{p}{p^*} + \frac{1}{p^*} + \frac{1}{2} = 1$, the embedding $\mathcal{D}(A_1^{1/2}) \hookrightarrow L^{p^*}(\Omega)$, and (3.16), we can estimate the term $\mathcal{J}_1$ as follows

$$\big|\mathcal{J}_1\big| \leq C_{f'}\big[\,|\Omega| + \|u^1(t)\|_{p^*}^{p^*} + \|u^2(t)\|_{p^*}^{p^*}\,\big]^{\frac{p}{p^*}}\|w(t)\|_{p^*}\|w_t(t)\|$$

$$\leq C\|w(t)\|_{p^*}\|w_t(t)\| \leq C\|A_1^{\frac{1}{2}}w(t)\|\|w_t(t)\| \leq C\mathcal{E}_w(t), \tag{3.17}$$

for some constant $C > 0$ depending on initial data. This is the precise moment where we should use $q \geq 1/2$ to the handling of $\Pi_2(t)$. Indeed, since $2q \geq 1$, we can apply Lemma 3.3 to obtain the following estimate

$$\big|\Pi_2(t)\big| \leq 2q\max\{\|U^1(t)\|_{\mathcal{H}^\alpha}, \|U^2(t)\|_{\mathcal{H}^\alpha}\}^{2q-1}\big[\mathcal{E}_w(t)\,\big]^{\frac{1}{2}},$$

and then,

$$\big|\mathcal{J}_2\big| \leq \frac{\gamma}{2}\big|\Pi_2(t)\big|\big[\,\|u_t^1(t)\| + \|u_t^2(t)\|\,\big]\|w_t(t)\| \leq C\mathcal{E}_w(t), \tag{3.18}$$

for some constant $C > 0$ depending on initial data. Replacing the above estimates (3.17)-(3.18) in (3.15), we obtain

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}_w(t) + \frac{\gamma}{2}\Pi_1(t)||w_t(t)||^2 \leq C\mathcal{E}_w(t), \tag{3.19}$$

for all $t \in [0, T]$ and some constant $C > 0$ depending on initial data. Hence, integrating (3.19) on $[0, t]$, applying Gronwall's inequality and using (3.16), we arrive at

$$||U^1(t) - U^2(t)||^2_{\mathcal{H}} \leq C_0 e^{C_1 t}||U_0^1 - U_0^2||^2_{\mathcal{H}},$$

for some constants $C_i = C_i(||(u_0^i, u_1^i)||_{\mathcal{H}}) > 0$, $i = 0, 1$, which proves (3.12) with non-decreasing function $\mathcal{Q}(t) := C_0 e^{C_1 t}$ as desired. □

Well-posedness: cases (k.2)-(k.3). Concerning the well-posedness of problem (1.1) with function $k(\cdot)$ given in cases (**k.2**) or (**k.3**), we can proceed verbatim as in the proofs of Theorems 3.2 and 3.4 by noting that such cases do not interfere in the locally Lipschitz property of operator $\mathcal{M}$ set by (3.7) nor in the computations for the continuous dependence (3.12). Thus, we shall omit the details here. □

For similar approaches in these cases, we refer to [29, Sect. 2] and [12, Subsect. 5.3.3].

3.2. **Generation of the dynamical system.** For every $h_\lambda = \lambda h \in H$, $\lambda \in [0, 1]$, Theorem 3.1 ensures the Hadamard well-posedness of problem (1.1) and, consequently, the definition of a family of nonlinear C_0-semigroups $S_\lambda(t) : \mathcal{H} \to \mathcal{H}$ given by

$$S_\lambda(t)(u_0, u_1) = (u^\lambda(t), u_t^\lambda(t)) := (u(t), u_t(t)), \quad t \geq 0, \tag{3.20}$$

where (u, u_t) is the unique solution of the abstract system (1.1). Moreover, through the condition (3.12) one sees that $S_\lambda(t)$ is locally Lipschitz continuous on the phase space $\mathcal{H}$.

Accordingly, we also note the change of the time variable $t \mapsto -t$ in problem (1.1) turns itself into the same problem unless the damping term, which is replaced by $-k\big(\mathcal{E}_\alpha(u,u_t)\big)u_t$ instead of $k\big(\mathcal{E}_\alpha(u,u_t)\big)u_t$ in (1.1). For such a "inverse-time" problem, we can use the same arguments as in Theorems 3.2 and 3.4 for all cases concerning $k(\cdot)$ to prove the well-posedness. Therefore, it allows us to set $S_\lambda(t)$ as an evolution C_0-group. This remark will be important for future discussions on full trajectories and invariance properties of bounded sets in $\mathcal{H}$.

Therefore, in what follows the dynamics of problem (1.1) shall be studied through its corresponding dynamical system $(\mathcal{H}, S_\lambda(t))$ originated by (3.20). We start with the case (**k.1**) and then we treat the other cases (**k.2**) and (**k.3**) as well.

To deal with computations in the next sections, it is worth keeping in mind the energy $E(t)$ given in (3.1) and, additionally, we set the notation $\omega := 1 - \frac{c_f}{\sigma_1} > 0$, as well as the following parameters and modified energy

$$K_\lambda = \big[C_f|\Omega| + \frac{1}{\sigma_1\omega}\|h_\lambda\|^2\big] > 0 \quad \text{and} \quad \widetilde{E}(t) = E(t) + K_\lambda, \quad t \geq 0. \tag{3.21}$$

4. Technical results: case (k.1)

In what follows, except in Subsection 4.4, we will continue using the simplified notation $u^\lambda := u$, by neglecting (whenever there is no confusion) the index $\lambda \in [0,1]$.

4.1. **Dissipativity and gradient property.** The first proposition gives us lower and upper estimates for $\widetilde{E}(t)$. It reads as follows:

Proposition 4.1. *Under the assumptions of Theorem* 3.2, *there exist constants* $C_{\alpha,q,\gamma} > 0$ *and* $C_{\widetilde{E}(0)} > 0$ *such that the modified energy* $\widetilde{E}(t)$ *satisfies*

$$\Big[\frac{q}{C_{\alpha,q,\gamma}}t + \frac{1}{\big[\widetilde{E}(0)\big]^q}\Big]^{-\frac{1}{q}} \leq \widetilde{E}(t) \leq \Big[\frac{q}{C_{\widetilde{E}(0)}}(t-1)^+ + \frac{1}{\big[\widetilde{E}(0)\big]^q}\Big]^{-\frac{1}{q}} + 8K_\lambda, \tag{4.1}$$

for all $t > 0$, *where we use the standard notation* $s^+ := (s + |s|)/2$.

Proof. We initially prove the first inequality of (4.1). Taking the scalar product in H of (3.4) with u_t and using that $\frac{d}{dt}\widetilde{E}(t) = \frac{d}{dt}E(t)$, we obtain

$$\frac{d}{dt}\widetilde{E}(t) = -\gamma||(u(t), u_t(t))||^{2q}_{\mathcal{H}^\alpha}\|u_t(t)\|^2, \quad t > 0. \tag{4.2}$$

Now, using the embedding $\mathcal{D}(A_1^{1/2}) \hookrightarrow \mathcal{D}(A) \hookrightarrow \mathcal{D}(A^\alpha)$, with constant $C_\alpha > 0$ to the second one, the expression for $\widetilde{E}(t)$ in (3.21), and also

(3.10), we get

$$\begin{aligned}\gamma||(u(t),u_t(t))||^{2q}_{\mathcal{H}^\alpha}\|u_t(t)\|^2 &\le \gamma C_\alpha^{2q}||(u(t),u_t(t))||^{2q}_{\mathcal{H}}\|u_t(t)\|^2\\ &= \gamma\Big[\frac{4C_\alpha}{\omega}\Big]^q\big[\,\widetilde{E}(t)\,\big]^q\big[2\widetilde{E}(t)\big] = \frac{2^{2q+1}C_\alpha^q\gamma}{\omega^q}\big[\,\widetilde{E}(t)\,\big]^{q+1}.\end{aligned}$$

Returning to (4.2), one sees that

$$\frac{d}{dt}\widetilde{E}(t)\big[\,\widetilde{E}(t)\,\big]^{-(q+1)} \ge -\frac{2^{2q+1}C_\alpha^q\gamma}{\omega^q},$$

or else,

$$-\frac{1}{q}\frac{d}{dt}\big[\,\widetilde{E}(t)\,\big]^{-q} \ge -\frac{2^{2q+1}C_\alpha^q\gamma}{\omega^q}. \tag{4.3}$$

Simply solving this differential inequality, we arrive at

$$\widetilde{E}(t) \ge \Big[\,\frac{2^{2q+1}C_\alpha^q\gamma q}{\omega^q}t + \frac{1}{\big[\widetilde{E}(0)\big]^q}\,\Big]^{-\frac{1}{q}},$$

which proves the first inequality in (4.1) with $C_{\alpha,q,\gamma} = \frac{\omega^q}{2^{2q+1}C_\alpha^q\gamma} > 0$.

Now, we are going to prove the second inequality of (4.1). To do so, we provide some proper estimates and then apply Nakao's method (cf. [40]).

Again from (3.10), we note that

$$\frac{\omega}{4}||(u(t),u_t(t)||^2_{\mathcal{H}} \le \widetilde{E}(t) \le E(t) + K_1, \quad t \ge 0,\ \lambda \in [0,1]. \tag{4.4}$$

We also observe that

$$\gamma||(u(t),u_t(t)||^{2q}_{\mathcal{H}^\alpha}\|u_t(t)\|^2 \ge \gamma\|u_t(t)\|^{2q}\|u_t(t)\|^2 = \gamma\|u_t(t)\|^{2(q+1)}, \tag{4.5}$$

and replacing (4.5) in (4.2), we get

$$\frac{d}{dt}\widetilde{E}(t) + \gamma\|u_t(t)\|^{2(q+1)} \le 0, \quad t > 0, \tag{4.6}$$

which implies that $\widetilde{E}(t)$ is non-increasing. Also, integrating (4.6) from t to $t+1$, we obtain

$$\gamma\int_t^{t+1}\|u_t(s)\|^{2(q+1)}\,ds \le \widetilde{E}(t) - \widetilde{E}(t+1) := [\,D(t)\,]^2. \tag{4.7}$$

Using Hölder's inequality with $\frac{q}{q+1} + \frac{1}{q+1} = 1$ and (4.7), we infer

$$\int_t^{t+1}\|u_t(s)\|^2 ds \le \Big[\int_t^{t+1} 1^{\frac{q+1}{q}}ds\Big]^{\frac{q}{q+1}}\Big[\int_t^{t+1}\|u_t(s)\|^{2(q+1)}ds\Big]^{\frac{1}{q+1}}$$

$$\leq \frac{1}{\gamma^{\frac{1}{q+1}}} [\, D(t)\,]^{\frac{2}{q+1}}. \tag{4.8}$$

From the Mean Value Theorem for integrals, there exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_t(t_i)\|^2 \leq 4 \int_t^{t+1} \|u_t(s)\|^2 ds \leq \frac{4}{\gamma^{\frac{1}{q+1}}} [\, D(t)\,]^{\frac{2}{q+1}}. \tag{4.9}$$

On the other hand, taking the scalar product in H of (3.4) with u and integrating over $[t_1, t_2]$, we have

$$\int_{t_1}^{t_2} \big[\, \|A_1^{\frac{1}{2}} u(s)\|^2 + \kappa \|A^{\frac{1}{2}} u(s)\|^2 + \big(f(u(s)), u(s) \big) - \big(h_\lambda, u(s) \big) \,\big] ds$$

$$= \int_{t_1}^{t_2} \|u_t(s)\|^2 ds + \sum_{i=1}^{2} F_i, \tag{4.10}$$

where

$$F_1 := \big(u_t(t_1), u(t_1)\big) - \big(u_t(t_2), u(t_2)\big),$$

$$F_2 := -\gamma \int_{t_1}^{t_2} ||(u(s), u_t(s))||_{\mathcal{H}^\alpha}^{2q} \big(u_t(s), u(s)\big) ds.$$

From assumption (3.3), we have

$$\int_{t_1}^{t_2} \big(f(u(s)), u(s)\big)\, ds \geq \int_{t_1}^{t_2} \big(\widehat{f}(u(s)), 1\big) ds - \frac{c_f}{2\sigma_1} \int_{t_1}^{t_2} \|A_1^{\frac{1}{2}} u(s)\|^2 ds.$$

Returning to (4.10) and using that $\omega = 1 - \frac{c_f}{\sigma_1}$, we get

$$\int_{t_1}^{t_2} \widetilde{E}(s)\, ds + \frac{1}{2} \int_{t_1}^{t_2} \big[\, \kappa \|A^{\frac{1}{2}} u(t)\|^2 + \omega \|A_1^{\frac{1}{2}} u(t)\|^2 \,\big] ds$$

$$\leq K_\lambda + \frac{3}{2} \int_{t_1}^{t_2} \|u_t(s)\|^2 ds + \sum_{i=1}^{2} F_i. \tag{4.11}$$

Let us estimate the terms F_1 and F_2 as follows. First, we note that through Hölder's inequality, (4.4), (4.9) and Young's inequality, we obtain

$$F_1 \leq \sum_{i=1}^{2} \|u_t(t_i)\| \|u(t_i)\| \leq \frac{1}{\sigma_1^{\frac{1}{2}}} \sum_{i=1}^{2} \|u_t(t_i)\| \|A_1^{\frac{1}{2}} u(t_i)\|$$

$$\leq \frac{8}{(\omega\sigma_1)^{\frac{1}{2}} \gamma^{\frac{1}{2(q+1)}}} [\, D(t)\,]^{\frac{1}{q+1}} \sup_{t_1 \leq s \leq t_2} [\widetilde{E}(s)]^{\frac{1}{2}}$$

$$\leq \frac{128}{\omega\sigma_1\gamma^{\frac{1}{q+1}}}[\, D(t)\,]^{\frac{2}{q+1}} + \frac{1}{8}\sup_{t_1\leq s\leq t_2}\widetilde{E}(s).$$

Additionally, using again that $\mathcal{D}(A^{1/2}) \hookrightarrow \mathcal{D}(A^{\alpha/2})$ with embedding constant C_α, and (3.10), we have

$$\|(u(t),u_t(t))\|_{\mathcal{H}^\alpha}^{2q} \leq C_\alpha^q\|(u(t),u_t(t))\|_{\mathcal{H}}^{2q} \leq \frac{2^q C_\alpha^q}{\omega^q}[\,\widetilde{E}(0)\,]^q.$$

From this, using (4.8), and proceeding as in the estimate for F_1, we also get

$$\begin{aligned}
F_2 &\leq \frac{\gamma 2^q C_\alpha^q}{\omega^q\sigma_1^{\frac{1}{2}}}[\,\widetilde{E}(0)\,]^q\int_{t_1}^{t_2}\|u_t(s)\|\|A_1^{\frac{1}{2}}u(s)\|\,ds\\
&\leq \frac{2^q\gamma^{\frac{2q+1}{2(q+1)}}C_\alpha^q}{\omega^q\sigma_1^{\frac{1}{2}}}[\,\widetilde{E}(0)]^q[\,D(t)\,]^{\frac{1}{q+1}}\sup_{t_1\leq s\leq t_2}\|A_1^{\frac{1}{2}}u(s)\|_2\\
&\leq \frac{2^{q+1}\gamma^{\frac{2q+1}{2(q+1)}}C_\alpha^q}{\omega^{q+1/2}\sigma_1^{\frac{1}{2}}}[\,\widetilde{E}(0)]^q[\,D(t)\,]^{\frac{1}{q+1}}\sup_{t_1\leq s\leq t_2}\widetilde{E}^{\frac{1}{2}}(s)\\
&\leq \frac{2^{2q+3}\gamma^{\frac{2q+1}{q+1}}C_\alpha^{2q}}{\omega^{2q+1}\sigma_1}[\,\widetilde{E}(0)\,]^{2q}[\,D(t)\,]^{\frac{2}{q+1}} + \frac{1}{8}\sup_{t_1\leq s\leq t_2}\widetilde{E}(s).
\end{aligned}$$

Regarding again (4.8) and replacing the estimates for F_i, $i=1,2$, in (4.11), we obtain

$$\int_{t_1}^{t_2}\widetilde{E}(s)\,ds \leq \overline{C}_{\widetilde{E}(0)}[\,D(t)\,]^{\frac{2}{q+1}} + \frac{1}{4}\sup_{t_1\leq s\leq t_2}\widetilde{E}(s) + K_\lambda, \tag{4.12}$$

where we set

$$\overline{C}_{\widetilde{E}(0)} := \Big[\,\frac{3}{2\gamma^{\frac{1}{q+1}}} + \frac{128}{\omega\sigma_1\gamma^{\frac{1}{q+1}}} + \frac{2^{2q+3}\gamma^{\frac{2q+1}{q+1}}C_\alpha^{2q}}{\omega^{2q+1}\sigma_1}[\,\widetilde{E}(0)\,]^{2q}\,\Big] > 0.$$

Using once more the Mean Value Theorem for integrals and the fact that $\widetilde{E}(t)$ is non-increasing, there exists $\zeta \in [t_1,t_2]$ such that

$$\int_{t_1}^{t_2}\widetilde{E}(s)\,ds = \widetilde{E}(\zeta)(t_2-t_1) \geq \frac{1}{2}\widetilde{E}(t+1),$$

and then

$$\sup_{t\leq s\leq t+1}\widetilde{E}(s) = \widetilde{E}(t) = \widetilde{E}(t+1) + [\,D(t)\,]^2 \leq 2\int_{t_1}^{t_2}\widetilde{E}(s)\,ds + [\,D(t)\,]^2.$$

Thus, from this and (4.12), we arrive at

$$\sup_{t\le s\le t+1} \widetilde{E}(s) \le [\,D(t)\,]^2 + 2\int_{t_1}^{t_2} \widetilde{E}(s)ds$$
$$\le [\,D(t)\,]^2 + 2\overline{C}_{\widetilde{E}(0)}[\,D(t)\,]^{\frac{2}{q+1}} + \frac{1}{2}\sup_{t\le s\le t+1} \widetilde{E}(s) + 2K_\lambda,$$

and since $0 < \frac{2}{q+1} \le 2$, we obtain

$$\sup_{t\le s\le t+1} \widetilde{E}(s) \le [\,D(t)\,]^{\frac{2}{q+1}} \big[\,2[\,D(t)\,]^{\frac{2q}{q+1}} + 4\overline{C}_{\widetilde{E}(0)}\big] + 4K_\lambda. \tag{4.13}$$

Observing that

$$2[\,D(t)\,]^{\frac{2q}{q+1}} \le 2\big[\widetilde{E}(t) + \widetilde{E}(t+1)\big]^{\frac{q}{q+1}} \le 2^{\frac{2q+1}{q+1}}\big[\widetilde{E}(0)\,\big]^{\frac{q}{q+1}},$$

and denoting by

$$C_{\widetilde{E}(0)} := 2^{q+1}\big[2^{\frac{2q+1}{q+1}}\big[\widetilde{E}(0)\,\big]^{\frac{q}{q+1}} + 4\overline{C}_{\widetilde{E}(0)}\big]^{q+1} > 0,$$

and also recalling the definition of $[D(t)]^2$ in (4.7), we obtain from (4.13) that

$$\sup_{t\le s\le t+1} [\widetilde{E}(s)]^{q+1} \le C_{\widetilde{E}(0)}[\,\widetilde{E}(t) - \widetilde{E}(t+1)\,] + [8K_\lambda]^{q+1}.$$

Hence, applying [40, Lemma 2.1] with $\widetilde{E} = \phi$, $C_{\widetilde{E}(0)} = C_0$, and $K = [8K_\lambda]^{q+1}$, we conclude

$$\widetilde{E}(t) \le \Big[\frac{q}{C_{\widetilde{E}(0)}}(t-1)^+ + \frac{1}{\big[\widetilde{E}(0)\big]^q}\Big]^{-\frac{1}{q}} + 8K_\lambda,$$

which ends the proof of the second inequality in (4.1).

The proof of Proposition 4.1 is therefore complete. □

Remark 4.1. It is worth point out that we always have

$$\Big[\frac{q}{C_{\alpha,q,\gamma}}t + \frac{1}{\big[\widetilde{E}(0)\big]^q}\Big]^{-\frac{1}{q}} \le \Big[\frac{q}{C_{\widetilde{E}(0)}}(t-1)^+ + \frac{1}{\big[\widetilde{E}(0)\big]^q}\Big]^{-\frac{1}{q}}, \tag{4.14}$$

so that it makes sense to express $\widetilde{E}(t)$ between the inequalities in (4.1). Indeed, from the definitions of $C_{\alpha,q,\gamma}$ and $C_{\widetilde{E}(0)}$ in the proof of Proposition 4.1, one can easily see that $C_{\alpha,q,\gamma} \le C_{\widetilde{E}(0)}$, from where one concludes directly (4.14).

Some prompt consequences of Proposition 4.1 are given below.

Corollary 4.2 (Dissipativity). *Under the assumptions of Proposition* 4.1, *the dynamical system* $(\mathcal{H}, S_\lambda(t))$ *given by* (3.20) *is dissipative, that is, it has a bounded absorbing set* $\mathcal{B} \subset \mathcal{H}$, *which is uniformly bounded with respect to* $\lambda \in [0,1]$. *In particular, there exists a positively invariant bounded absorbing set.*

Proof. From (4.1) and (4.4), we obtain

$$||(u(t), u_t(t))||_{\mathcal{H}}^2 \leq \frac{4}{\omega}\Big[\frac{q}{C_{\widetilde{E}(0)}}(t-1)^+ + \frac{1}{\big[\widetilde{E}(0)\big]^q}\Big]^{-\frac{1}{q}} + \frac{32K_1}{\omega}, \quad t>0,$$

for all $\lambda \in [0,1]$. Thus, given an arbitrary bounded set $B \subset \mathcal{H}$ and taking $(u_0, u_1) \in B$, there exists a time $t_B > 0$ such that

$$||(u(t), u_t(t))||_{\mathcal{H}}^2 \leq \frac{64K_1}{\omega}, \quad \forall\, t \geq t_B. \tag{4.15}$$

Therefore, $\mathcal{B} = \overline{B\big(0, 8\sqrt{\frac{K_1}{\omega}}\big)}^{\mathcal{H}}$ constitutes a bounded absorbing set (uniformly with respect to $\lambda \in [0,1]$) for $(\mathcal{H}, S_\lambda(t))$. The construction of a positively invariant bounded absorbing set is standard. $\square$

Corollary 4.3 (Uniform Global Boundedness). *Under the assumptions of Proposition* 4.1, *the trajectory solutions of problem* (3.4) *are globally bounded in time* (*uniformly with respect to* $\lambda \in [0,1]$) *for initial data lying in bounded sets. More precisely, given a bounded set* $B \subset \mathcal{H}$ *and initial data* $(u_0, u_1) \in B$, *then there exists a constant* $C_B > 0$ (*depending only on* B) *such that*

$$||S_\lambda(t)(u_0, u_1)||_{\mathcal{H}} = ||(u(t), u_t(t))||_{\mathcal{H}} \leq C_B, \quad \forall\, t \geq 0.$$

Proof. It is a direct consequence of Corollary 4.2 (see (4.15)) and Theorem 3.2-(i). $\square$

Corollary 4.4 (Gradient Property). *Under the assumptions of Proposition* 4.1, *the dynamical system* $(\mathcal{H}, S_\lambda(t))$ *given by* (3.20) *is gradient, that is, there exists a strict Lyapunov functional* $\Phi_\lambda := \Phi$ *for* $(\mathcal{H}, S_\lambda(t))$. *Moreover, the Lyapunov functional* Φ *is bounded from above on any bounded subset of* $\mathcal{H}$ *and the set* $\overline{\Phi}_R = \{U \in \mathcal{H}\ ;\ \Phi(U) \leq R\}$ *is bounded in* $\mathcal{H}$ *for every* $R > 0$.

Proof. Let us define $\Phi := E$. From (3.8) one sees that the mapping

$$t \mapsto E(u(t), u_t(t)) = \Phi(S_\lambda(t)U_0)$$

is non-increasing for every $U_0 := (u_0, u_1) \in \mathcal{H}$. Additionally, from (3.9) and (4.5), one gets

$$\Phi(S_\lambda(t)U_0) + \gamma \int_0^t \|u_t(\tau)\|^{2(q+1)}\, d\tau = \Phi(U_0), \quad t > 0, \tag{4.16}$$

for every $U_0 \in \mathcal{H}$. From (4.16), it easily concludes that

$$\Phi(S_\lambda(t)U_0) = \Phi(U_0) \quad \Rightarrow \quad U_0 \in \mathcal{N}_\lambda, \quad t > 0,$$

where $\mathcal{N}_\lambda$ is defined in (5.1). Since we know that

$$U_0 \in \mathcal{N}_\lambda \quad \Leftrightarrow \quad S_\lambda(t)(U_0) = U_0, \quad t > 0,$$

then Φ is a strict Lyapunov functional for the dynamical system $(\mathcal{H}, S_\lambda(t))$.

Moreover, from (4.16), we have $\Phi(S_\lambda(t)U_0) \le \Phi(U_0)$ and, therefore, it is trivial to conclude that Φ is bounded from above on bounded subsets of $\mathcal{H}$. Finally, if $\Phi(S_\lambda(t)U_0) \le R$, then in view of (4.4), we obtain that $S_\lambda(t)U_0 = (u(t), u_t(t))$ satisfies

$$\|S_\lambda(t)U_0\|_{\mathcal{H}}^2 \le \frac{4}{\omega}(R + K_1), \quad t \ge 0, \ \lambda \in [0,1].$$

Hence, $\overline{\Phi}_R$ is a bounded set of $\mathcal{H}$ for every $R > 0$. □

Corollary 4.5 (Polynomial Decay Range)**.** *Under the assumptions of Proposition* 4.1, *and additionally assuming that $h \equiv 0$ and $C_f = 0$ in* (3.3), *then energy $E(t)$ satisfies*

$$\left[\frac{q}{C_{\alpha,q,\gamma}}t + \frac{1}{\left[E(0)\right]^q}\right]^{-\frac{1}{q}} \le E(t) \le \left[\frac{q}{C_{E(0)}}(t-1)^+ + \frac{1}{\left[E(0)\right]^q}\right]^{-\frac{1}{q}}, \quad (4.17)$$

for all $t > 0$.

Proof. It is a promptly consequence of (4.1) with $K_\lambda = 0$ and $\widetilde{E}(t) = E(t)$ in (3.21). □

Remark 4.2. Under the conditions of Corollary 4.5, the homogeneous problem related (3.4) is polynomially stable with rate $1/q$. Moreover, (4.17) shows that the energy $E(t)$ is squeezed in a polynomial decay range with optimal rate $1/q$ in the sense that it cannot be improved (by using (4.17)). Therefore, under these circumstances, problem (3.4) is never exponential stable in the homogeneous scenario with respect to mild (weak) solutions.

4.2. **The set of stationary solutions.**

Lemma 4.6. *Under the assumptions of Theorem* 3.2, *the set*

$$\mathcal{N}_\lambda = \left\{(u,0) \in \mathcal{H};\ \kappa A u + A_1 u + f(u) = h_\lambda\right\}, \quad \lambda \in [0,1], \quad (4.18)$$

is bounded in $\mathcal{H}$, uniformly with respect to $\lambda \in [0,1]$. In particular, if $h \equiv 0$ and $C_f = 0$ in (3.3), *then $\mathcal{N}_0 = \{(0,0)\}$.*

Proof. The existence of (at least one non-trivial) solution $u^\lambda := u$ for the equation

$$\kappa A u + A_1 u + f(u) = h_\lambda \quad \text{in} \quad H, \tag{4.19}$$

will be given in the next result. Taking the inner product in H of (4.19) with u, we have

$$\|A_1^{\frac{1}{2}} u\|^2 + \kappa \|A^{\frac{1}{2}} u\|^2 = -\big(f(u), u\big) + \big(h_\lambda, u\big). \tag{4.20}$$

From assumption (3.3), we have

$$-\big(f(u), u\big) \leq C_f |\Omega| + c_f \|u\|^2 \leq C_f |\Omega| + \frac{c_f}{\sigma_1} \|A_1^{\frac{1}{2}} u\|^2.$$

Also, from Young's inequality with $\omega = 1 - \frac{c_f}{\sigma_1} > 0$, we infer

$$\big(h_\lambda, u\big) \leq \frac{\omega}{2} \|A_1^{\frac{1}{2}} u\|^2 + \frac{2}{\sigma_1 \omega} \|h_\lambda\|^2.$$

Going back to (4.20), we obtain

$$\frac{\omega}{2} \|A_1^{\frac{1}{2}} u\|^2 + \kappa \|A^{\frac{1}{2}} u\|^2 \leq C_f |\Omega| + \frac{2}{\sigma_1 \omega} \|h_\lambda\|^2, \tag{4.21}$$

from where one concludes that $\mathcal{N}_\lambda$ is bounded in $\mathcal{H}$, uniformly with respect to $\lambda \in [0,1]$. In particular, if $h \equiv 0$ and $C_f = 0$, then (4.21) also implies that $u = 0$ and thus $\mathcal{N}_0$ is the trivial null set. □

Lemma 4.7. *Under the above notations and assumptions of Theorem* 3.2*, if $C_f > 0$ in* (3.3) *and even if $h \equiv 0$, then it is possible to show that the set*

$$\mathcal{N}_0 = \big\{(u, 0) \in \mathcal{H};\ \kappa A u + A_1 u + f(u) = 0\big\}$$

has a nontrivial (weak) solution $u \neq 0$ in H. Therefore, it is possible to conclude that the set $\mathcal{N}_0$ (and more generally $\mathcal{N}_\lambda$) has at least two stationary solutions.

Proof. Since $f(0) = 0$, then obviously $u = 0$ is the trivial solution of

$$\kappa A u + A_1 u + f(u) = 0 \quad \text{in} \quad H. \tag{4.22}$$

In what follows, let us deal with the case of nontrivial weak solution for u for (4.22). To fix the ideas, we are going to assume, by simplicity, the following concrete example for f

$$f(s) = |s|^\delta s - \sigma |s|^r s, \quad \sigma > 0,\ 0 < r < \delta \leq p. \tag{4.23}$$

The Euler-Lagrange functional $I_f : \mathcal{D}(A_1^{1/2}) \to \mathbb{R}$ corresponding to (4.22)-(4.23) is given by

$$I_f(u) = \frac{1}{2}\|A_1^{\frac{1}{2}}u\|^2 + \frac{\kappa}{2}\|A^{\frac{1}{2}}u\|^2 + \frac{1}{\delta+2}\|u\|_{\delta+2}^{\delta+2} - \frac{\sigma}{r+2}\|u\|_{r+2}^{r+2}.$$

We claim that for all $\sigma > 0$, I_f is coercive and bounded from below. In fact, since $0 < r < \delta$, then from the embedding $L^\delta \hookrightarrow L^r$ (with constant $C > 0$), we get

$$I_f(u) \geq \frac{1}{2}\|A_1^{\frac{1}{2}}u\|^2 + \frac{1}{\delta+2}\|u\|_{\delta+2}^{\delta+2} - \frac{\sigma C}{r+2}\|u\|_{\delta+2}^{r+2} \geq \frac{1}{2}\|A_1^{\frac{1}{2}}u\|^2 + D_0, \quad (4.24)$$

where

$$D_0 = \inf_{\tau \geq 0}\Big\{\frac{\tau^{\delta+2}}{\delta+2} - \sigma C\frac{\tau^{r+2}}{r+2}\Big\}.$$

Then, because of (4.24), one has that I_f is clearly coercive[1] and bounded from below.

On the other hand, by fixing $0 \neq u \in \mathcal{D}(A_1^{1/2})$, and regarding the embedding chain $\mathcal{D}(A_1^{1/2}) \hookrightarrow \mathcal{D}(A) \hookrightarrow L^\delta \hookrightarrow L^r$, we observe that there exists a parameter $\sigma_0 > 0$ such that $I_f(u) < 0$. For such a number σ_0, we consider a minimizing sequence, namely, a sequence $(u_n) \subset \mathcal{D}(A_1^{1/2})$ such that

$$\lim_{n\to+\infty} I_f(u_n) \;=\; \inf_{v\in\mathcal{D}(A_1^{\frac{1}{2}})} I_f(v) := \xi.$$

From the coerciveness of I_f, we have that (u_n) is bounded, and passing to a sub-sequence if necessary, we infer $u_n \to u$ weakly in $\mathcal{D}(A_1^{1/2})$. Due to the compactness of the embeddings $\mathcal{D}(A_1^{1/2}) \hookrightarrow L^{\delta+2}$ and $\mathcal{D}(A_1^{1/2}) \hookrightarrow \mathcal{D}(A) \hookrightarrow L^{r+2}$ (once $r+2 < \delta+2 \leq p^*$), then

$$\xi \;\leq\; I_f(u) \;\leq\; \liminf_{n\to+\infty} I_f(u_n) \;=\; \xi.$$

This implies that u is a global minimizer and, therefore, a nontrivial critical point of I_f, which in turn corresponds to a weak solution of (4.22) as desired. $\square$

[1] Given a Banach space $(X, \|\cdot\|)$, we recall that a functional $I : X \to \mathbb{R}$ is *coercive* if $\|u_n\| \to \infty$ implies $I(u_n) \to \infty$.

4.3. **Key estimates with respect to critical parameters α and p^*.** The next result is a generalized version the one presented in [40, Lemma 2.1]. It will be useful in the next key stability inherent to our dynamical system.

Proposition 4.8 (Nakao's Generalized Lemma)**.** *Let $\phi(t)$ be a non-negative continuous and $K(t)$ be a non-negative non-decreasing functions on $[0,T)$, $T>1$, possibly $T=\infty$, such that*

$$\sup_{t\leqslant s\leqslant t+1}[\phi(s)]^{1+\rho}\leqslant C_0(\phi(t)-\phi(t+1))+K(t),\quad 0\leqslant t\leqslant T-1, \tag{4.25}$$

for some $C_0>0$ and $\rho\geq 0$. Then, the following estimates hold:

(**N.1**) *For $\rho>0$, we have*

$$\phi(t)\leqslant \big(C_0^{-1}\rho(t-1)^+ + \big(\sup_{0\leqslant s\leqslant 1}\phi(s)\big)^{-\rho}\big)^{-\frac{1}{\rho}}+\big[K(t)\big]^{\frac{1}{\rho+1}},\quad 0\leqslant t<T, \tag{4.26}$$

where we consider usual notation $s^+ := (s+|s|)/2$.

(**N.2**) *For $\rho=0$, we have*

$$\phi(t)\leqslant \sup_{0\leqslant s\leqslant 1}\phi(s)\big(\frac{C_0}{1+C_0}\big)^{[t]}+K(t),\quad 0\leqslant t<T, \tag{4.27}$$

where $[s]$ stands for the largest integer less than or equal to $s\geq 0$.

Proof. It follows similar patterns as done in [40, Lemma 2.1] with proper modifications in what concerns the function $K(t)$, which in [40] is only assumed as a positive constant $K(t):=K>0$. Here, we address a more general class of functions by assuming that $K(t)$ can be a non-decreasing function, instead of a positive constant only.

For the sake of completeness, we present a detailed proof in Appendix A.2. $\square$

Remark 4.3. Given a measurable function $g:[0,t+1]\to\mathbb{R}$, $t\geq 0$, we observe that for any $a,b\in[t,t+1]$ with $a\leq b$, it holds the following estimate

$$\Big|\int_a^b g(s)\,ds\Big|\leq \sup_{s\in[0,t+1]}\sup_{r\in[0,s]}\Big|\int_r^s g(\tau)\,d\tau\Big|. \tag{4.28}$$

Indeed, to reach it we simply note that it holds the chain of inequalities

$$\Big|\int_a^b g(\tau)\,d\tau\Big|\leq \sup_{s\in[a,t+1]}\Big|\int_a^s g(\tau)\,d\tau\Big|\leq \sup_{s\in[0,t+1]}\sup_{r\in[0,s]}\Big|\int_r^s g(\tau)\,d\tau\Big|.$$

The next result provides a key inequality in the present article. It gives a suitable estimate encompassing the difference of two trajectory solutions of (3.4). To its proof, we combine refined and new arguments along with the above Nakao generalized result and (4.28).

Before stating such a key result, let us remark that by virtue of Corollary 4.3 any trajectory solution is globally bounded in time on bounded sets (uniformly with respect to $\lambda \in [0,1]$). This fact will be highly used in the next result. Also, in what follows, we will denote by $C_B > 0$ several different constants depending on a general bounded set $B \subset \mathcal{H}$.

Proposition 4.9 (Key Inequality)**.** *Under the assumptions of Theorem* 3.2*, let us also consider a bounded set $B \subset \mathcal{H}$. Given $U^i = (u_0^i, u_1^i) \in B$, $i = 1, 2$, we denote by $S_\lambda(t)U^i = (u^i(t), u_t^i(t)), i = 1, 2$, the respective trajectory solution corresponding to the dynamical system* (3.20)*. Then, there exists a constant $C_B > 0$ such that*

$$||S_\lambda(t)U^1 - S_\lambda(t)U^2||_{\mathcal{H}}^2 \leq \big[C_B^{-1} q(t-1)^+ + \big(\sup_{0\leq s\leq 1} ||(w(s), w_t(s))||_{\mathcal{H}}^2\big)^{-q}\big]^{-\frac{1}{q}}$$
$$+ C_B \sup_{0\leq s\leq t+1} \big[\|A^\alpha w(s)\|^{\frac{2(q+1)}{2q+1}} + \|w(s)\|_{p+2}^2\big]^{\frac{1}{q+1}} + \big[\, J_f(w(t), w_t(t))\,\big]^{\frac{1}{q+1}}. \tag{4.29}$$

for all $t > 0$ and $\lambda \in [0,1]$, where we set $w = u^1 - u^2$, $F(w) = f(u^1) - f(u^2)$, and

$$J_f(w(t), w_t(t)) = 2 \sup_{s\in[0,t+1]} \sup_{r\in[0,s]} \Big| \int_r^s (F(w(\tau)), w_t(\tau))d\tau \Big|$$
$$+ 4^{q+1} \sup_{s\in[0,t+1]} \sup_{r\in[0,s]} \Big| \int_r^s (F(w(\tau)), w_t(\tau))d\tau \Big|^{q+1}. \tag{4.30}$$

Proof. Let $S_\lambda(t)U^1 - S_\lambda(t)U^2 := (w(t), w_t(t))$. Since we are dealing with the difference of two trajectory solutions of problem (3.4), then we are going to borrow some notations used in Theorem 3.4. Indeed, we first note that (w, w_t) is a solution (in the weak and strong sense) of problem (3.13). Also, the functional $\mathcal{E}_w$ defined in (3.14) satisfies

$$||(w(t), w_t(t))||_{\mathcal{H}}^2 \leq \mathcal{E}_w(t) \leq \mu_\kappa ||(w(t), w_t(t))||_{\mathcal{H}}^2, \tag{4.31}$$

form some constant $\mu_\kappa := 1 + \kappa\mu_0 > 0$, see (3.16), and the identity

$$\frac{d}{dt}\mathcal{E}_w(t) + \gamma\Pi_1(t)\|w_t(t)\|^2 \tag{4.32}$$
$$= -\gamma\Pi_2(t)\big(u_t^1(t) + u_t^2(t), w_t(t)\big) + 2\big(F(w(t)), w_t(t)\big),$$

which is essentially (3.15). In what follows, our first goal is to estimate the right-hand side of (4.32).

First, using that $[a^{2q}+b^{2q}] \geq [a-b]^{2q}$, we get

$$\gamma\Pi_1(t)\|w_t(t)\|^2 \geq \gamma\big[\,\|u_t^1(t)\|^{2q}+\|u_t^2(t)\|^{2q}\,\big]\|w_t(t)\|^2 \geq \frac{\gamma}{2^{2q}}\|w_t(t)\|^{2(q+1)}. \tag{4.33}$$

Now, for $q \geq 1/2$, we claim that

$$-\gamma\Pi_2(t)\big(u_t^1(t)+u_t^2(t),w_t(t)\big) \leq C_B\|A^\alpha w(t)\|^{\frac{2(q+1)}{2q+1}} + \frac{\gamma}{2^{2(q+1)}}\|w_t(t)\|^{2q+1}, \tag{4.34}$$

for some constant $C_B>0$. Indeed, if $||S_\lambda(t)U^1||_{\mathcal{H}^\alpha} = ||S_\lambda(t)U^2||_{\mathcal{H}^\alpha} = 0$, there is nothing to do. Let us suppose then $||S_\lambda(t)U^1||_{\mathcal{H}^\alpha}+||S_\lambda(t)U^2||_{\mathcal{H}^\alpha} > 0$. From the Mean Value Theorem (MVT for short), there exists a number

$$\xi_\vartheta = \vartheta||S_\lambda(t)U^1||_{\mathcal{H}^\alpha} + (1-\vartheta)||S_\lambda(t)U^2||_{\mathcal{H}^\alpha}, \quad \vartheta\in(0,1),$$

such that

$$\begin{aligned}
\Pi_2(t) &= 2q\int_0^1 \big|\xi_\vartheta\big|^{2q-1}d\vartheta\big[\,||S_\lambda(t)U^1||_{\mathcal{H}^\alpha} - ||S_\lambda(t)U^2||_{\mathcal{H}^\alpha}\,\big] \\
&= 2q\int_0^1 \big|\xi_\vartheta\big|^{2q-1}d\vartheta\frac{\big[\,\|u_t^1(t)\|^2-\|u_t^2(t)\|^2\,\big]}{||S_\lambda(t)U^1||_{\mathcal{H}^\alpha}+||S_\lambda(t)U||_{\mathcal{H}^\alpha}} \\
&\quad + 2q\int_0^1 \big|\xi_\vartheta\big|^{2q-1}d\vartheta\frac{\big[\,\|A^\alpha u^1(t)\|^2-\|A^\alpha u^2(t)\|^2\,\big]}{||S_\lambda(t)U^1||_{\mathcal{H}^\alpha}+||S_\lambda(t)U||_{\mathcal{H}^\alpha}}.
\end{aligned}$$

Thus, we can write

$$-\gamma\Pi_2(t)\big(u_t^1(t)+u_t^2(t),w_t(t)\big) = \eta(t)+\chi(t), \tag{4.35}$$

where

$$\begin{aligned}
\eta(t) &= -2q\gamma\int_0^1 \big|\xi_\vartheta\big|^{2q-1}d\vartheta\frac{\big[\,\|u_t^1(t)\|^2-\|u_t^2(t)\|^2\,\big]^2}{||S_\lambda(t)U^1||_{\mathcal{H}^\alpha}+||S_\lambda(t)U||_{\mathcal{H}^\alpha}} \\
\chi(t) &= -2q\gamma\int_0^1 \big|\xi_\vartheta\big|^{2q-1}d\vartheta\frac{\big[\,\|A^\alpha u^1(t)\|^2-\|A^\alpha u^2(t)\|^2\,\big]}{||S_\lambda(t)U^1||_{\mathcal{H}^\alpha}+||S_\lambda(t)U||_{\mathcal{H}^\alpha}}\big(u_t^1(t),w_t(t)\big) \\
&\quad - 2q\gamma\int_0^1 \big|\xi_\vartheta\big|^{2q-1}d\vartheta\frac{\big[\,\|A^\alpha u^1(t)\|^2-\|A^\alpha u^2(t)\|^2\,\big]}{||S_\lambda(t)U^1||_{\mathcal{H}^\alpha}+||S_\lambda(t)U||_{\mathcal{H}^\alpha}}\big(u_t^2(t),w_t(t)\big).
\end{aligned}$$

Using Young inequality with $\frac{2q+1}{2(q+1)}+\frac{1}{2(q+1)}=1$, the term $\chi(t)$ can be estimated as follows

$$|\chi(t)| \leq C_B\frac{\big[\,||S_\lambda(t)U^1||^2_{\mathcal{H}^\alpha}+||S_\lambda(t)U^2||^2_{\mathcal{H}^\alpha}\,\big]}{||S_\lambda(t)U^1||_{\mathcal{H}^\alpha}+||S_\lambda(t)U||_{\mathcal{H}^\alpha}}\|A^\alpha w(t)\|\|w_t(t)\|$$

$$\leq C_B\|A^\alpha w(t)\|^{\frac{2(q+1)}{2q+1}} + \frac{\gamma}{2^{2(q+1)}}\|w_t(t)\|^{2(q+1)},$$

for some constant $C_B > 0$, once $2q-1 \geq 0$. Noting that $\eta(t) \leq 0$ and connecting the last estimate in (4.35), we arrive at (4.34).

Thus, collecting (4.33) and (4.34) with (4.32) , we obtain

$$\frac{d}{dt}\mathcal{E}_w(t) + \frac{\gamma}{2^{2q+1}}\|w_t(t)\|^{2(q+1)} \leq C_B\big[\,\|A^\alpha w(t)\|^{\frac{2(q+1)}{2q+1}} + 2\big(F(w(t)), w_t(t)\big)\,\big]. \tag{4.36}$$

Integrating (4.36) from t to $t+1$, we have

$$\begin{aligned} &\frac{\gamma}{2^{2q+1}}\int_t^{t+1}\|w_t(s)\|^{2(q+1)}ds \\ &\leq \mathcal{E}_w(t) - \mathcal{E}_w(t+1) + C_B\int_t^{t+1}\|A^\alpha w(s)\|^{\frac{2(q+1)}{2q+1}}ds \\ &\quad + 2\Big|\int_t^{t+1}(F(w(s)), w_t(s))ds\Big| := \big[G(t)\big]^2. \end{aligned} \tag{4.37}$$

Now, from Hölder's inequality with $\frac{q}{q+1} + \frac{1}{q+1} = 1$ and (4.37), we have

$$\int_t^{t+1}\|w_t(s)\|^2 ds \leq \Big[\int_t^{t+1}\|w_t(s)\|^{2(q+1)}ds\Big]^{\frac{1}{q+1}} \leq \frac{2^{\frac{2q+1}{q+1}}}{\gamma^{\frac{1}{q+1}}}\big[\,G(t)\,\big]^{\frac{2}{q+1}}, \tag{4.38}$$

which implies (from the MVT for Integrals) that there exists $t_1 \in [t, t+\frac{1}{4}]$, $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|w_t(t_i)\|^2 \leq 4\Big[\int_t^{t+1}\|w_t(s)\|^2 ds\Big] \leq \frac{2^{\frac{4q+3}{q+1}}}{\gamma^{\frac{1}{q+1}}}\big[\,G(t)\,\big]^{\frac{2}{q+1}}. \tag{4.39}$$

Let us keep the above estimates in mind to apply them properly in the next computations.

Working on the other hand, we take now the multiplier w in (3.13) and integrating from t_1 to t_2, we have

$$\int_{t_1}^{t_2}\mathcal{E}_w(t)\,ds = 2\int_{t_1}^{t_2}\|w_t(s)\|^2 ds + \sum_{i=1}^{4}\mathcal{L}_i, \tag{4.40}$$

where

$$\begin{aligned} \mathcal{L}_1 &= -\int_{t_1}^{t_2}\big(F(w(s)), w(s)\big)ds, \\ \mathcal{L}_2 &= -\big[\,\big(w_t(t_2), w(t_2)\big) - \big(w_t(t_1), w(t_1)\big)\,\big], \end{aligned}$$

$$\mathcal{L}_3 = -\frac{\gamma}{2}\int_{t_1}^{t_2} \Pi_1(s)\big(w_t(s), w(s)\big)ds,$$
$$\mathcal{L}_4 = -\frac{\gamma}{2}\int_{t_1}^{t_2} \Pi_2(s)\big(\big[\,u_t^1(s) + u_t^2(s)\,\big], w(s)\big)ds.$$

The terms $\mathcal{L}_i$, $i = 1, \cdots, 4$ can be estimated as follows. First, from MVT, the hypothesis (3.2), Hölder's inequality and the embedding $\mathcal{D}(A_1^{1/2}) \hookrightarrow L^{p+2}$, we have

$$\begin{aligned}
\mathcal{L}_1 &\leq \int_{t_1}^{t_2} \big(\,\big[f(u^1) - f(u^2)\big],\, w\big)ds \\
&\leq C_{f'}\int_{t_1}^{t_2}\int_\Omega \big[\,(1 + |u_1| + |u^2|)^p\,\big]|w|^2 dxds \\
&\leq C_{f'}\int_{t_1}^{t_2}\Big[\int_\Omega (1 + |u^1|^p + |u^2|^p)^{\frac{p+2}{p}}\Big]^{\frac{p}{p+2}}\|w(s)\|_{p+2}^2 ds \\
&\leq C_B\int_{t_1}^{t_2}\|w(s)\|_{p+2}^2 ds,
\end{aligned}$$

for some constant $C_B > 0$. Also, using now the embedding $\mathcal{D}(A_1^{1/2}) \hookrightarrow H$, Young's inequality, (4.38)-(4.39), we get

$$\begin{aligned}
\mathcal{L}_2 &\leq \frac{1}{\sigma_1^{\frac{1}{2}}}\sum_{i=1}^{2}\|w_t(t_i)\|\|A_1^{\frac{1}{2}}w(s)\| \leq \frac{2}{\sigma_1^{\frac{1}{2}}}\Big\{\frac{2^{\frac{4q+3}{2(q+1)}}}{\gamma^{\frac{1}{2(q+1)}}}\big[\,G(t)\,\big]^{\frac{1}{q+1}}\Big\}\sup_{t_1\leq s\leq t_2}\|A_1^{\frac{1}{2}}w(s)\| \\
&\leq \frac{2^{\frac{4q+3}{q+1}}}{\delta\sigma_1\gamma^{\frac{1}{q+1}}}\big[\,G(t)\,\big]^{\frac{2}{q+1}} + \delta\sup_{t_1\leq s\leq t_2}\big[\mathcal{E}_w(t)\big],
\end{aligned}$$

and

$$\mathcal{L}_3 \leq C_B\Big(\int_{t_1}^{t_2}\|w_t(s)\|ds\Big)\sup_{t_1\leq s\leq t_2}\|A_1^{\frac{1}{2}}w(s)\| \leq C_B\big[G(t)\big]^{\frac{2}{q+1}} + \delta\sup_{t_1\leq s\leq t_2}\big[\mathcal{E}_w(t)\big],$$

for some constant $C_B > 0$. Third, using Lemma 3.3 and noting $\mathcal{D}(A_1^{1/2}) \hookrightarrow \mathcal{D}(A^\alpha)$, $L^{p+2}(\Omega) \hookrightarrow H$, we obtain for $q \geq 1/2$ that

$$\begin{aligned}
\mathcal{L}_4 &\leq \gamma q\int_{t_1}^{t_2}\max\{||S_\lambda(s)U^1||_{\mathcal{H}^\alpha}^2, ||S_\lambda(s)U^1||_{\mathcal{H}^\alpha}^2\}^{2q-1}[\mathcal{E}_w(t)]^{\frac{1}{2}}\|w(s)\|ds \\
&\leq C_B\int_{t_1}^{t_2}[\mathcal{E}_w(t)]^{\frac{1}{2}}\|w(s)\|ds
\end{aligned}$$

$$\leq \delta \sup_{t_1 \leq s \leq t_2} [\mathcal{E}_w(s)] + C_B \int_{t_1}^{t_2} \|w(s)\|_{p+2}^2 ds,$$

for some constant $C_B > 0$ and $\delta > 0$.

Replacing the latter estimates for $\mathcal{L}_1, \cdots, \mathcal{L}_4$ in (4.40), we arrive at

$$\int_{t_1}^{t_2} \big[\mathcal{E}_w(s)\big] ds \leq C_B \big[\, G(t)\,\big]^{\frac{2}{q+1}} + 3\delta \sup_{t_1 \leq s \leq t_2} \big[\mathcal{E}_w(s)\big] + C_B \int_t^{t+1} \|w(s)\|_{p+2}^2 \, ds,$$

for some constant $C_B > 0$. Again from the MVT, there exists $\tau_1 \in [t_1, t_2] \subset [t, t+1]$ such that

$$\mathcal{E}_w(\tau_1) \leq C_B \big[G(t)\big]^{\frac{2}{q+1}} + 3\delta \sup_{t_1 \leq s \leq t_2} \big[\mathcal{E}_w(s)\big] + C_B \int_t^{t+1} \|w(s)\|_{p+2}^2 ds. \quad (4.41)$$

Let us also consider $\tau_2 \in [t, t+1]$ such that

$$\mathcal{E}_w(\tau_2) := \sup_{t \leq s \leq t+1} \big[\mathcal{E}_w(s)\big].$$

Now, integrating (4.36) over $[t, \tau_2]$ and over $[\tau_1, t+1]$, using (4.41) and noting that

$$\mathcal{E}_w(t) \leq \mathcal{E}_w(t+1) + \big[\, G(t)\,\big]^2,$$

we obtain

$$\begin{aligned}
\mathcal{E}_w(\tau_2) &\leq \mathcal{E}_w(t) + C_B \int_t^{t+1} \|A^\alpha w(s)\|^{\frac{2(q+1)}{2q+1}} ds + 2\Big| \int_t^{\tau_2} (F(w(s)), w_t(s)) ds \Big| \\
&\leq \mathcal{E}_w(t+1) + \big[\, G(t)\,\big]^2 + C_B \int_t^{t+1} \|A^\alpha w(s)\|^{\frac{2(q+1)}{2q+1}} ds \\
&\quad + 2\Big| \int_t^{\tau_2} (F(w(s)), w_t(s)) ds \Big| \\
&\leq \mathcal{E}_w(\tau_1) + \big[\, G(t)\,\big]^2 + C_B \int_t^{t+1} \|A^\alpha w(s)\|^{\frac{2(q+1)}{2q+1}} ds \\
&\quad + 2\Big| \int_{\tau_1}^{t+1} (F(w(s)), w_t(s)) ds \Big| + 2\Big| \int_t^{\tau_2} (F(w(s)), w_t(s)) ds \Big| \\
&\leq \big[\, G(t)\,\big]^2 + C_B \big[\, G(t)\,\big]^{\frac{2}{q+1}} + 3\delta \mathcal{E}_w(\tau_2) \\
&\quad + C_B \int_t^{t+1} \|A^\alpha w(s)\|^{\frac{2(q+1)}{2q+1}} ds + C_B \int_{t_1}^{t_2} \|w(s)\|_{p+2}^2 \, ds \\
&\quad + 2\Big| \int_{\tau_1}^{t+1} (F(w(s)), w_t(s)) ds \Big| + 2\Big| \int_t^{\tau_2} (F(w(s)), w_t(s)) ds \Big|,
\end{aligned}$$

for some constant $C_B > 0$. Choosing $\delta > 0$ small enough, noting that $\big[\, G(t) \,\big]^{q+1} \leq C_B$, for some constant $C_B > 0$, and using (4.28), we infer

$$\begin{aligned}
\mathcal{E}_w(\tau_2) \leq & \, C_B \big[\, G(t) \,\big]^{\frac{2}{q+1}} + C_B \int_t^{t+1} \|A^\alpha w(s)\|^{\frac{2(q+1)}{2q+1}} ds \\
& + C_B \int_t^{t+1} \|w(s)\|_{p+2}^2 \, ds + 2 \Big| \int_{\tau_1}^{t+1} (F(w(s)), w_t(s)) ds \Big| \\
& + 2 \Big| \int_t^{\tau_2} (F(w(s)), w_t(s)) ds \Big| \\
\leq & \, C_B \big[\, G(t) \,\big]^{\frac{2}{q+1}} + C_B \sup_{t \leq s \leq t+1} \big[\|A^\alpha w(s)\|^{\frac{2(q+1)}{2q+1}} + \|w(s)\|_{p+2}^2 \big] \\
& + 4 \sup_{s \in [0,t+1]} \sup_{r \in [0,s]} \Big| \int_r^s (F(w(\tau)), w_t(\tau)) d\tau \Big|. \qquad (4.42)
\end{aligned}$$

In this way, from (4.42), the definition of $G(t)$ in (4.37), and using again (4.28), we arrive at

$$\sup_{t \leq s \leq t+1} \big[\mathcal{E}_w(s) \big]^{q+1} \leq C_B \big[\mathcal{E}_w(t) - \mathcal{E}_w(t+1) \,\big] + K_f(t), \qquad (4.43)$$

where $K_f(t) := K_f(w(t), w_t(t))$ is set by

$$\begin{aligned}
K_f(t) = & \, 2 \sup_{s \in [0,t+1]} \sup_{r \in [0,s]} \Big| \int_r^s (F(w(s)), w_t(s)) ds \Big| \\
& + 4^{q+1} \sup_{s \in [0,t+1]} \sup_{r \in [0,s]} \Big| \int_r^s (F(w(s)), w_t(s)) ds \Big|^{q+1} \\
& + C_B \sup_{0 \leq s \leq t+1} \big[\|A^\alpha w(s)\|^{\frac{2(q+1)}{2q+1}} + \|w(s)\|_{p+2}^2 \big] \\
= & \, J_f(t) + C_B \sup_{0 \leq s \leq t+1} \big[\|A^\alpha w(s)\|^{\frac{2(q+1)}{2q+1}} + \|w(s)\|_{p+2}^2 \big],
\end{aligned}$$

and $J_f(t) := J_t(w(t), w_t(t))$ is defined in (4.30). Hence, applying Proposition 4.8 with $\phi := \mathcal{E}_w$ and $K = K_f$ in (4.43), by noting that $K_f(t)$ is a non-decreasing function, and regarding (4.31), we conclude

$$\mathcal{E}_w(t) \leqslant \big[C_B^{-1} q (t-1)^+ + \big(\sup_{0 \leqslant s \leqslant 1} ||(w(s), w_t(s))||_{\mathcal{H}}^2 \big)^{-q} \big]^{-\frac{1}{q}} + \big[K_f(t) \big]^{\frac{1}{q+1}}.$$

Finally, since $0 < \frac{1}{q+1} < 1$, we have $|a+b|^{\frac{1}{q+1}} \leq |a|^{\frac{1}{q+1}} + |b|^{\frac{1}{q+1}}$, and then (4.29) holds true with $J_f(t)$ given in (4.30). $\square$

Corollary 4.10 (Stabilizability Estimate)**.** *Under the same assumptions and statements of Proposition* 4.9*, there exists a constant* $C_B > 0$ *such that*

$$\|S_\lambda(t)U^1 - S_\lambda(t)U^2\|_{\mathcal{H}}^2 \leq \Big[C_B^{-1} q(t-1)^+ + \big(\sup_{0\leq s\leq 1} \|(w(s), w_t(s))\|_{\mathcal{H}}^2\big)^{-q}\Big]^{-\frac{1}{q}}$$

$$+ C_B \sup_{0\leq s\leq t+1} \Big[\|A^\alpha w(s)\|^{\frac{2(q+1)}{2q+1}} + \|w(s)\|_{p^*}^{\frac{2(q+1)}{2q+1}}\Big]^{\frac{1}{q+1}}. \tag{4.44}$$

for all $t > 0$ *and* $\lambda \in [0,1]$.

Proof. By means of the condition (3.2), MVT, Hölder's inequality with $\frac{p}{p^*} + \frac{1}{p^*} + \frac{1}{2} = 1$, Young's inequality with $\frac{2q+1}{2(q+1)} + \frac{1}{2(q+1)} = 1$, and the embedding $\mathcal{D}(A_1^{1/2}) \hookrightarrow L^{p^*}$, one can estimate

$$|\big(F(w(t)), w_t(t)\big)| \leq C_B \|w(t)\|_{p^*}^{\frac{2(q+1)}{2q+1}} + \frac{\gamma}{2^{2(q+1)}}\big\|w_t(t)\big\|^{2(q+1)}, \quad t \geq 0,$$

for some constant $C_B > 0$. Therefore, we go back to (4.36) and proceed verbatim the proof of Proposition 4.9 to conclude (4.44), where we also note that $L^{p^*} \hookrightarrow L^{p+2}$. □

Remark 4.4. Although Corollary 4.10 provides a "milder" stabilizability estimate with respect to L^{p^*}-norm, it will be useful to reach an estimate for Kolmogorov's ε-entropy of the attractor $\mathfrak{A}_\lambda$ corresponding to the dynamical system $(\mathcal{H}, S_\lambda(t))$ in sub-critical aspects with respect to parameters α and p^*.

The next result is an extended version of some results presented in [32] for second-order wave problems. We have raised it to our concerns in higher-order Sobolev spaces on bounded domains, namely, in $\mathcal{H} = \mathcal{D}(A_1^{\frac{1}{2}}) \times H$, but we notify that its proof follows the same lines as in [32, Section 2]. It will be helpful in our next result on asymptotic smoothness.

Proposition 4.11. *Let* f *be a function satisfying Assumption* 3.1*. Let us also consider* $\big\{\big(u^n, u_t^n\big)\big\}$ *be a weakly-star convergent sequence in* $L^\infty\big(s, T; \mathcal{H}\big)$ *with* $0 \leq s < T$*. Then*

$$\lim_{n\to\infty} \lim_{m\to\infty} \int_s^T \big(f\big(u^n(t)\big) - f\big(u^m(t)\big), u_t^n(t) - u_t^m(t)\big) dt = 0. \tag{4.45}$$

Proof. It is similar to the statements provided in [32, Lemmas 2.1 and 2.2], but with proper modification on the functional spaces and additional minor adjustments.

For the sake of the reader, and in order to guarantee, we can do state such a result in our case, we present the detailed proof in Appendix A.3. □

4.4. **Lipschitz property with respect to index λ.** Below, we are going to prove the λ-Lipschitz property, and therefore continuity, of the following mapping $[0,1] \ni \lambda \mapsto S_\lambda(t)U_0 \in \mathcal{H}$, for all given $t \geq 0$ and $U_0 \in B$, where $B \subset \mathcal{H}$ is bounded set. To analyze such a Lipschitz continuity in terms of the parameter λ, we turn ourselves back to the notation u^λ to the solution of (3.4), that is, to the dynamical system $S_\lambda(t)U_0 = (u^\lambda(t), u_t^\lambda(t))$ given in (3.20).

Proposition 4.12 (λ-Lipschitz Property)**.** *Under the assumptions of Theorem* 3.2, *let us consider an arbitrary bounded set $B \subset \mathcal{H}$ and denote by $S_\lambda(t)U_0 = (u^\lambda(t), u_t^\lambda(t))$ the trajectory solution corresponding to initial data $U_0 = (u_0, u_1) \in B$. Then, for any given index $\lambda_0 \in [0,1]$, there exists a positive non-decreasing function $\hat{\mathcal{Q}}(t) = \hat{\mathcal{Q}}\big(B, ||h||, t\big)$ such that*

$$\|S_\lambda(t)U_0 - S_{\lambda_0}(t)U_0\|_{\mathcal{H}} \leq \hat{\mathcal{Q}}(t)|\lambda - \lambda_0|, \quad t \geq 0. \tag{4.46}$$

Proof. Let us fix $\lambda_0 \in [0,1]$ and set $w^\lambda := u^\lambda - u^{\lambda_0}$. Then, w^λ is a solution (in the weak and strong sense) of the following problem

$$\begin{cases} w_{tt}^\lambda + \kappa A w^\lambda + A_1 w^\lambda + \dfrac{\gamma}{2}\Pi_1^\lambda w_t^\lambda + \dfrac{\gamma}{2}\Pi_2^\lambda\big[u_t^\lambda + u_t^{\lambda_0}\big] + F(w^\lambda) = h_{\lambda-\lambda_0}, \\ w^\lambda(0) = 0, \quad w_t^\lambda(0) = 0, \end{cases} \tag{4.47}$$

where hereafter we take the advantage of notations and estimates introduced in the proof of Theorem 3.4, namely, we first set $F(w^\lambda) = f(u^\lambda) - f(u^{\lambda_0})$ and

$$\Pi_i^\lambda(t) = \|S_\lambda(t)U_0\|_{\mathcal{H}^\alpha}^{2q} + (-1)^{1-i}\|S_{\lambda_0}(t)U_0\|_{\mathcal{H}^\alpha}^{2q}, \quad i = 1, 2.$$

Taking the multiplier w_t^λ in (4.47), we have

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}_w^\lambda(t) + \frac{\gamma}{2}\Pi_1^\lambda(t)\|w_t^\lambda(t)\|^2 = \mathcal{J}_1^\lambda(t) + \mathcal{J}_2^\lambda(t) + (\lambda - \lambda_0)(h, w_t^\lambda(t)), \tag{4.48}$$

where

$$\begin{aligned} \mathcal{E}_w^\lambda(t) &= \|w_t^\lambda(t)\|^2 + \kappa\|A^{\frac{1}{2}}w^\lambda(t)\|^2 + \|A_1^{\frac{1}{2}}w^\lambda(t)\|^2, \\ \mathcal{J}_1^\lambda(t) &= -\big(F(w^\lambda(t)), w_t^\lambda(t)\big), \\ \mathcal{J}_2^\lambda(t) &= -\frac{\gamma}{2}\Pi_2^\lambda(t)\big(u_t^\lambda(t) + u_t^{\lambda_0}(t), w_t^\lambda(t)\big). \end{aligned}$$

Repeating the same arguments as in (3.16), (3.17), and (3.18), we infer

$$||S_\lambda(t)U_0 - S_{\lambda_0}(t)U_0||_{\mathcal{H}}^2 \leq \mathcal{E}_w(t) \leq \mu_\kappa ||S_\lambda(t)U_0 - S_{\lambda_0}(t)U_0||_{\mathcal{H}}^2, \quad t \geq 0, \tag{4.49}$$

and

$$|\mathcal{J}_1^\lambda(t)|, \, |\mathcal{J}_2^\lambda(t)| \leq C_B\mathcal{E}_w^\lambda(t), \quad t \geq 0,$$

for some constant $C_B > 0$. Additionally, using Hölder and Young's inequalities and since $\mathcal{D}(A_1^{\frac{1}{2}}) \hookrightarrow H$, we get

$$|(\lambda - \lambda_0)(h, w_t^\lambda(t))| \leq \frac{1}{2}|\lambda - \lambda_0|^2 \|h\|^2 + \frac{1}{2}\mathcal{E}_w^\lambda(t).$$

Replacing the latter two estimates in (4.48), we have

$$\frac{d}{dt}\mathcal{E}_w^\lambda(t) \leq C_B \mathcal{E}_w^\lambda(t) + |\lambda - \lambda_0|^2 \|h\|^2,$$

for some constant $C_B > 0$, and integrating it on $(0, t)$, we arrive at

$$\mathcal{E}_w^\lambda(t) \leq C_B \int_0^t \mathcal{E}_w^\lambda(s)\, ds + t|\lambda - \lambda_0|^2 \|h\|^2, \quad t > 0.$$

Therefore, from Gronwall's inequality and (4.49), we finally conclude that (4.46) holds true with $\hat{\mathcal{Q}}(t) := t^{\frac{1}{2}} e^{C_B t/2} \|h\|$. □

5. Long-time dynamics: case (k.1)

Before proceeding with the main results of this article, we notify for the reader's guidance that all abstract concepts and results on dynamical systems are reminded in Appendix B, by following e.g. the references [3, 7, 12, 16, 17, 18, 21, 26, 27, 35, 43, 47].

5.1. Attractors and continuity: cases $\alpha \in [0,1)$ and $p^* \leq \frac{2n}{n-4}$. We initially remark that all results up to now hold for any critical parameters $\alpha \in [0,1]$ and $p^* \leq \frac{2n}{n-4}$. However, due to compactness issues inside this case (**k.1**), the next theorem dealing with the existence of a family of attractors and its continuity will require the subcritical assumption with respect to to fractional powers α, namely, $\alpha \in [0,1)$, but we still work in the critical scenario with respect to the source growth exponent $p^* \leq \frac{2n}{n-4}$.

Theorem 5.1. *Let us take on the same assumptions of Theorem* 3.2, *with the additional condition $\alpha \in [0,1)$, and let $(\mathcal{H}, S_\lambda(t))$ be the dynamical system given by* (3.20). *Then. we have*

- (**I.1**) **Asymptotic Smoothness/Compactness.** *For every $\lambda \in [0,1]$, the dynamical system $(\mathcal{H}, S_\lambda(t))$ is asymptotically smooth/compact.*
- (**I.2**) **Family of Attractors.** *For every $\lambda \in [0,1]$, the dynamical system $(\mathcal{H}, S_\lambda(t))$ possesses a global attractor $\mathfrak{A}_\lambda \subset \mathcal{H}$, which is compact and connected.*

(**I.3**) **Geometrical Structure.** *The family of global attractors* $\{\mathfrak{A}_\lambda\}_{\lambda\in[0,1]}$ *is characterized by the unstable manifold emanating from the set of stationary solutions, namely, we have* $\mathfrak{A}_\lambda = \mathbf{M}^u(\mathcal{N}_\lambda)$ *with*

$$\mathcal{N}_\lambda = \big\{(u,0)\in\mathcal{H};\ \kappa Au + A_1u + f(u) = h_\lambda\big\}, \quad \lambda\in[0,1]. \tag{5.1}$$

(**I.4**) **Equilibria Set.** *Every trajectory stabilizes to the set* $\mathcal{N}_\lambda$ *in the sense that*

$$\lim_{t\to+\infty} \operatorname{dist}(S_\lambda(t)U_0, \mathcal{N}_\lambda) = 0, \quad \forall\, U_0\in\mathcal{H}.$$

In particular, the set $\mathcal{N}_\lambda := \mathfrak{A}_\lambda^{\min}$ *is a global minimal attractor for every* $\lambda\in[0,1]$. *Moreover, any trajectory from* $\mathfrak{A}_\lambda$ *has an upper bound in terms of* $\mathfrak{A}_\lambda^{\min}$, *namely,*

$$\sup\big\{||(u,u_t)||_{\mathcal{H}};\ (u,u_t)\in\mathfrak{A}_\lambda\big\} \le \sup\big\{||(u,0)||_{\mathcal{H}};\ (u,0)\in\mathfrak{A}_\lambda^{\min}\big\}. \tag{5.2}$$

(**I.5**) **Non-triviality.** *The family of minimal attractors* $\{\mathfrak{A}_\lambda^{\min}\}_{\lambda\in[0,1]}$ *is nontrivial. In other words, even if* $h\equiv 0$, *the minimal attractor* $\mathfrak{A}_\lambda^{\min}$ *has at least two stationary solutions for every* $\lambda\in[0,1]$.

(**I.6**) **Triviality.** *If we additionally suppose that* $h\equiv 0$ *and* $C_f = 0$ *in* (3.3), *the attractor* $\mathfrak{A}_0$ *is trivial. More precisely,* $\mathfrak{A}_0 = \{(0,0)\}$ *with polynomial attraction depending on the exponent* $q\ge\frac{1}{2}$ *as follows*

$$\operatorname{dist}_{\mathcal{H}}(S_\lambda(t)B,\mathfrak{A}_0) = \sup_{U_0\in B}\|S_\lambda(t)U_0\|_{\mathcal{H}} \le \frac{C_B}{[c_B + q\hat{c}_B\, t]^{1/2q}}, \tag{5.3}$$

$t\to\infty$, *for any initial data* U_0 *lying in bounded sets* $B\subset\mathcal{H}$, *where* C_B, c_B, $\hat{c}_B > 0$ *are constants depending on* B.

(**I.7**) **Upper Semicontinuity.** *The family of global attractors* $\{\mathfrak{A}_\lambda\}_{\lambda\in[0,1]}$ *is upper semicontinuous at any fixed* $\lambda_0\in[0,1]$, *that is,*

$$\lim_{\lambda\to\lambda_0} \operatorname{dist}_{\mathcal{H}}\big(\mathfrak{A}_\lambda,\mathfrak{A}_{\lambda_0}\big) = 0.$$

(**I.8**) **Residual Continuity.** *The family of global attractors* $\{\mathfrak{A}_\lambda\}_{\lambda\in[0,1]}$ *is continuous in a residual*[2] *set* $I\subset[0,1]$, *that is, for any* $\lambda_0\in I$,

$$\lim_{\lambda\to\lambda_0}\big[\operatorname{dist}_{\mathcal{H}}\big(\mathfrak{A}_{\lambda_0},\mathfrak{A}_\lambda\big) + \operatorname{dist}_{\mathcal{H}}\big(\mathfrak{A}_\lambda,\mathfrak{A}_{\lambda_0}\big)\big] = 0.$$

In particular, the set of continuity points of $\mathfrak{A}_\lambda$ *is dense in* $[0,1]$.

---

[2]Let X be a complete metric space and $Y\subset X$. We recall that Y is *residual* in X if $X\backslash Y$ is a countable union of nowhere dense sets.

Proof. We gather the ingredients coming from Section 4 along with the abstract results reminded in Appendix B.

(**I.1**) Let us initially consider a bounded positively invariant set $B \subset \mathcal{H}$, take two trajectory solutions $S_\lambda(t)U^i = (u^i(t), u^i_t(t))$, $i = 1, 2$, corresponding to initial data $U^i = (u^i_0, u^i_1) \in B$, $i = 1, 2$, and consider any $\varepsilon > 0$.

From the key inequality (4.29), there exists a time large enough $T := T_{\lambda,B} > 0$ such that

$$\|S_\lambda(T)U^1 - S_\lambda(T)U^2\|_{\mathcal{H}} \quad \leq \quad \varepsilon + \psi_{\varepsilon,B,T}(U^1, U^2), \tag{5.4}$$

where we set $\psi_{\varepsilon,B,T} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$\begin{aligned}\psi_{\varepsilon,B,T}(U^1, U^2) : = C_B \sup_{0\leq s\leq T+1} \Big[\|A^\alpha u^1(s) - A^\alpha u^2(s)\|^{\frac{2(q+1)}{2q+1}} \\ + \|u^1(s) - u^2(s)\|^2_{p+2}\Big]^{\frac{1}{2(q+1)}} + \big[J_f(T)\big]^{\frac{1}{2(q+1)}}\end{aligned} \tag{5.5}$$

for some constant $C_B > 0$ and $J_f(t)$ given in (4.30).

Now, given a sequence of initial data $U^n = (u^n_0, u^n_1) \in B$, as before, we write $S_\lambda(t)U^n = (u^n(t), u^n_t(t))$. Since B is invariant by $S_\lambda(t)$, $t \geq 0$, it follows that $(u^n(t), u^n_t(t))$ are uniformly bounded in $\mathcal{H} = \mathcal{D}(A_1^{\frac{1}{2}}) \times H$ from Corollary 4.3. Thus, (u^n, u^n_t) is bounded in $C([0, T + 1], \mathcal{H})$.

Below is the precise moment we invoke the assumption $\alpha \in [0, 1)$. Indeed, for such fractional powers and any $p^* \leq \frac{2n}{n-4}$, we use the fact that $\mathcal{D}(A_1^{\frac{1}{2}}) \hookrightarrow \mathcal{D}(A^\alpha)$ and $\mathcal{D}(A_1^{\frac{1}{2}}) \hookrightarrow L^{p+2}(\Omega)$ are compact embeddings. Then, by virtue of [45, Corollary 4] there exists a subsequence, still denoted by (u^n), such that

$$(u^n) \;\; \text{converges strongly in} \;\; C([0, T + 1], \mathcal{D}(A^\alpha) \cap L^{p+2}(\Omega)). \tag{5.6}$$

Therefore,

$$\lim_{n\to\infty} \lim_{m\to\infty} \Big[\|A^\alpha u^n(s) - A^\alpha u^m(s)\|^{\frac{2(q+1)}{2q+1}} + \|u^n(s) - u^m(s)\|^2_{p+2}\Big]^{\frac{1}{2(q+1)}} = 0, \tag{5.7}$$

for every $s \in [0, T + 1]$. Additionally, from (5.6) and (4.45), and the expression for $J_f(t)$ in (4.30), we also have

$$\lim_{n\to\infty} \lim_{m\to\infty} J_f(s) = \lim_{n\to\infty} \lim_{m\to\infty} J_f(u^n(s) - u^m(s), u^n_t(s) - u^m_t(s)) = 0. \tag{5.8}$$

Then, from (5.7) and (5.8), we conclude

$$\lim_{n\to\infty} \lim_{m\to\infty} \psi_{\varepsilon,B,T}(U^n, U^m) = 0, \tag{5.9}$$

for every sequence $U^n \in B$, which proves that $\psi_{\lambda,B,T}$ is a contractive function on $B \times B$.

Therefore, from (5.4) and (5.9), we can apply Theorem B.1 to conclude that $(\mathcal{H}, S_\lambda(t))$ is asymptotically smooth. It is also asymptotically compact in view of Proposition B.2.

(**I.2**) It follows from Corollary 4.2 and step (**I.1**), in combination with Theorem B.3.

(**I.3**) It follows from Corollary 4.4 along with Theorem B.4.

(**I.4**) It follows from (**I.3**), Corollary 4.4, and Theorem B.5. Moreover, the upper bound (5.2) follows from the first limit in Theorem B.4, Corollary 4.4, and Lemma 4.6, once we have that the Lyapunov function $\Phi := E$ is topologically equivalent to the norm of the phase space $\mathcal{H}$ (see also [18, Remark 7.5.8]).

(**I.5**) It follows directly from Lemma 4.7.

(**I.6**) It follows directly from the second part of Lemma 4.6, by applying Corollary 4.5 for initial data lying in bounded sets $B \subset \mathcal{H}$.

(**I.7**) From Corollary 4.2, the dynamical system $(\mathcal{H}, S_\lambda(t))$ has a bounded absorbing set $\mathcal{B} \subset \mathcal{H}$ uniformly bounded with respect to $\lambda \in [0,1]$. Thus, the attractors $\mathfrak{A}_\lambda \subset \mathcal{B}$ are uniformly bounded for all $\lambda \in [0,1]$. Additionally, from Proposition 4.12, we have

$$\lim_{\lambda\to\lambda_0} \sup_{U_0\in\mathcal{B}} \|S_\lambda(t)U_0 - S_{\lambda_0}(t)U_0\|_{\mathcal{H}} = 0, \quad t \geq t_0, \tag{5.10}$$

for every given $t_0 \geq 0$. Therefore, employing Theorem B.6, we conclude that the family $\{\mathfrak{A}_\lambda\}_{\lambda\in[0,1]}$ is upper semicontinuous at any fixed $\lambda_0 \in [0,1]$.

(**I.8**) By noting that (5.10) is valid for all $t_0 > 0$ and any bounded set $B \subset \mathcal{H}$ (see again Proposition 4.12), then we are under the assumptions of Theorem B.7. Therefore, the desired conclusion on residual continuity follows. □

5.2. **Kolmogorov ε-Entropy: cases $\alpha \in [0,1)$ and $p^* < \frac{2n}{n-4}$.** As observed in Remark 4.4, we are going to appeal to Corollary 4.10 in order to reach an estimate for the Kolmogorov's ε-entropy of the attractor $\mathfrak{A}_\lambda$ corresponding to the dynamical system $(\mathcal{H}, S_\lambda(t))$. For this reason (see the "weaker" stabilizability inequality (4.44)), we must employ the subcritical case with respect to parameters α and p^*. Our second main result reads as follows.

Theorem 5.2 (**Kolmogorov ε-Entropy).** *Let us consider the assumptions of Theorem* 5.1*, with the additional condition $p < \frac{4}{n-4}$ in Assumption* 3.1*. Then, there exists $0 < \varepsilon_0 < 1$ such that for all $\varepsilon \leq \varepsilon_0 < 1$, the Kolmogorov*

$\varepsilon-$entropy $H_\varepsilon(\mathfrak{A}_\lambda)$ of the existing global attractor $\mathfrak{A}_\lambda$, $\lambda \in [0,1]$, satisfies the following estimate for arbitrary $\delta \in (0,1)$

$$H_\varepsilon(\mathfrak{A}_\lambda) \leq \frac{2}{1-\delta}\int_\varepsilon^{\varepsilon_0} \frac{\ln m\big(g_\delta^{-1}(s), z(s)\big)}{s} ds + H_{g_\delta\left(\varepsilon_0\right)}(\mathfrak{A}_\lambda), \tag{5.11}$$

where $g_\delta(s) = \frac{1+\delta}{2}s$ and $z(s) = \frac{1}{2}(\delta s)^{2(q+1)}$, $0 < s < \varepsilon_0$, and

$$m(r,a) = \sup\{m(B,a);\ B \subseteq \mathfrak{A}_\lambda, \operatorname{diam} B \leq 2r\},$$

with $m(B,a)$ being the maximal number of elements $u_j^B \in B$ such that, for any $a>0$, we have $\varrho\big(S_\lambda(T^\star)u_j^B, S_\lambda(T^\star)u_i^B\big) > a, i \neq j, i,j = 1,\dots,m(B,a)$, for $T^\star > 0$ large enough, and ϱ is a pseudometric on $\mathcal{H}$.

Proof. Let us consider two trajectory solutions $S_\lambda(t)U^i = (u^i(t), u_t^i(t))$, $i = 1,2$, corresponding to initial data $U^i = (u_0^i, u_1^i) \in \mathfrak{A}_\lambda$, $i = 1,2$, $\lambda \in [0,1]$, and still denote $w := u^1 - u^2$. Since $\mathfrak{A}_\lambda$ is compact (and then bounded) and invariant $S_\lambda(t)\mathfrak{A}_\lambda = \mathfrak{A}_\lambda$, then $S_\lambda(t)U^i \in \mathfrak{A}_\lambda$ for all $t>0$. Additionally, from Corollary 4.10, there exists a time $T^\star := T^\star(\mathfrak{A}_\lambda) > 0$ such that

$$\begin{aligned} &||S_\lambda(T^\star)U^1 - S_\lambda(T^\star)U^2||_{\mathcal{H}} \\ &\quad\leq \frac{1}{2}\|U^1 - U^2\|_{\mathcal{H}} + \Big[C_{\mathfrak{A}_\lambda} \sup_{0\leq s\leq T^\star+1}\big(\|A^\alpha w(s)\| + \|w(s)\|_{p^*}\big)\Big]^{\frac{1}{2(q+1)}}, \end{aligned} \tag{5.12}$$

for some constant $C_{\mathfrak{A}_\lambda} > 0$ and all $U^1, U^2 \in \mathfrak{A}_\lambda$, where (by virtue of Corollary 4.3 and the identity $\frac{2(q+1)}{2q+1} = 1 + \frac{1}{2q+1}$), we have used

$$\|A^\alpha w(s)\|^{\frac{2(q+1)}{2q+1}} + ||w(s)||_{p^*}^{\frac{2(q+1)}{2q+1}} \leq C_{\mathfrak{A}_\lambda}\big(\|A^\alpha w(s)\| + ||w(s)||_{p^*}\big), \quad s \geq 0. \tag{5.13}$$

Moreover, from Theorem 3.4, there exists a constant $L^\star > 0$ such that

$$||S_\lambda(T^\star)U^1 - S_\lambda(T^\star)U^2||_{\mathcal{H}} \leq L^\star\|U^1 - U^2\|_{\mathcal{H}}, \quad \forall\, U^1, U^2 \in \mathfrak{A}_\lambda. \tag{5.14}$$

Therefore, one can see from (5.12)-(5.14) that assumptions 1 to 4 of Theorem B.8 are fulfilled with

$$\begin{aligned} &\mathcal{M} := \mathfrak{A}_\lambda, \quad V := S_\lambda(T^\star), \quad g(s) = \frac{1}{2}s, \quad h(s) := s^{\frac{1}{2(q+1)}}, \quad \varrho_1 \equiv 0, \\ &\varrho_2(S_\lambda(T^\star)U^1, S_\lambda(T^\star)U^2) := C_{\mathfrak{A}_\lambda} \sup_{0\leq s\leq T^\star+1}\big(\|A^\alpha w(s)\| + \|w(s)\|_{p^*}\big). \end{aligned} \tag{5.15}$$

It is worth mentioning that due to the compactness embedding $\mathcal{D}(A_1^{\frac{1}{2}}) \hookrightarrow \mathcal{D}(A^\alpha) \cap L^{p^*}$, then $\varrho := \varrho_2$ is a compact seminorm[3] on $\mathcal{D}(A_1^{\frac{1}{2}})$, and item 4

[3] We recall that a seminorm $n_X(\cdot)$ defined on a Banach space X is *compact* if whenever a sequence $x_j \to 0$ weakly in X one has $n_X(x_j) \to 0$.

of Theorem B.8 follows by a standard argument, see e.g. [16, p. 55] for a generic proof of this fact.

Hence, the estimate for the Kolmogorov ε-entropy $H_\varepsilon(\mathfrak{A}_\lambda)$ provided in (5.11) follows from the conclusion of Theorem B.8. □

Remark 5.1. As final information in this subsection, we note that both Theorems 5.1 and 5.2 remain unchanged when we neglect the potential energy in the damping coefficient $\mathcal{E}_\alpha(u, u_t)$ given in (1.2). In this case the equation in (3.4) reduces to the particular one with nonlinear averaged damping

$$u_{tt} + \kappa A u + A_1 u + \gamma \|u_t\|^{2q} u_t + f(u) = h_\lambda, \quad t > 0, \ \lambda \in [0, 1].$$

Therefore, all results in the present section provide a generalization of the ones in [49, 50] when one considers a constant coefficient of extensibility κ. Additionally, the same happens if one takes a slightly more general situation with respect to $\kappa(\cdot)$ as a nonlocal function under suitable properties like in [29, 30, 49, 50].

5.3. **Attempts for finite dimensionality.** Below, we try to clarify how hard (if not impossible) is to achieve the finiteness of the fractal dimension ($\dim_{\mathcal{H}}^f \mathfrak{A}_\lambda$) of the attractors $\mathfrak{A}_\lambda$, $\lambda \in [0, 1]$, corresponding to the dynamical system $(\mathcal{H}, S_\lambda(t))$ defined in (3.20) in case (**k.1**). A first attempt is trying to invoke Theorem B.9. In this way, in view of notations in (5.15), we can rewrite (5.12) as

$$\|VU^1 - VU^2\|_{\mathcal{H}} \leq \frac{1}{2}\|U^1 - U^2\|_{\mathcal{H}} + \left[\varrho(VU^1, VU^2)\right]^{\frac{1}{2(q+1)}}. \tag{5.16}$$

Therefore, all hypotheses of Theorem B.9 are satisfied except for item (ii) that requires the linearity of function $h(s) = s_0 s$. Indeed, this fact is impossible in our case since $h(s) = s^{\frac{1}{2(q+1)}}$ for $q \geq \frac{1}{2}$. The only chance to achieve (5.16) with linear function $h(s)$ is to consider the particular (and already known) case $q = 0$. But this latter especial case reduces the damping term in (3.4) to the linear one γu_t, $\gamma > 0$, which in turn is a particular case of (**k.2**) in what concerns function $k(\cdot)$. For such a case, we show later that $\dim_{\mathcal{H}}^f \mathfrak{A}_\lambda < \infty$, the regularity of any trajectory from the attractors and (generalized) fractal exponential attractors.

Going back to this "worse" case (**k.1**), another attempt in trying to achieve (5.16) (or else (5.12) which comes from Corollary 4.10) with a proper power concerning its last term is to regard perturbed energy computations instead of Nakao's method as in the proof of Proposition 4.9 and Corollary 4.10. In such a way, the following result can be proved.

Proposition 5.3. *Under the same assumptions and statements of Proposition* 4.9*, and given any* $\epsilon > 0$*, there exist constants* $c_B, C_B > 0$ *depending on* B *such that*

$$||S_\lambda(t)U^1 - S_\lambda(t)U^2||_{\mathcal{H}}^2 \leq C_B \big\| U^1 - U^2 \big\|_{\mathcal{H}}^2 e^{-c_B t} + \epsilon^2$$
$$+ C_B \int_0^t e^{-c_B(t-s)} \Big(\|A^\alpha w(s)\|^{\frac{2(q+1)}{2q+1}} + ||w(s)||_{p^*}^{\frac{2(q+1)}{2q+1}} \Big) ds, \quad (5.17)$$

for every $t > 0$*, where we still denote* $w = u^1 - u^2$*.*

Proof. The proof relies on energy perturbation and similar technical estimates as used in the proof of Proposition 4.9 and Corollary 4.10, along with proper Young's inequality. Thus, it will be omitted. □

Hence, as a consequence of (5.17) on $B := \mathfrak{A}_\lambda$, and using again (5.13), we arrive at

$$\|S_\lambda(T^\star)U^1 - S_\lambda(T^\star)U^2\|_{\mathcal{H}} \leq \frac{1}{2}\|U^1 - U^2\|_{\mathcal{H}} + \epsilon + \big[\varrho(S_\lambda(T^\star)U^1, S_\lambda(T^\star)U^2)\big]^{\frac{1}{2}}, \quad (5.18)$$

for some time $T^\star := T^\star(\mathfrak{A}_\lambda) > 0$ large enough, some compact seminorm ϱ, and any $\epsilon > 0$. Nonetheless, (5.18) is not enough to achieve the finiteness of the fractal dimension $\dim_{\mathcal{H}}^f \mathfrak{A}_\lambda$ by means of Theorem B.9.

The above approaches (5.16) and (5.18) for studying the dimensionality of the attractors $\mathfrak{A}_\lambda$ raise similar issues as presented in [16] in terms of (more) general stabilizability estimates. Indeed, this is the exact moment where we explore the difficulty imposed by function $k(s) = \gamma s^q$ for any $q \geq \frac{1}{2}$ in case (**k.1**) because under this structure the character of the functions $h(s) = s^{\frac{1}{2(q+1)}}$ and $h(s) = s^{\frac{1}{2}}$ present in the lower order terms (LOT)

$$\mathrm{LOT}\big(U^1, U^2\big) := \varrho(S_\lambda(T^\star)U^1, S_\lambda(T^\star)U^2)$$

are determined from the behavior of the nonlinear damping term

$$k\big(\mathcal{E}_\alpha(u, u_t)\big)u_t = \gamma\big[\mathcal{E}_\alpha(u, u_t)\big]^q u_t, \quad q \geq \frac{1}{2}, \quad (5.19)$$

with $\mathcal{E}_\alpha(u, u_t)$ being set in (1.2). Moreover, due to the computations in the proof of Proposition 4.9 (see (4.33)-(4.35)) it seems that the nonlocal nonlinear damping term (5.19) does not even provide a suitable coercivity property as usual for nonlinear damping like $D(u_t)$, where $D : \mathbb{R} \to \mathbb{R}$ is a real function with growth exponent q, namely,

$$(D(s) - D(r))(s - r) \geq c_q |s - r|^{q+2}, \quad \forall\ s, r \in \mathbb{R},\ q \geq 0, \quad (5.20)$$

or

$$(D(s)-D(r))(s-r) \geq c_q\big(|s|^q+|r|^q\big)|s-r|^2, \quad \forall\, s,r \in \mathbb{R},\ q \geq 0, \tag{5.21}$$

or else, for any given $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$C_\epsilon(D(s)-D(r))(s-r) \geq |s-r|^2-\epsilon, \quad \forall\, s,r \in \mathbb{R}. \tag{5.22}$$

It is worth mentioning that the above assumptions (5.20)-(5.22) have been extensively regarded in the literature in what concerns long-time dynamics of hyperbolic-type second-order evolution problems with nonlinear damping, see for instance the works [2, 11, 13, 14, 15, 17, 18, 19, 20, 24, 30, 33, 36, 40, 42, 46] where (at least) some of them do play an important role in finding the asymptotic smoothness of the corresponding nonlinear infinite-dimensional dynamical system.

Although conditions (5.20)-(5.22) have shown to be very effective in building the existence of compact global attractors, the picture is much more delicate when one deals with respect to regularity and especially finite-dimensionality of such global attractors. Indeed, among the above-mentioned works with nonlinear damping, those that proved the finiteness of fractal (or Hausdorff) dimension strongly used more hypotheses on D and its derivatives D', D'', see for instance [14, Theorem 1.4], [15, Theorem 1.5], [17, Theorem 5.8], [18, Theorem 9.2.6], [19, Assumption 3 and Theorem 3.5], [30, Proposition 4 and Theorem 3.2], [32, Theorems 3.1 and 3.2], [36, Theorem 8], and [42, Theorem 1.1], just to quote a few. Nonetheless, such damping controlling by means of its derivative does not seem to be applicable to the nonlinear damping (5.19) due to its nonlocal structure. This is similar to the nonlocal damping addressed in [2], where the author involves differential operators in space and covers a wide class of average nonlocal damping. Therefore, we conclude that (5.19) represents a generalization of the linear damping in a different way of the existing literature with respect to the nonlinear damping terms $D(u_t)$.

In conclusion, to our best knowledge, there is no keen theory to conclude the finiteness of the fractal (or at least Hausdorff) dimension for problems with non-local damping like (5.19) where the behavior of lower order terms are determined, in general, by the possibly degenerate function $k(s) = \gamma s^q$ over the linear energy coefficient $\mathcal{E}_\alpha(u, u_t)$. A way of circumvent this situation is when k is bounded from below but such a case is covered by the case (**k.2**) to be analyzed next.

6. Long-time dynamics: case (k.2)

As we are going to show below, this case is much more touchable in the sense that it provides the finiteness and regularity of the global attractor $\mathfrak{A}_\lambda \subset \mathcal{H}, \lambda \in [0,1]$, related to the dynamical system $(\mathcal{H}, S_\lambda(t))$ given by (3.20) in case (**k.2**). Moreover, in this case, we also prove the existence of (generalized) fractal exponential attractors $\mathfrak{A}_\lambda^{\exp} \subset \mathcal{H}$. Such statements are due to the positiveness of function $k(\cdot)$ in case (**k.2**), which allows us to control the behavior of the nonlocal damping (especially at the origin) and this makes this case "smoother" than the previous one.

We also note that due to the structure of damping in case (**k.2**), we will be able to work with critical parameters $\alpha \in [0,1]$ and $p^* \leq \frac{2n}{n-4}$ for obtaining technical estimates and the computations rely on similar techniques as previously used in the literature, see e.g. [10, 17, 25, 29, 30].

In this case, the essential technical results proved in Section 4 can be refined as follows.

Proposition 6.1 (Dissipativity). *Under the assumptions of Theorem* 3.1 *with* $k(\cdot)$ *given in case* (**k.2**), *there exist positive constants* $c = c_{\widetilde{E}(0)}$, $C = C_{\widetilde{E}(0)}$ *(which may depend on initial data) such that* $\widetilde{E}(t)$ *given in* (3.21) *satisfies*

$$\widetilde{E}(t) \leq C\widetilde{E}(0)e^{-ct} + 8K_\lambda, \quad t > 0. \tag{6.1}$$

In particular, the dynamical system $(\mathcal{H}, S_\lambda(t))$ *given by* (3.20) *is dissipative, say with (positively invariant) bounded absorbing set* $\mathcal{B} \subset \mathcal{H}$*, which is uniformly bounded with respect to* $\lambda \in [0,1]$*.*

Proof. Since $k(\cdot)$ is a C^1-function on $[0,\infty)$ such that $k(s) > 0$, $s \geq 0$, then the proof follows exactly the same lines as in [30, Proposition 1]. □

Additionally, Corollaries 4.3 to 4.5 can be adapted to this case as well. Also, and much more important, the following stability inequality can be reached.

Proposition 6.2 (Stabilizability Estimate). *Under the assumptions of Theorem* 3.1 *with* $k(\cdot)$ *given in case* (**k.2**), *let us consider a bounded set* $B \subset \mathcal{H}$ *with initial data* $U^i = (u_0^i, u_1^i) \in B$, $i = 1, 2$. *Still denoting by* $S_\lambda(t)U^i = (u^i(t), u_t^i(t)), i = 1, 2$, *the corresponding dynamical system* (3.20), *then there exist constants* $c_B, C_B > 0$ *(depending on* B*) such that*

$$\begin{aligned} ||S_\lambda(t)U^1 - S_\lambda(t)U^2||_{\mathcal{H}}^2 &\leq C_B e^{-c_B t}||U^1 - U^2||_{\mathcal{H}}^2 \\ &\quad + C_B \int_0^t e^{-c_B(t-s)}\Big(\|A^\alpha w(s)\|^2 + ||w(s)||_{p+2}^2\Big) ds, \end{aligned} \tag{6.2}$$

for all $t > 0$ *and* $\lambda \in [0,1]$, *where* $w = u^1 - u^2$.

Proof. The proof is analogous to [29, Proposition 1] with help of similar arguments as in [10, Lemma 4.9] to handle critical exponent. □

The motivation for getting (6.2) with respect to critical growth exponent p came from the results in [17, Proposition 4.13] and [25, Lemma 7.1]. Finally, we also note that Proposition 4.12 can be proved in this case in a very similar way. Therefore, we are able to state our main results in the present section as follows.

6.1. **Attractors, continuity, finite dimensionality, and regularity: cases** $\alpha \in [0,1)$ **and** $p^* \leq \frac{2n}{n-4}$**.** As in Subsection 5.1, the above technical estimates in the present case hold for any critical parameters $\alpha \in [0,1]$ and $p^* \leq \frac{2n}{n-4}$, but due to the compactness issues involving the parameter α (which comes now from (6.2)), we must work in the subcritical case with respect to it. Nonetheless, nothing changes with respect to the source growth exponent $p^* \leq \frac{2n}{n-4}$.

Theorem 6.3. *Let us take on the same assumptions of Theorem* 3.1 *with* $k(\cdot)$ *given in case* (**k.2**)*. We additionally assume that* $\alpha \in [0,1)$*. Then, the dynamical system* $(\mathcal{H}, S_\lambda(t))$ *set in* (3.20) *has the following properties:*

- (**J.1**) **Quasi-Stability.** *The dynamical system* $(\mathcal{H}, S_\lambda(t))$ *is asymptotically quasi-stable on any positively invariant bounded set* $B \subset \mathcal{H}$*, for every* $\lambda \in [0,1]$*. In particular, it is asymptotically smooth/compact.*
- (**J.2**) **Family of Attractors.** *For every* $\lambda \in [0,1]$*, the dynamical system* $(\mathcal{H}, S_\lambda(t))$ *possesses a global attractor* $\mathfrak{A}_\lambda \subset \mathcal{H}$*, which is compact and connected.*
- (**J.3**) **Geometrical Structure.** *The family of global attractors* $\{\mathfrak{A}_\lambda\}_{\lambda\in[0,1]}$ *is characterized by the unstable manifold emanating from the set of stationary solutions, namely, we have* $\mathfrak{A}_\lambda = \mathbf{M}^u(\mathcal{N}_\lambda)$ *with*

$$\mathcal{N}_\lambda = \big\{(u,0) \in \mathcal{H};\ \kappa A u + A_1 u + f(u) = h_\lambda\big\}, \quad \lambda \in [0,1].$$

- (**J.4**) **Equilibria Set.** *Every trajectory stabilizes to the set* $\mathcal{N}_\lambda$ *in the sense that*

$$\lim_{t\to+\infty} \operatorname{dist}(S_\lambda(t)U_0, \mathcal{N}_\lambda) = 0, \quad \forall\, U_0 \in \mathcal{H}.$$

In particular, the set $\mathcal{N}_\lambda := \mathfrak{A}_\lambda^{\min}$ *is a global minimal attractor for every* $\lambda \in [0,1]$*. Moreover, any trajectory from* $\mathfrak{A}_\lambda$ *has an upper bound in terms of* $\mathfrak{A}_\lambda^{\min}$*, namely,*

$$\sup\big\{||(u,u_t)||_{\mathcal{H}};\ (u,u_t) \in \mathfrak{A}_\lambda\big\} \leq \sup\big\{||(u,0)||_{\mathcal{H}};\ (u,0) \in \mathfrak{A}_\lambda^{\min}\big\}.$$

(**J.5**) **Non-triviality.** *The family of minimal attractors* $\{\mathfrak{A}_\lambda^{\min}\}_{\lambda\in[0,1]}$ *is nontrivial. In other words, even if* $h \equiv 0$, *the minimal attractor* $\mathfrak{A}_\lambda^{\min}$ *has at least two stationary solutions for every* $\lambda \in [0,1]$.

(**J.6**) **Triviality.** *If we additionally suppose that* $h \equiv 0$ *and* $C_f = 0$ *in* (3.3), *the attractor* $\mathfrak{A}_0 = \{(0,0)\}$ *is trivial with exponential attraction as follows*

$$\operatorname{dist}_{\mathcal{H}}(S_\lambda(t)B, \mathfrak{A}_0) = \sup_{U_0\in B} \|S_\lambda(t)U_0\|_{\mathcal{H}} \le C_B e^{-c_B t}, \quad t \to \infty,$$

for any initial data $U_0 \in B \subset \mathcal{H}$, *where* C_B, $c_B > 0$ *are constants depending on* B.

(**J.7**) **Upper Semicontinuity.** *The family of global attractors* $\{\mathfrak{A}_\lambda\}_{\lambda\in[0,1]}$ *is upper semicontinuous at any fixed* $\lambda_0 \in [0,1]$, *that is,*

$$\lim_{\lambda\to\lambda_0} \operatorname{dist}_{\mathcal{H}}\big(\mathfrak{A}_\lambda, \mathfrak{A}_{\lambda_0}\big) = 0.$$

(**J.8**) **Residual Continuity.** *The family of global attractors* $\{\mathfrak{A}_\lambda\}_{\lambda\in[0,1]}$ *is continuous in a residual set* $J \subset [0,1]$, *that is, for any* $\lambda_0 \in J$,

$$\lim_{\lambda\to\lambda_0} \Big[\operatorname{dist}_{\mathcal{H}}\big(\mathfrak{A}_{\lambda_0}, \mathfrak{A}_\lambda\big) + \operatorname{dist}_{\mathcal{H}}\big(\mathfrak{A}_\lambda, \mathfrak{A}_{\lambda_0}\big)\Big] = 0.$$

In particular, the set of continuity points of $\mathfrak{A}_\lambda$ *is dense in* $[0,1]$.

(**J.9**) **Finite Dimensionality.** *The compact global attractor* $\mathfrak{A}_\lambda$ *has finite fractal dimension*

$$dim_{\mathcal{H}}^f\big(\mathfrak{A}_\lambda\big) < \infty, \quad \lambda \in [0,1].$$

(**J.10**) **Regularity.** *Any trajectory* $\Gamma = \{(u(t); u_t(t));\ t \in \mathbb{R}\} \subset \mathfrak{A}_\lambda$ *has the following regularity*

$$(u_t, u_{tt}) \in L^\infty(\mathbb{R}; \mathcal{H}). \tag{6.3}$$

Moreover, there exists a constant $R > 0$ *such that*

$$\sup_{\Gamma\subset\mathfrak{A}_\lambda} \sup_{t\in\mathbb{R}} \|(u_t(t), u_{tt}(t))\|_{\mathcal{H}}^2 \le R^2. \tag{6.4}$$

(**J.11**) **Generalized Fractal Exponential Attractor.** *The dynamical system* $(\mathcal{H}, S_\lambda(t))$ *possesses a generalized fractal exponential attractor* $\mathfrak{A}_\lambda^{\exp}$ *with finite fractal dimension* $\big(dim_{\mathcal{H}^{-s}}^f\big(\mathfrak{A}_\lambda^{\exp}\big) < \infty\big)$ *in the extended space*

$$\mathcal{H}^{-s} := \mathcal{D}\big(A_1^{(1-s)/2}\big) \times \mathcal{D}\big(A_1^{-s/2}\big), \quad 0 < s \le 1. \tag{6.5}$$

Proof. The proof is a consequence of Propositions 6.1 and 6.2 in combination with the abstract results reminded in Appendix B. It follows analogously to Theorem 5.1 except for the item (**J.1**) and the further items (**J.9**)-(**J.11**).

(**J.1**) Let us consider a bounded positively invariant set $B \subset \mathcal{H}$ and two trajectory solutions $S_\lambda(t)U^i = (u^i(t), u_t^i(t))$, $i = 1, 2$, with initial data $U^i = (u_0^i, u_1^i) \in B$, $i = 1, 2$.

Now, in view of the stabilizability estimate (6.2) we have

$$||S_\lambda(t)U^1 - S_\lambda(t)U^2||_{\mathcal{H}}^2 \leq a_1(t)||U^1 - U^2||_{\mathcal{H}}^2 + a_2(t) \sup_{0<s<t} \left[n(u^1(s) - u^2(s))\right]^2, \quad (6.6)$$

where

$$a_1(t) := C_B e^{-c_B t}, \quad a_2(t) := C_B \int_0^t e^{-c_B(t-s)}\, ds, \quad t > 0,$$

and

$$n(u) := \|A^\alpha u\|^2 + \|u\|_{p+2}^2 = \|u\|_{\mathcal{D}(A^\alpha) \cap L^{p+2}}$$

is a compact seminorm once here the embedding $\mathcal{D}(A_1^{\frac{1}{2}}) \hookrightarrow \mathcal{D}(A^\alpha) \cap L^{p^*}$ is compact.

From (3.12) and (6.6) one can see that the dynamical system $(\mathcal{H}, S_\lambda(t))$ satisfies the required conditions (B.5)-(B.8) with

$$\begin{array}{l} X := \mathcal{D}(A_1^{\frac{1}{2}}), \quad Y := H, \quad Z := \{0\}, \\ a(t) := \mathcal{Q}(t), \quad b(t) := a_1(t), \quad c(t) := a_2(t). \end{array} \quad (6.7)$$

Therefore, $(\mathcal{H}, S_\lambda(t))$ is asymptotically quasi-stable on $B \subset \mathcal{H}$. In particular, by Proposition B.11 it is also asymptotically smooth.

(**J.2**)-(**J.8**) It follows verbatim the same arguments as in (**I.2**)-(**I.8**).

(**J.9**)-(**J.10**) From the first items (**J.1**) and (**J.2**), $(\mathcal{H}, S_\lambda(t))$ is asymptotically quasi-stable on the global attractor $\mathfrak{A}_\lambda \subset \mathcal{H}$ for every $\lambda \in [0, 1]$. Thus, from Theorem B.12 one gets $\dim_{\mathcal{H}}^f(\mathfrak{A}_\lambda) < \infty$ as desired. Moreover, since $a_\infty = \sup_{t \in \mathbb{R}^+} a_2(t) < \infty$, then the properties (6.3)-(6.4) follows from Theorem B.13.

(**J.11**) From Proposition 6.1 and item (**J.1**), the dynamical system $(\mathcal{H}, S_\lambda(t))$ is asymptotically quasi-stable on the positively invariant bounded absorbing set $\mathcal{B}$. In what follows, for any $U_0 \in \mathcal{B}$, we are going to prove that mapping

$$t \mapsto S_\lambda(t)U_0 := (u(t), u_t(t)) \quad (6.8)$$

is Hölder continuous in $\mathcal{H}^{-s}$ in (6.5) for any $0 < s \le 1$. Let us start with $s = 1$. From the well-posedness result, one can infer $(u_t, u_{tt}) \in L^\infty_{\text{loc}}\,(\mathcal{H}^{-1})$. Thus, by taking $U_0 \in \mathcal{B}$, $T > 0$, and any t_1, $t_2 \in [0, T]$, we get

$$\begin{aligned}\|S_\lambda(t_2)U_0 - S_\lambda(t_1)U_0\|_{\mathcal{H}^{-1}} &\le \int_{t_1}^{t_2} \big\|\frac{d}{ds}(u(s), u_t(s))\big\|_{\mathcal{H}^{-1}} ds\\ &\le \Big(\int_0^T \big\|(u_t(s), u_{tt}(s))\big\|^2_{\mathcal{H}^{-1}} ds\Big)^{\frac12} |t_2 - t_1|^{\frac12} \le C_{\mathcal{B},T}|t_2 - t_1|^{\frac12},\end{aligned}$$

that is, $t \mapsto S(t)U_0$ is Hölder continuous in $\mathcal{H}^{-1}$. Besides, for $0 < s < 1$, one has from the above case and interpolation theorem that

$$\|S_\lambda(t_2)U_0 - S_\lambda(t_1)U_0\|_{\mathcal{H}^{-s}} \le C_{s,\mathcal{B},T}|t_2 - t_1|^{s/2}, \quad t_1,\, t_2 \in [0, T],$$

for some constant $C_{s,\mathcal{B},T} > 0$, which proves the Hölder continuity in $\mathcal{H}^{-s}$.

Hence, from Theorem B.14 the dynamical system $(\mathcal{H}, S_\lambda(t))$ has a generalized fractal exponential attractor $\mathfrak{A}^{\exp}_\lambda$ with finite fractal dimension in $\mathcal{H}^{-s}$ for $0 < s \le 1$, that is,

$$\dim^f_{\mathcal{H}^{-s}}\big(\mathfrak{A}^{\exp}_\lambda\big) < \infty, \quad 0 < s \le 1.$$

This completes the proof of Theorem 6.3. □

Although Theorem 6.3 - (**J.11**) provides the existence of a generalized fractal exponential attractor whose fractal dimension is finite in the extended space $\mathcal{H}^{-s}$, $0 < s \le 1$, one sees from its proof that the same methodology cannot be extended to the lower limit case $s = 0$, say in $\mathcal{H}^0 = \mathcal{H}$. However, among all possibilities for the function $k(\cdot)$ in case (**k.2**), in the constant scenario we are supposed to reach exponential attractors, that is, with finite fractal dimensional in $\mathcal{H}$. But even so, the above approach seems to be not applicable and to circumvent the difficulty in obtaining the Hölder continuity of the mapping (6.8) in $\mathcal{H}$, we are going to replace it by a Lipschitz continuous property on a suitable space. This is exactly the goal of the next section.

6.2. **A special case: constant $k(\cdot)$ and $p^* < \frac{2n}{n-4}$.** It is worth pointing out that case (**k.2**) covers the class of constant functions $k(s) = \gamma > 0$ for all $s \ge 0$, which in turn reflects to the case (**k.1**) with $q = 0$ in (3.4). However, in this very special case of linear damping γu_t and commutative patterns $A = A_1^{\frac12}$ (see (2.13)), we can go further. Indeed, in the next result, we prove that the dynamical system $(\mathcal{H}, S_\lambda(t))$ has a time-dependent exponential attractor $\mathfrak{A}^{\exp}_\lambda = \big\{\mathfrak{A}^{\exp}_\lambda(t);\, t \in \mathbb{R}\big\} \subset \mathcal{H}$ for every $\lambda \in [0, 1]$, whose sections $\mathfrak{A}^{\exp}_\lambda(t)$ have finite fractal dimension in $\mathcal{H}$ for all $t \in \mathbb{R}$. This provides, in particular, the existence of an exponential attractor $\widetilde{\mathfrak{A}^{\exp}_\lambda}$ for the dynamical system

$(\mathcal{H}, S_\lambda(t))$. To this purpose, we shall work with the decomposition method motivated by the works [7, 8, 21, 22, 23, 39, 47].

For $k \equiv \gamma > 0$ and requiring the case (2.13), then problem (1.1) turns into

$$\begin{cases} u_{tt} + \kappa A_1^{\frac{1}{2}} u + A_1 u + \gamma u_t + f(u) = h_\lambda, \quad t > 0, \\ (u(0), u_t(0)) = (u_0, u_1) := U_0. \end{cases} \tag{6.9}$$

As in (3.20), we still denote by $(\mathcal{H}, S_\lambda(t))$ the dynamical system associated with (6.9) in case (**k.2**) under the assumption of constant function $k(\cdot)$. Moreover, for each $\lambda \in [0,1]$ we split the semigroup $S_\lambda(t) = S_\lambda^1(t) + S_\lambda^2(t)$ with $S_\lambda^1(t), S_\lambda^2(t)$, $t \geq 0$, given as follows.

Let us consider the evolution operator

$$S_\lambda^1(t) : \mathcal{H} \to \mathcal{H}, \quad S_\lambda^1(t) U_0 := (v(t), v_t(t)), \tag{6.10}$$

where v the solution of the linear problem

$$\begin{cases} v_{tt} + \kappa A_1^{\frac{1}{2}} v + A_1 v + \gamma\, v_t = h_\lambda, \quad t > 0, \\ (v(0), v_t(0)) = U_0. \end{cases} \tag{6.11}$$

Then, we set $S_\lambda^2(t) : \mathcal{H} \to \mathcal{H}$ as

$$S_\lambda^2(t) U_0 = S_\lambda(t) U_0 - S_\lambda^1(t) U_0 := (z(t), z_t(t)), \tag{6.12}$$

where z solves the following problem

$$\begin{cases} z_{tt} + \kappa A_1^{\frac{1}{2}} z + A_1 z + \gamma\, z_t = -f(u), \quad t > 0, \\ (z(0), z_t(0)) = (0,0). \end{cases} \tag{6.13}$$

Proposition 6.4. *Under the above setting* (6.9)-(6.13)*, let us consider a bounded set $B \subset \mathcal{H}$ with initial data $U_0^1, U_0^2 \in B$. Then, there exist constants $c_B, C_B > 0$ (depending on B) such that*

$$\|S_\lambda^1(t) U_0^1 - S_\lambda^1(t) U_0^2\|_{\mathcal{H}} \leq C_B e^{-c_B t} \|U_0^1 - U_0^2\|_{\mathcal{H}}, \quad t > 0. \tag{6.14}$$

Proof. Since the difference $S_\lambda^1(t) U_0^1 - S_\lambda^1(t) U_0^2 := (\tilde{v}(t), \tilde{v}_t(t))$ satisfies the homogeneous linear problem related to (6.11)

$$\begin{cases} \tilde{v}_{tt} + \kappa A_1^{\frac{1}{2}} \tilde{v} + A_1 \tilde{v} + \gamma\, \tilde{v}_t = 0, \quad t > 0, \\ (\tilde{v}(0), \tilde{v}_t(0)) = U_0^1 - U_0^2, \end{cases}$$

then the proof is a particular case of Proposition 6.2 neglecting the precompact component. $\square$

Proposition 6.5. *Under the above setting* (6.9)-(6.13)*, let us also consider Assumption* 3.1 *with subcritical exponent* $p < \frac{4}{n-4}$. *Then, there exists a Banach space* $\mathcal{W}$ *such that the embedding* $\mathcal{H} \hookrightarrow \mathcal{W}$ *is compact and*

$$\|S^2_\lambda(t)U^1_0 - S^2_\lambda(t)U^2_0\|_{\mathcal{H}} \leq \hat{\mathcal{Q}}(t)\|U^1_0 - U^2_0\|_{\mathcal{W}}, \quad t > 0, \tag{6.15}$$

for all $U^1_0, U^2_0 \in \mathcal{H}$*, where* $\tilde{\mathcal{Q}}(t) = \tilde{\mathcal{Q}}(\|U^1_0\|_{\mathcal{H}}, \|U^2_0\|_{\mathcal{H}}, t)$ *a positive non-decreasing function.*

Proof. Given $U^i_0 = (u^i_0, u^i_1) \in \mathcal{H}$ and denoting $S^2_\lambda(t)U^i_0 = (z^i(t), z^i_t(t))$, $i = 1, 2$, then the function $z = z^1 - z^2$ satisfies

$$z_{tt} + \kappa A_1^{\frac{1}{2}} z + A_1 z + \gamma\, z_t = f(u^2) - f(u^1), \quad (z(0), z_t(0)) = (0, 0). \tag{6.16}$$

Taking the multiplier z_t in (6.16), we have

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}_z(t) = -\gamma\|z_t(t)\|_2^2 + (f(u^2(t)) - f(u^1(t)), z_t(t)), \tag{6.17}$$

where

$$\mathcal{E}_z(t) = \|z_t(t)\|^2 + \kappa\|A_1^{1/4} z(t)\|^2 + \|A_1^{\frac{1}{2}} z(t)\|^2, \quad t \geq 0.$$

Let us estimate the second term on the right-hand side of (6.17) as follows.

We first claim that there exists a power $s \in (0, 2)$ such that

$$\|f(z) - f(w)\| \leq C_f(1 + \|z\|^p_{p^*} + \|w\|^p_{p^*})\|A_1^{s/4} z - A_1^{s/4} w\|, \tag{6.18}$$

$z, w \in \mathcal{D}(A_1^{\frac{1}{2}})$, for some constant $C_f > 0$. Indeed, Given $z, w \in \mathcal{D}(A_1^{\frac{1}{2}}) \hookrightarrow L^{p^*}(\Omega)$, from Assumption 3.1 and Hölder's inequality with $\frac{p}{p+1} + \frac{1}{p+1} = 1$, we get

$$\begin{aligned}
\|f(z) - f(w)\|^2 &= \int_\Omega \Big(\int_0^1 f'(\theta z + (1-\theta)w)(z - w)\, d\theta\Big)^2 dx \\
&\leq C_{f'} \int_\Omega \big(1 + |z|^{2p} + |w|^{2p}\big)|z - w|^2\, dx \\
&\leq C_{f'}\big(|\Omega| + \|z\|^{2p}_{p^*} + \|w\|^{2p}_{p^*}\big)\|z - w\|^2_{p^*}.
\end{aligned}$$

Also, since $p < \frac{4}{n-4}$, then $s := \frac{n}{2} - \frac{n}{p^*}$ satisfies $0 < s < 2 < \frac{n}{2}$, that is, the number s satisfies the conditions of [1, Theorem 5.1.5], and thus

$$\mathcal{D}(A_1^{\frac{1}{2}}) \hookrightarrow \mathcal{D}(A_1^{s/4}) \hookrightarrow H^s(\Omega) \hookrightarrow L^{p^*}(\Omega),$$

from where it follows (6.18) for some constant $C_f > 0$. Now, using Cauchy-Schwarz's inequality and (6.18), we obtain

$$(f(u^2(t)) - f(u^1(t)), z_t(t)) \leq C_0\|A_1^{s/4}(u^1 - u^2)(t)\|[\mathcal{E}_z(t)]^{\frac{1}{2}}, \tag{6.19}$$

for some constant $C_0 = C_0(\|U_0^1\|_{\mathcal{H}}, \|U_0^2\|_{\mathcal{H}}) > 0$. Replacing (6.19) in (6.17) arrive at

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}_z(t) \leq -\gamma\mathcal{E}_z(t) + C_0\|A_1^{s/4}(u^1-u^2)(t)\|[\mathcal{E}_z(t)]^{\frac{1}{2}}. \tag{6.20}$$

From (6.20) and regarding [44, Lemma 4.1] with $\phi := \mathcal{E}_z$ set on $[0,t]$, and also taking into account that $\mathcal{E}_z(0) = 0$, we have

$$[\mathcal{E}_z(t)]^{\frac{1}{2}} \leq C_0 \int_0^t \|A_1^{s/4}(u^1-u^2)(\tau)\|\, d\tau, \tag{6.21}$$

for some constant $C_0 > 0$ depending on initial data.

On the other hand, by setting $u := u^1 - u^2$, then the function $\tilde{u} := A_1^{(s-2)/4}u$ fulfills the problem

$$\begin{cases} \tilde{u}_{tt} + \kappa A_1^{\frac{1}{2}}\tilde{u} + A_1\tilde{u} + \gamma\,\tilde{u}_t = A_1^{(s-2)/4}(f(u^2) - f(u^1)), \quad t > 0, \\ (\tilde{u}(0), \tilde{u}_t(0)) = (A_1^{(s-2)/4}u_0, A_1^{(s-2)/4}u_1). \end{cases} \tag{6.22}$$

Taking the multiplier $\tilde{u}_t$ in (6.22), and integrating the resulting expression on $(0,\tau)$, $\tau \in (0,t)$, we get

$$\frac{1}{2}\mathcal{G}_u(\tau) \leq \frac{1}{2}\mathcal{G}_u(0) + \int_0^\tau (A_1^{(s-2)/4}(f(u^2(r)) - f(u^1(r))), \tilde{u}_t(r))dr, \tag{6.23}$$

where

$$\mathcal{G}_u(t) = \|A_1^{(s-2)/4}u_t(t)\|^2 + \kappa\|A_1^{(s-1)/4}u(t)\|^2 + \|A_1^{s/4}u(t)\|^2.$$

From Cauchy-Schwarz's inequality, since $L^2(\Omega) \hookrightarrow D(A_1^{(s-2)/4})$, and using again (6.18), we infer

$$\Big|\int_0^\tau (A_1^{(s-2)/4}(f(u^2(r)) - f(u^1(r))), \tilde{u}_t(r))dr\Big| \leq C_0 \int_0^\tau \mathcal{G}_u(r)dr,$$

for some constant $C_0 = C_0(\|U_0^1\|_{\mathcal{H}}, \|U_0^2\|_{\mathcal{H}}) > 0$. Plugging the last estimate in (6.23),

$$\frac{1}{2}\mathcal{G}_u(\tau) \leq \frac{1}{2}\mathcal{G}_u(0) + C_0\int_0^\tau \mathcal{G}_u(r)dr,$$

and from Gronwall's inequality, we obtain

$$\mathcal{G}_u(\tau) \leq e^{C_0\tau}\mathcal{G}_u(0), \tag{6.24}$$

for some $C_0 = C_0(\|U_0^1\|_{\mathcal{H}}, \|U_0^2\|_{\mathcal{H}}) > 0$. Combining the estimates (6.21) and (6.24), we finally conclude

$$\|S_\lambda^2(t)U_0^1 - S_\lambda^2(t)U_0^2\|_{\mathcal{H}} = \|(z(t), z_t(t))\|_{\mathcal{H}} \leq [\mathcal{E}_z(t)]^{\frac{1}{2}}$$

$$\leq C_0 \int_0^t [\mathcal{G}_u(\tau)]^{\frac{1}{2}}\, d\tau \leq te^{C_0 t}[\mathcal{G}_u(0)]^{\frac{1}{2}} \leq Cte^{C_0 t}\|U_0^1 - U_0^2\|_{\mathcal{W}},$$

for some constant $C > 0$ and $C_0 = C_0(\|U_0^1\|_{\mathcal{H}}, \|U_0^2\|_{\mathcal{H}}) > 0$, where we set

$$\mathcal{W} := D(A_1^{s/4}) \times D(A_1^{(s-2)/4}).$$

Therefore, once the embedding $\mathcal{H} \hookrightarrow \mathcal{W}$ is compact, the estimate (6.15) follows by taking $\tilde{\mathcal{Q}}(t) = Cte^{C_0 t}$. $\square$

We are now in a position to state the result dealing with exponential attractor for the dynamical system $(\mathcal{H}, S_\lambda(t))$ corresponding to (6.9).

Theorem 6.6 (**Exponential Attractor**). *Let us assume that the assumptions of Proposition* 6.5 *holds. Then, the dynamical system* $(\mathcal{H}, S_\lambda(t))$ *has a time-dependent exponential attractor* $\mathfrak{A}_\lambda^{\exp} = \{\mathfrak{A}_\lambda^{\exp}(t)\}_{t\in\mathbb{R}}$, *for every* $\lambda \in [0,1]$, *whose sections* $\mathfrak{A}_\lambda^{\exp}(t)$ *have finite fractal dimension in* $\mathcal{H}$ *for all* $t \in \mathbb{R}$, *that is,*

$$dim_{\mathcal{H}}^f\big(\mathfrak{A}_\lambda^{\exp}(t)\big) < \infty, \quad t \in \mathbb{R}.$$

In particular, there exists a time $T^\star > 0$ *such that*

$$\widetilde{\mathfrak{A}_\lambda^{\exp}} := \bigcup_{t\in[T^\star, 2T^\star]} S_\lambda(t)\overline{\mathfrak{A}_\lambda^{\exp}},$$

is an exponential attractor for $(\mathcal{H}, S_\lambda(t))$, *where* $\overline{\mathfrak{A}_\lambda^{\exp}}$ *denotes the exponential attractor for the corresponding discrete semigroup* $\{S_\lambda(nT^\star)\}_{n\in\mathbb{N}}$.

Proof. In order to employ Theorem B.15, we note that the next conditions are verified.

(S1) From Proposition 6.1, the dynamical system $(\mathcal{H}, S_\lambda(t))$ possesses a bounded (uniformly with respect to $\lambda \in [0,1]$) absorbing set $\mathcal{B} \subset \mathcal{H}$.

(S2) From Proposition 6.4, there exist a time $T^\star > 0$ and a constant $a := a_{T^\star,\mathcal{B}} < \frac{1}{2}$ such that $S_\lambda^1(t)$ satisfies the contraction on $\mathcal{B}$

$$\|S_\lambda^1(T^\star)U_0^1 - S_\lambda^1(T^\star)U_0^2\|_{\mathcal{H}} \leq a\|U_0^1 - U_0^2\|_{\mathcal{H}}, \quad U_0^1, U_0^2 \in \mathcal{B}.$$

(S3) From Proposition 6.5, there exists a constant given by $b = \hat{\mathcal{Q}}(T^\star) > 0$ such that $S_\lambda^2(t)$ satisfies the smoothing condition on $\mathcal{B}$

$$\|S_\lambda^2(T^\star)U_0^1 - S_\lambda^2(T^\star)U_0^2\|_{\mathcal{H}} \leq b\|U_0^1 - U_0^2\|_{\mathcal{W}}, \quad U_0^1, U_0^2 \in \mathcal{B},$$

where $\mathcal{H} \hookrightarrow \mathcal{W}$ is compactly embedded.

(S4) From Theorem 3.1 (see (3.12)), the semigroup $S_\lambda(t)$ is Lipschitz on $\mathcal{B}$ with constant $L_t = \mathcal{Q}(t) > 0$, that is,

$$\|S_\lambda(t)U_0^1 - S_\lambda(t)U_0^2\|_{\mathcal{H}} \leq L_t\|U_0^1 - U_0^2\|_{\mathcal{H}}, \quad U_0^1, U_0^2 \in \mathcal{B},\ t \geq 0.$$

Therefore, by means of Theorem B.15 and Remark B.1, the conclusion of Theorem 6.6 is ensured. □

7. LONG-TIME DYNAMICS: CASE (**k.3**)

We have finally arrived at the critical case with respect to power $\alpha = 1$. In this case, by virtue of (2.12), the k-function argument $\mathcal{E}_1(u, u_t)$ set in (1.2) can be written as

$$\mathcal{E}_1(u, u_t) = \|Au\|^2 + \|u_t\|^2 = \|A_1^{\frac{1}{2}}u\|^2 + \|u_t\|^2, \tag{7.1}$$

from where one sees why we lose compactness of the damping coefficient in phase space $\mathcal{H} = \mathcal{D}(A_1^{\frac{1}{2}}) \times H$. Therefore, this case motivated us to keep the closedness property in the definition of global attractors instead of compactness, because here the system has a degenerate damping coefficient (in the $\mathcal{H}$-topology) without control in a neighborhood of origin and, consequently, a noncompact global attractor comes into play.

To deal with problem in this case, let us consider problem (1.1) with critical power $\alpha = 1$ and vanishing functions $f = h = 0$. In this way, (1.1) with notation (7.1) can be expressed as follows

$$\begin{cases} u_{tt} + \kappa Au + A_1 u + k\big(\|A_1^{\frac{1}{2}}u\|^2 + \|u_t\|^2\big)u_t = 0, \\ u(0) = u_0, \quad u_t(0) = u_1, \end{cases} \tag{7.2}$$

where we notice that in case (**k.3**), we assume that $k(\cdot)$ is a bounded Lipschitz function on $[0, \infty)$ such that $k \equiv 0$ on $[0, 1]$ and $k(s)$ is strictly increasing for $s > 1$.

Also, the energy functional $E(t) := E\big(u(t), u_t(t)\big)$ set in (3.1) boils down to the following

$$E(t) = \frac{1}{2}\big[\, \|u_t(t)\|^2 + \|A_1^{\frac{1}{2}}u(t)\|^2 + \kappa\|A^{\frac{1}{2}}u(t)\|^2 \,\big], \tag{7.3}$$

and satisfies the relation

$$E(t) + \int_0^t k\big(||(u(\tau), u_t(\tau))||^2_{\mathcal{H}}\big)\big\|u_t(\tau)\big\|^2 d\tau = E(0), \quad t > 0. \tag{7.4}$$

It is worth mentioning that, under the above conditions in the setting of problem (7.2), the dynamical system related to both cases (**k.1**) and (**k.2**) possesses a trivial attractor, see Theorem 5.1-(**I.6**) and Theorem 6.3-(**J.6**). Nonetheless, in the present case, even under this particular scenario concerning the source f and external force h, we are going to show below that the dynamical system corresponding to problem (7.2) has a nontrivial noncompact global attractor.

7.1. **A noncompact attractor: the critical case $\alpha = 1$.** Since the critical problem (7.2) arises independently of the parameter $\lambda \in [0, 1]$, then we denote by $(\mathcal{H}, S(t))$ the dynamical system associated with (7.2). We also remind from Subsection 3.2 that $\{S(t)\}_{t\in\mathbb{R}}$ can be seen as an evolution C_0-group.

Here, our main result concerning the dynamical system $(\mathcal{H}, S(t))$ reads as follows.

Theorem 7.1 (**Noncompact Global Attractor).** *Under the setting of problem* (7.2) *with function $k(\cdot)$ in case* (**k.3**)*, the corresponding dynamical system $(\mathcal{H}, S(t))$ has the following global attractor*

$$\mathfrak{A}_\kappa = \big\{\big(u_0, u_1\big) \in \mathcal{H};\ \big\|u_1\big\|^2 + \big\|A_1^{1/2}u_0\big\|^2 + \kappa\|A^{\frac{1}{2}}u_0\|^2 \leq 1\big\}, \tag{7.5}$$

for $\kappa > 0$ (possibly zero) small enough.

Proof. The proof will be done in two steps by appealing to the definition of the global attractor.

Step 1. Fully Invariance. We start by noting that the linear problem

$$u_{tt} + \kappa Au + A_1 u = 0, \quad u(0) = u_0,\ u_t(0) = u_1, \tag{7.6}$$

has a unique solution $(u(t), u_t(t))$ such that the energy set in (7.3) satisfies

$$E(u(t), u_t(t)) = E(u_0, u_1), \quad t \in \mathbb{R}.$$

From this, since $k(s) = 0$ for $s \in [0, 1]$, and uniqueness of the solution, it is easy to verify that $\mathfrak{A}_\kappa$ is a fully invariant set with respect to $S(t)$ and, consequently, $\mathcal{H}\backslash\mathfrak{A}_\kappa$ so is it. Moreover, from the energy identity, it is also easy to check that the set $\mathcal{B}_{R,\kappa} := \mathcal{B}_R$ given by

$$\mathcal{B}_R = \big\{(u_0, u_1) \in \mathcal{H};\ \|u_1\|^2 + \|A_1^{1/2}u_0\|^2 + \kappa\|A^{\frac{1}{2}}u_0\|^2 \leq R^2\big\}$$

is forward invariant with respect to $S(t)$ for every $R > 1$.

Step 2. Uniform Attracting. Given any bounded set $B \subset \mathcal{H}$, there exists $R > 1$ such that $B \subset \mathcal{B}_R$. Hence, below, we only need to verify that $S(t)\mathcal{B}_R$ goes to $\mathfrak{A}_\kappa$ uniformly with respect to $\mathcal{B}_R$, for every $R > 1$.

Let us consider $U_0 = (u_0, u_1) \in \mathcal{B}_R$, $R > 1$, and $S(t)U_0 = \big(u(t), u_t(t)\big)$ the corresponding semigroup solution. There are only two possibilities $U_0 \in \mathfrak{A}_\kappa$ or else $U_0 \in \mathcal{B}_R\backslash\mathfrak{A}_\kappa$. If $U_0 \in \mathfrak{A}_\kappa$, then $S(t)U_0 \in S(t)\mathfrak{A}_\kappa = \mathfrak{A}_\kappa$, and the conclusion follows trivially. Thus, in what follows, we take $U_0 \in \mathcal{B}_R \backslash \mathfrak{A}_\kappa$. We first claim that

$$\lim_{t\to+\infty} \big[2E(S(t)U_0)\big] = 1. \tag{7.7}$$

Indeed, let us suppose that it does not hold. Thus, due to the invariance of $\mathcal{H}\backslash\mathfrak{A}_\kappa$ and since the mapping $t \mapsto 2E(S(t)U_0)$ is non-increasing, there exists $R_0 > 1$ such that

$$2E(S(t)U_0) \geq R_0, \ t > 0, \quad \text{and} \quad \lim_{t\to+\infty} \big[2E(S(t)U_0)\big] = R_0. \tag{7.8}$$

Now, we remember (analogously to (3.16)) that

$$||S(t)U_0||^2_{\mathcal{H}} \leq 2E(S(t)U_0) \leq (1+\kappa\mu_0)||S(t)U_0||^2_{\mathcal{H}}, \tag{7.9}$$

and taking $\kappa > 0$ small enough (or possibly zero) so that $\kappa < \frac{R_0-1}{\mu_0}$, then $R_\kappa := \frac{R_0}{1+\kappa\mu_0} > 1$. From (7.8)-(7.9) one sees that $\big\|S(t)U_0\big\|^2_{\mathcal{H}} \geq R_\kappa > 1$ for all $t \geq 0$, and from the assumption on $k(\cdot)$ in the range $(1,\infty)$, we infer

$$k\big(\big\|S(t)U_0\big\|^2_{\mathcal{H}}\big) \geq k(R_\kappa) := k_0 > 0 \ \text{ for all } \ t \geq 0. \tag{7.10}$$

Going back to problem (7.2), taking the multiplier u_t, and using (7.10), we get

$$\frac{d}{dt}E(S(t)U_0) + 2k_0\big\|u_t(t)\big\|^2 \leq 0, \quad t > 0. \tag{7.11}$$

Using (7.11) and since $k_1 = \sup\{k(s);\, s \geq 1\} < \infty$, then problem (7.2) behaves like in case (**k.2**) and similar to Proposition 6.1 (see [29, Remark 7] for more details) one shows that there exists a constant $c > 0$ (which may depend on U_0 and k_0) such that

$$E(S(t)U_0) \leq 3E(U_0)e^{-c\,t}, \quad t > 0,$$

whenever $2E(S(t)U_0) \geq R_0 > 1$, which is a contraction to (7.8). Therefore, (7.7) holds true and is uniform with respect to $U_0 \in \mathcal{B}_R\backslash\mathfrak{A}_\kappa$. Additionally,

$$\lim_{t\to+\infty} \big[2E(S(t)U_0)\big]^{\frac{1}{2}} = 1, \quad U_0 \in \mathcal{B}_R\backslash\mathfrak{A}_\kappa.$$

Finally, for every $U_0 \in \mathcal{B}_R\backslash\mathfrak{A}_\kappa$, we observe that

$$\begin{aligned} \operatorname{dist}\big(S(t)U_0, \mathfrak{A}_\kappa\big) &\leq \Big\|S(t)U_0 - \frac{S(t)U_0}{[2E(S(t)U_0)]^{\frac{1}{2}}}\Big\|_{\mathcal{H}} \\ &= \frac{\big\|S(t)U_0\big\|_{\mathcal{H}}\big[[2E(S(t)U_0)]^{\frac{1}{2}} - 1\big]}{[2E(S(t)U_0)]^{\frac{1}{2}}} \leq \big[[2E(S(t)U_0)]^{\frac{1}{2}} - 1\big], \end{aligned}$$

where in the last inequality, we use (7.9), from where one concludes

$$\lim_{t\to+\infty} \mathrm{dist}_{\mathcal{H}}\big(S(t)(\mathcal{B}_R\backslash\mathfrak{A}_\kappa),\mathfrak{A}_\kappa\big) = \lim_{t\to+\infty}\Big[\sup_{U_0\in\mathcal{B}_R\backslash\mathfrak{A}_\kappa} \mathrm{dist}\big(S(t)U_0,\mathfrak{A}_\kappa\big)\Big] = 0.$$

This completes the proof that the bounded closed set $\mathfrak{A}_\kappa$ given in (7.5) uniformly attracts $\mathcal{B}_R$ for any $R > 1$, and bounded subsets $B \subset \mathcal{H}$ as well.

Therefore, $\mathfrak{A}_\kappa$ is a global attractor for the dynamical system $(\mathcal{H}, S(t))$ generated by problem (7.2). □

Corollary 7.2. *Under the hypotheses of Theorem* 7.1*, the dynamical system* $(S(t),\mathcal{H})$ *associated with problem* (7.2) *does not have a compact global attractor.*

Proof. It is directly a consequence of the uniqueness of a global attractor. □

Corollary 7.3. *Under the hypotheses of Theorem* 7.1 *with* $\kappa = 0$*, then the closed ball*

$$\mathfrak{B}_0 = \big\{\big(u_0,u_1\big)\in\mathcal{H};\ \big\|u_1\big\|^2 + \big\|A_1^{1/2}u_0\big\|^2 \le 1\big\}$$

is the global attractor for $(S(t),\mathcal{H})$*.*

Remark 7.1. The result stated in Corollary 7.3 corresponds to the one approached [12, Proposition 5.3.9]. Additionally, we note that in this limit case $\kappa = 0$, one can relax the assumption of $k(\cdot)$ on the interval $[0, 1]$, for instance with arbitrary behavior on $[0, 1]$ instead vanishing on it, see [12, Subsect. 5.3.3].

Appendix A. Auxiliary proofs

A.1. **Completion of the proof of Theorem 3.2.** By using Assumption 3.1, we prove now that operator $\mathcal{M} : \mathcal{H} \to \mathcal{H}$ defined in (3.7) is a locally Lipschitz continuous operator on $\mathcal{H}$.

In fact, let us consider $R > 0$ and $U = (u, v), \widetilde{U} = (\widetilde{u}, \widetilde{v}) \in \mathcal{H}$ such that $||U||_{\mathcal{H}}, ||\widetilde{U}||_{\mathcal{H}} \le R$. From definition (3.7) and the $\mathcal{H}$-norm, we have

$$\begin{aligned}
&||\mathcal{M}(U) - \mathcal{M}(\widetilde{U})\,||_{\mathcal{H}}\\
&= \big\|\kappa[Au - A\widetilde{u}] + \gamma\big[||U||^{2q}_{\mathcal{H}^\alpha} v - ||\widetilde{U}||^{2q}_{\mathcal{H}^\alpha}\widetilde{v}\big] - [f(u) - f(\widetilde{u})]\big\|\\
&\le \kappa\|U - \widetilde{U}\|_{\mathcal{H}} + \gamma\,\underbrace{\big\|\,||U||^{2q}_{\mathcal{H}^\alpha} v - ||\widetilde{U}||^{2q}_{\mathcal{H}^\alpha}\widetilde{v}\big\|}_{\mathcal{I}_1} + \underbrace{\|f(u) - f(\widetilde{u})\|}_{\mathcal{I}_2}.
\end{aligned} \tag{A.1}$$

In what follows, we are going to estimate the terms $\mathcal{I}_1$ and $\mathcal{I}_2$ above. To estimate the term $\mathcal{I}_1$, we first note that

$$\begin{aligned}&||U||^{2q}_{\mathcal{H}^\alpha} v - ||\widetilde{U}||^{2q}_{\mathcal{H}^\alpha}\widetilde{v}\\ &= \frac{1}{2}\big[||U||^{2q}_{\mathcal{H}^\alpha} + ||\widetilde{U}||^{2q}_{\mathcal{H}^\alpha}\big][v-\widetilde{v}] + \frac{1}{2}\big[||U||^{2q}_{\mathcal{H}^\alpha} - ||\widetilde{U}||^{2q}_{\mathcal{H}^\alpha}\big][v+\widetilde{v}]. \qquad (A.2)\end{aligned}$$

Using that $\mathcal{D}(A_1^{\frac{1}{2}}) \subseteq \mathcal{D}(A) \subseteq \mathcal{D}(A^\alpha)$ and $\|A_1^{\frac{1}{2}}u\|^2 = \|Au\|$ for $u \in \mathcal{D}(A_1^{\frac{1}{2}})$, we can estimate the first term of the sum in (A.2) as follows

$$\frac{1}{2}\big\|\big[||U||^{2q}_{\mathcal{H}^\alpha} + ||\widetilde{U}||^{2q}_{\mathcal{H}^\alpha}\big][v-\widetilde{v}]\,\big\| \le C_R\|v-\widetilde{v}\| \le C_R||U-\widetilde{U}||_{\mathcal{H}}.$$

We emphasize again that this is the precise moment where the assumption $q \ge 1/2$ is crucial in our computations. Indeed, once we have $2q \ge 1$, then we can use Lemma 3.3 to obtain

$$\begin{aligned}&\frac{1}{2}\big\|\big[||U||^{2q}_{\mathcal{H}^\alpha} - ||\widetilde{U}||^{2q}_{\mathcal{H}^\alpha}\big][v+\widetilde{v}]\big\|\\ &\le q\max\big\{||U||_{\mathcal{H}^\alpha}, ||\tilde{U}||_{\mathcal{H}^\alpha}\big\}^{2q-1}\|v+\widetilde{v}\;\|||U-\widetilde{U}||_{\mathcal{H}^\alpha} \le C_R||U-\widetilde{U}||_{\mathcal{H}}\end{aligned}$$

for some $C_R > 0$. Therefore, collecting the last two estimates, we deduce from (A.2) that $\mathcal{I}_1 \le C_R||U-\widetilde{U}||_{\mathcal{H}}$. Lastly, we are going to estimate the term $\mathcal{I}_2$. Using Mean Value Theorem, conditions (3.2), Hölder's inequality with $\frac{p}{p+1} + \frac{1}{p+1} = 1$ and the embedding $\mathcal{D}(A_1^{\frac{1}{2}}) \hookrightarrow L^{p^*}(\Omega)$, we have

$$\begin{aligned}\mathcal{I}_2 &\le 2^{2p}C_{f'}\big[|\Omega| + \|u\|^{p^*}_{p^*} + \|\widetilde{u}\|^{p^*}_{p^*}\big]^{\frac{p}{p^*}}\|u-\widetilde{u}\|_{p^*}\\ &\le 2^{2p}C_{f'}C_{|\Omega|}\big[|\Omega| + C^{p^*}_{|\Omega|}\big(\|A_1^{\frac{1}{2}}u\|^{p^*} + \|A_1^{\frac{1}{2}}\widetilde{u}\|^{p^*}\big)\big]^{\frac{p}{p^*}}\|A_1^{\frac{1}{2}}u - A_1^{\frac{1}{2}}\widetilde{u}\|\\ &\le C_R||U-\widetilde{U}||_{\mathcal{H}}.\end{aligned}$$

replacing $\mathcal{I}_1$ and $\mathcal{I}_2$ in (A.1), we obtain

$$||\mathcal{M}(U) - \mathcal{M}(\widetilde{U})\,||_{\mathcal{H}} \le C_R||U-\widetilde{U}||_{\mathcal{H}},$$

for some $C_R > 0$, which proves the desired. □

A.2. **Proof of Proposition 4.8. Proof of** (**N.1**)**.** In this case, under the assumptions of Proposition 4.8, we are going to prove the inequality (4.26). Indeed, we first observe that it holds true for $0 \le t \le 1$. Then, let us prove it for $t > 1$. Setting the function

$$\beta(t) := \big(C_0^{-1}\rho(t-1)^+ + \big(\sup_{0\leqslant s\leqslant 1}\phi(s)\big)^{-\rho}\big)^{-1/\rho}, \quad t \ge 0,$$

we initially claim that the following property holds: if for some $t \geq 0$, the inequality holds

$$\phi(t) \leqslant \beta(t) + [K(t)]^{1/(\rho+1)}, \tag{A.3}$$

then the next inequality holds as well

$$\phi(t+1) \leqslant \beta(t+1) + [K(t+1)]^{1/(\rho+1)}. \tag{A.4}$$

Indeed, let us suppose that (A.4) does not hold, that is

$$\phi(t+1) > \beta(t+1) + [K(t+1)]^{1/(\rho+1)}. \tag{A.5}$$

Now, if $\phi(t) \leqslant [K(t)]^{1/(\rho+1)}$, then (A.5) and (A.3) imply

$$\phi(t+1) > [K(t+1)]^{1/(\rho+1)} \geq [K(t)]^{1/(\rho+1)} > \phi(t),$$

where we have used that $K(t)$ is non-increasing. Thus, from (4.25), we obtain

$$\phi(t+1) < [K(t+1)]^{1/(\rho+1)},$$

which is a contradiction with (A.5). Thus, we can infer that

$$\phi(t) > [K(t)]^{1/(\rho+1)}$$

and, consequently,

$$\begin{cases} \varphi(t) := \phi(t) - [K(t)]^{1/(\rho+1)} > 0, \\ \varphi(t+1) := \phi(t+1) - [K(t+1)]^{1/(\rho+1)} > 0. \end{cases}$$

From this, one sees

$$[\varphi(t)]^{\rho+1} + K(t) \leqslant \left(\varphi(t) + [K(t)]^{1/(\rho+1)}\right)^{\rho+1} = [\phi(t)]^{1+\rho}, \tag{A.6}$$

and using again (4.25), one gets

$$[\varphi(t)]^{1+\rho} \leqslant C_0(\varphi(t) - \varphi(t+1)). \tag{A.7}$$

We additionally set $\psi(t) := \varphi^{-\rho}(t)$ and $\psi(t+1) := \varphi^{-\rho}(t+1)$. Then, a straightforward integral computation along with (A.7) lead to

$$\begin{aligned} \psi(t+1) - \psi(t) &= -\int_0^1 \frac{d}{d\tau}(\tau\varphi(t) + (1-\tau)\varphi(t+1))^{-\rho} d\tau \\ &= \rho \int_0^1 (\tau\varphi(t) + (1-\tau)\varphi(t+1))^{-(1+\rho)} d\tau (\varphi(t) - \varphi(t+1)) \\ &\geqslant \rho\varphi^{-(1+\rho)}(t)(\varphi(t) - \varphi(t+1)) \geqslant \rho C_0^{-1}. \end{aligned}$$

Thus, we have

$$\varphi^{-\rho}(t+1) = \psi(t+1) \geqslant \psi(t) + \rho C_0^{-1} = \varphi^{-\rho}(t) + \rho C_0^{-1},$$

that is,

$$\varphi(t+1) \leq \big(\varphi^{-\rho}(t) + \rho C_0^{-1}\big)^{-1/\rho}, \tag{A.8}$$

From (A.8) and (A.3), we finally arrive at

$$\begin{aligned}
\phi(t+1) &= \varphi(t+1) + [K(t+1)]^{1/(\rho+1)} \\
&\leqslant \big(\varphi^{-\rho}(t) + \rho C_0^{-1}\big)^{-1/\rho} + [K(t+1)]^{1/(\rho+1)} \\
&\leqslant \big(\rho C_0^{-1}(t-1)^+ + \sup_{0\leqslant s\leqslant 1}[\phi(s)]^{-\rho} + \rho C_0^{-1}\big)^{-1/\rho} + [K(t+1)]^{1/(\rho+1)} \\
&\leqslant \beta(t+1) + [K(t+1)]^{1/(\rho+1)},
\end{aligned}$$

which contradicts (A.5). This concludes the proof of our initially assertion.

To conclude the proof, we note that for any $t > 1$ real, we can write $t = n + r$ with $n \in \mathbb{N}$ and $0 \leq r < 1$. From the beginning, (A.3) holds true for $0 \leq r < 1$ and from (A.4),

$$\phi(r+1) \leqslant \beta(r+1) + [K(r+1)]^{1/(\rho+1)}.$$

Using this assertion n times, we infer

$$\phi(t) = \phi(r+n) \leqslant \beta(r+n) + [K(r+n)]^{1/(\rho+1)} = \beta(t) + [K(t)]^{1/(\rho+1)},$$

as desired. This concludes the proof of (4.26).

Proof of (N.2). To the proof of (4.27), we initially set

$$\beta(t) := \sup_{0\leqslant s\leqslant 1} \phi(s)\Big(\frac{C_0}{1+C_0}\Big)^{[t]}, \quad t \geq 0,$$

and proceed verbatim as in the first case. □

A.3. **Proof of Proposition 4.11.** From the hypothesis of Proposition 4.11, one sees

$$\begin{cases} u^n \to u \text{ weakly-star in } & L^\infty\big(s,T;\mathcal{D}(A_1^{\frac{1}{2}})\big), \\ u_t^n \to u_t \text{ weakly-star in } & L^\infty(s,T;H), \end{cases} \tag{A.9}$$

and from the Aubin-Lions compactness theorem (see e.g. Simon [45]), we also have

$$u^n \to u \quad \text{strongly in} \quad C([s,T];H). \tag{A.10}$$

Additionally, by using Lemma 8.1 in Lions and Magenes [38] (see on p. 275 therein), (A.9) also implies that u^n is bounded in $C_s\big(s,T;\mathcal{D}(A_1^{\frac{1}{2}})\big)$, and then $u^n(t)$ is bounded in $\mathcal{D}(A_1^{\frac{1}{2}})$ for all $t \in [s,T]$. From this and (A.10), one gets

$$u^n(t) \to u(t) \quad \text{weakly in} \quad \mathcal{D}(A_1^{\frac{1}{2}}),\ s \leq t \leq T, \tag{A.11}$$

and due to the compact embedding theorem, we infer

$$\widehat{f}\big(u^n(t)\big) \to \widehat{f}(u(t)) \quad \text{strongly in} \quad L^1(\Omega), \ s \le t \le T. \tag{A.12}$$

where we notice that $\widehat{f}(u) = \int_0^u f(\tau)d\tau$. Also, from (A.9), assumptions on f and again (A.10), we have

$$\big(f\big(u^n\big), u_t^n\big) \to \big(f(u), u_t\big) \quad \text{strongly in} \quad L^1(s,T). \tag{A.13}$$

Now, regarding

$$\frac{\partial}{\partial t}\int_\Omega \widehat{f}\big(u^n(x,t)\big)dx = \big(f\big(u^n(t)\big), u_t^n(t)\big),$$

we get

$$\int_s^t \big(f(u^n(\tau)), u_t^n(\tau)\big)d\tau = \int_\Omega \widehat{f}(u^n(x,t))dx - \int_\Omega \widehat{f}(u^n(s,x))dx.$$

From this identity (which also holds true for u) and from the limits (A.12)-(A.13), we finally arrive at

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T \big(f\big(u^n(t)\big) - f\big(u^m(t)\big), u_t^n(t) - u_t^m(t)\big)dt \\
&= \lim_{n\to\infty}\int_\Omega \widehat{f}\big(u^n(x,T)\big)dx + \lim_{m\to\infty}\int_\Omega \widehat{f}\big(u^m(x,T)\big)dx \\
&\quad - \lim_{n\to\infty}\int_\Omega \widehat{f}\big(u^n(x,s)\big)dx - \lim_{m\to\infty}\int_\Omega \widehat{f}\big(u^m(x,s)\big)dx \\
&\quad - \lim_{n\to\infty}\lim_{n\to\infty}\int_s^T\int_\Omega f\big(u^n(t,x)\big)u_t^m(x,t)dxdt \\
&\quad - \lim_{n\to\infty}\lim_{n\to\infty}\int_s^T\int_\Omega f\big(u^m(x,t)\big)u_t^n(x,t)dxdt \\
&= 2\int_\Omega f(u(x,T))dx - 2\int_\Omega f(u(x,s))dx - 2\int_s^T \big(f(u(t)), u_t(t)\big)dxdt \\
&= 0,
\end{aligned}$$

which proves the desired in (4.45). □

Appendix B. Auxiliary facts on dynamical systems

The concepts and results on dynamical systems reminded below can be found e.g. in [3, 7, 12, 16, 17, 18, 21, 26, 27, 35, 43, 47]. Hereafter, (H, S_t) stands for a dynamical system consisting of a C_0-semigroup $S_t,\ t \ge 0$, defined on a Banach space $(H, \|\cdot\|)$.

• The dynamical system (H, S_t) is said to be *dissipative* if it possesses a bounded *absorbing set*, that is, a bounded set $\mathcal{B} \subset H$ such that for any bounded set $B \subset H$ there exists $t_B \geq 0$ satisfying $S_t B \subset \mathcal{B}$, $\forall\, t \geq t_B$.

• One says that (H, S_t) is *asymptotically smooth* if for any bounded positively invariant set $B \subset H$ ($S_t B \subseteq B$ for all $t \geq 0$), there exists a compact set $\mathcal{M} \subset \overline{B}$ such that

$$\lim_{t \to +\infty} \operatorname{dist}_H(S_t B, \mathcal{M}) = 0, \tag{B.1}$$

where $\operatorname{dist}_H(\cdot, \cdot)$ stands for the *Hausdorff semidistance*[4] in H.

• $\big(H, S_t\big)$ is said to be *asymptotically compact* if and only if there exists an *attracting compact set* $\mathcal{M}$, that is, for any bounded set B one has that (B.1) holds.

• A *global attractor* for (H, S_t) is a bounded closed set $\mathcal{A} \subset H$ which is fully invariant and uniformly attracting, that is, $S_t \mathcal{A} = \mathcal{A}$ for all $t \geq 0$ and for every bounded subset $B \subset H$,

$$\lim_{t \to +\infty} \operatorname{dist}_H(S_t B, \mathcal{A}) = 0.$$

• A *global minimal attractor* for (H, S_t) is a bounded closed set $\mathcal{A}_{\min} \subset H$ which is positively invariant ($S_t \mathcal{A}_{\min} \subseteq \mathcal{A}_{\min}$) and attracts uniformly every point, that is,

$$\lim_{t \to +\infty} \operatorname{dist}(S_t U_0, \mathcal{A}_{\min}) = 0, \quad \text{for any} \quad U_0 \in \mathcal{H},$$

and $\mathcal{A}_{\min}$ has no proper subsets possessing these two properties.

• The *unstable manifold* emanating from a set $\mathcal{N}$, denoted by $\mathbf{M}^u(\mathcal{N})$, is a set of H such that for each $U_0 \in \mathbf{M}^u(\mathcal{N})$ there exists a full trajectory $\Gamma = \{\mathbf{U}(t);\ t \in \mathbb{R}\}$ satisfying

$$\mathbf{U}(0) = U_0 \quad \text{and} \quad \lim_{t \to -\infty} \operatorname{dist}(\mathbf{U}(t), \mathcal{N}) = 0.$$

• The dynamical system (H, S_t) is said to be *gradient* if and only if there exists a *strict Lyapunov functional* on H, that is, there exists a continuous functional $\Phi(z)$ such that the function $t \mapsto \Phi(S_t z)$ is nonincreasing for any $z \in H$, and the equation $\Phi(S_t z) = \Phi(z)$ for all $t > 0$ and some $z \in H$ implies that $S_t z = z$ for all $t > 0$.

[4]The Hausdorff semidistance of two non-empty subsets $A, B \subset H$ is given by

$$\operatorname{dist}_H(A, B) := \sup_{x \in A} \operatorname{dist}(x, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|.$$

• The *Kolmogorov* $\varepsilon-$*entropy* $H_\varepsilon(\mathcal{M})$ of a compact set $\mathcal{M} \subset H$ is given by

$$H_\varepsilon(\mathcal{M}) = \ln N(\mathcal{M}, \varepsilon), \quad \varepsilon > 0, \tag{B.2}$$

where $N(\mathcal{M}, \varepsilon)$ is the minimal number of closed sets of the diameter not greater than 2ε which cover the compact $\mathcal{M}$. The *fractal dimension* $\dim_f \mathcal{M}$ of $\mathcal{M}$ is defined by the formula

$$\dim_H^f \mathcal{M} = \limsup_{\varepsilon \to 0} \frac{H_\varepsilon(\mathcal{M})}{\ln(1/\varepsilon)}.$$

• A compact set $\mathcal{A}_{\exp} \subset H$ is said to be a *fractal exponential attractor* of the dynamical system (H, S_t) if $\mathcal{A}_{\exp}$ is a positively invariant set of finite fractal dimension and for every bounded set $B \subset H$ there exist positive constants t_B, C_B and σ_B such that

$$\operatorname{dist}_H(S_t B, \mathcal{A}_{\exp}) \le C_B\, e^{-\sigma_B (t - t_B)}, \quad t \ge t_B.$$

If the exponential attractor has finite fractal dimension in some extended space $\widetilde{H} \supseteq H$, one calls this exponentially attracting set as a *generalized fractal exponential attractor*.

The first results below deal with the existence and characterization of global attractors. To their statements, we follow more closely the works [17, 18].

Theorem B.1 ([17, Proposition 2.10])**.** *Assume that for any bounded positively invariant set $B \subset H$ and for any $\varepsilon > 0$, there exists $T = T(\varepsilon, B)$ such that*

$$\|S_t z_1 - S_t z_2\| \le \varepsilon + \phi_T(z_1, z_2), \quad \forall\, z_1, z_2 \in B,$$

where $\phi_T : B \times B \to \mathbb{R}$ satisfies

$$\liminf_{n \to \infty} \liminf_{m \to \infty} \phi_T(z_n, z_m) = 0, \tag{B.3}$$

for any sequence (z_n) in B. Then (H, S_t) is an asymptotically smooth dynamical system.

Proposition B.2 ([18, Proposition 7.1.4])**.** *Let $\big(H, S_t\big)$ be a dissipative dynamical system. Then, $\big(H, S_t\big)$ is asymptotically compact if and only if $\big(H, S_t\big)$ is asymptotically smooth.*

Theorem B.3 ([17, Theorem 2.3])**.** *Let $\big(H, S_t\big)$ be a dissipative dynamical system. Then $\big(H, S_t\big)$ possesses a compact global attractor $\mathcal{A}$ if and only if $\big(H, S_t\big)$ is asymptotically smooth.*

Theorem B.4 ([17, Theorem 2.28]). *Let $\mathcal{N}$ be the set of stationary points*[5] *of (H, S_t) and assume that (H, S_t) possesses a compact global attractor $\mathcal{A}$. If there exists a strict Lyapunov functional on $\mathcal{A}$, then $\mathcal{A} = \mathbf{M}^u(\mathcal{N})$. Moreover, the global attractor $\mathcal{A}$ consists of full trajectories $\Gamma = \{U(t) : t \in \mathbb{R}\}$ such that*

$$\lim_{t\to-\infty} \operatorname{dist}(U(t), \mathcal{N}) = 0 \qquad \text{and} \qquad \lim_{t\to+\infty} \operatorname{dist}(U(t), \mathcal{N}) = 0.$$

Theorem B.5 ([18, Theorem 7.5.10]). *Assume that a gradient dynamical system (H, S_t) possesses a compact global attractor $\mathcal{A}$. Then for any $z \in H$, we have*

$$\lim_{t\to+\infty} \operatorname{dist}(S_t z, \mathcal{N}) = 0,$$

that is, any trajectory stabilizes to the set $\mathcal{N}$ of stationary points. In particular, $\mathcal{A}_{\min} = \mathcal{N}$.

In the next results, we deal with a family of attractors $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda} \subset H$.[6] To their statements, we follow the references [27, 43]. We first remind the concepts of upper semicontinuity and (residual) continuity as follows.

- The family of attractors $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ is *upper semicontinuous* at the point λ_0 if and only if
$$\lim_{\lambda\to\lambda_0} \operatorname{dist}_H\big(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}\big) = 0,$$
where, as above, $\operatorname{dist}_H(\cdot,\cdot)$ stands for the Hausdorff semidistance in H.
- The family of attractors $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ is *continuous* at the point λ_0 when
$$\lim_{\lambda\to\lambda_0} \big[\operatorname{dist}_H\big(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda\big) + \operatorname{dist}_H\big(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}\big)\big] = 0.$$

Theorem B.6 ([43, Theorem 10.16]). *Let $\{S_t^\lambda\}_{\lambda\in\Lambda}$ be a family of semigroups on H possessing global attractors $\mathcal{A}_\lambda$ for $\lambda \in \Lambda$. Let us additionally assume:*

(a) *the attractors $\mathcal{A}_\lambda$ are uniformly bounded, i.e., there exists a bounded set $\mathcal{B}_0 \subset H$ such that $\mathcal{A}_\lambda \subset \mathcal{B}_0$, for all $\lambda \in \Lambda$;*
(b) *there exists $t_0 \geq 0$ such that*

$$\lim_{\lambda\to\lambda_0} \sup_{z\in\mathcal{B}_0} ||S_t^\lambda z - S_t^{\lambda_0}|| = 0, \quad \forall\, t \geq t_0.$$

Then, the family of attractors $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ is upper semicontinuous at the point λ_0.

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[5] $\mathcal{N} = \big\{v \in H : S_t v = v \text{ for all } t \geq 0\big\}$.
[6] Here, Λ stands for complete metric space.

Theorem B.7 ([27, Theorem 5.2])**.** *Let $\{S_t^\lambda\}_{\lambda\in\Lambda}$ be a family of semigroups on H. Let us additionally suppose:*

(a) *S_t^λ has a global attractor $\mathcal{A}_\lambda$ for every $\lambda \in \Lambda$;*
(b) *there is a bounded subset $\mathcal{B} \subset H$ such that $\mathcal{A}_\lambda \subset \mathcal{B}$ for every $\lambda \in \Lambda$;*
(c) *for $t > 0, S_t^\lambda x$ is continuous in λ, uniformly for x in bounded subsets of X.*

Then, the family $\mathcal{A}_\lambda$ is continuous in λ for all λ_0 in a residual subset of Λ. In particular, the set of continuity points of $\mathcal{A}_\lambda$ is dense in Λ.

The next result deals with an estimate for Kolmogorov's ε-entropy $H_\varepsilon(\mathcal{M})$ of a compact set $\mathcal{M} \subset H$, where now $(H, \|\cdot\|)$ means a Hilbert space. For its proof, we refer to [16].

Theorem B.8 ([16, Theorem 4.2])**.** *Let H be a separable Hilbert space and $\mathcal{M}$ be a bounded closed set in H. Assume that there exists a mapping $V : \mathcal{M} \mapsto H$ such that:*

1. $\mathcal{M} \subseteq V\mathcal{M}$;
2. *V is Lipschitz on $\mathcal{M}$, that is, there exists $L > 0$ such that*
$$\|Vz_1 - Vz_2\| \le L\|z_1 - z_2\|, \quad z_1, z_2 \in \mathcal{M};$$
3. *There exist pseudometrics ϱ_1 and ϱ_2 on H such that*
$$\|Vz_1 - Vz_2\| \le g\big(\|z_1 - z_2\|\big) + h\Big(\big[\varrho_1(z_1, z_2)^2 + \varrho_2(Vz_1, Vz_2)^2\big]^{1/2}\Big)$$
for all $z_1, z_2 \in \mathcal{M}$, where $g, h : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous non-decreasing functions such that
$$g(0) = 0,\ g(s) < s,\ s > 0,\ s - g(s) \text{ is nondecreasing,}$$
and the function $h(s)$ is strictly increasing in the interval $\big[0, s_0\big]$ for some $s_0 > 0$ with $h(0) = 0$.
4. *For any $q > 0$ and for any closed bounded set $B \subset \mathcal{M}$ the maximal number $m(B, q)$ of elements $x_j^B \in B$ such*
$$\varrho_1\big(x_j^B, x_i^B\big)^2 + \varrho_2\big(Vx_j^B, Vx_i^B\big)^2 > q^2, \quad i \ne j,\ i, j = 1, \dots, m(B, q),$$
is finite.

Then $\mathcal{M}$ is a compact set and there exists $0 < \varepsilon_0 < 1$ such that for all $\varepsilon \le \varepsilon_0 < 1$, Kolmogorov's ε-entropy $H_\varepsilon(\mathcal{M})$ admits the following estimate

$$H_\varepsilon(\mathcal{M}) \le \int_\varepsilon^{\varepsilon_0} \frac{\ln m\big(g_\delta^{-1}(s), q(s)\big)}{s - g_\delta(s)} ds + H_{g_0(\varepsilon_0)}(\mathcal{M}),$$

where $g_\delta(s) = \frac{1-\delta}{2} g(2s) + \delta s$ *with arbitrary* $\delta \in (0,1)$*, the function* $q(s)$ *is defined by the formula*

$$q(s) = \frac{1}{2} h^{-1}\{\delta[2s - g(2s)]\}, 0 < s < \varepsilon_0,$$

and

$$m(r,q) = \sup\{m(B,q) : B \subseteq M, \operatorname{diam} B \leq 2r\}.$$

Theorem B.9 ([16, Theorem 4.5])**.** *Under the hypotheses of Theorem B.8 with*

(i) $\lim_{s\to 0} \frac{g(s)}{s} = g_0 < 1$;
(ii) $h(s)$ *being a linear function* ($h(s) = h_0 \cdot s$)*;*
(iii) $\varrho_i := n_i, i = 1,2,$ *being a precompact seminorm on* H *(item 4 can be neglected).*

Then $\mathcal{M}$ *is a compact set in* H *with finite fractal dimension* $\left(dim_H^f \mathcal{M} < \infty\right)$.

Proof. See also [17, Theorem 2.15], [18, Theorem 7.3.3], or [12, Theorem 3.1.15] for slightly new versions of this result. □

Below we recall the notion on quasi-stable dynamical systems by following [12, 18] and then some results on regularity, finite dimensionality, and exponential attractors.

• The dynamical system (H, S_t) is said to be *quasi-stable* on a set $\mathcal{B} \subset H$ (at time t^*) if there exist time $t^* > 0$, a Banach space Z, a globally Lipschitz mapping $V : \mathcal{B} \mapsto Z$, and a compact seminorm $n_Z(\cdot)$ on the space Z, such that

$$\left\|S_{t^*} z_1 - S_{t^*} z_2\right\| \leq q \cdot \left\|z_1 - z_2\right\| + n_H\left(V z_1 - V z_2\right), \tag{B.4}$$

for every $z_1, z_2 \in \mathcal{B}$ with $0 \leq q < 1$. Here, the space Z, the operator V, the seminorm n_Z, and the time moment t^* may depend on $\mathcal{B}$.

• Let H be decomposed as $H = X \times Y \times Z$ where X, Y, Z are reflexive Banach spaces with X compactly embedded in Y, and endowed with the usual norm. Additionally, let the dynamical system (H, S_t) be given by an evolution operator like

$$S_t z = (u(t), u_t(t), \zeta(t)), \quad z = (u_0, u_1, \zeta_0) \in H, \tag{B.5}$$

where the functions u and ζ possess the properties

$$u \in C(\mathbb{R}^+; X) \cap C^1(\mathbb{R}^+; Y), \quad \zeta \in C(\mathbb{R}^+; Z). \tag{B.6}$$

Under this structure, one says that (H, S_t) is *asymptotically quasi-stable* on a set $\mathcal{B} \subset H$ if there exist a compact seminorm $n_X(\cdot)$ on X, non-negative scalar functions $a(t)$ and $c(t)$ locally bounded in $[0,\infty)$, and a function $b \in L^1(\mathbb{R}^+)$ with $\lim_{t\to\infty} b(t) = 0$, such that

$$\|S_t z_1 - S_t z_2\|^2 \leq a(t)\|z_1 - z_2\|^2, \tag{B.7}$$

and

$$\|S_t z_1 - S_t z_2\|^2 \leq b(t)\|z_1 - z_2\|^2 + c(t) \sup_{0<s<t} \big[n_X(u_1(s) - u_2(s))\big]^2, \tag{B.8}$$

for any $z_1, z_2 \in \mathcal{B}$, where we denote $S_t z_i = \big(u_i(t), u_{t,i}(t), \zeta_i(t)\big), i = 1, 2$.

The next result shows that the quasi-stability notion generalizes the concept of asymptotically quasi-stability for structural systems like (B.5)-(B.6). To avoid repetitions, whenever we (only) state below that (H, S_t) is asymptotically quasi-stable, it is implicit that (H, S_t) is given by (B.5)-(B.6).

Proposition B.10 ([12, Proposition 3.4.17])**.** *If (H, S_t) is asymptotically quasi-stable on some set $\mathcal{B} \subset H$, then it is quasi-stable on $\mathcal{B}$ at every time $T > 0$ such that $b(T) < 1$.*

The following result is a direct consequence of [12, Proposition 3.4.3 and Corollary 3.4.4] and [18, Proposition 7.9.4 and Corollary 7.9.5].

Proposition B.11. *Let us assume that the dynamical system (H, S_t) is dissipative and (asymptotically) quasi-stable on every bounded forward invariant set $\mathcal{B} \subset H$. Then, $\big(H, S_t\big)$ is asymptotically smooth and, consequently, it possesses a compact global attractor $\mathcal{A} \subset H$.*

Theorem B.12. *Let us assume that the dynamical system (H, S_t) possesses a compact global attractor $\mathcal{A}$ and is (asymptotically) quasi-stable on $\mathcal{A}$ at some point $t^* > 0$. Then, $\mathcal{A}$ has finite fractal dimension $dim_H^f \mathcal{A} < \infty$.*

Proof. See [18, Theorem 7.9.6] for asymptotically quasi-stable systems and [12, Theorem 3.4.5] for more general quasi-stable systems. ☐

For asymptotically quasi-stable systems one can reach the following regularity of trajectories from the attractor.

Theorem B.13 ([18], Theorem. 7.9.8)**.** *Let us assume that (H, S_t) satisfies the structure* (B.5)-(B.6)*, possesses a compact global attractor $\mathcal{A}$ and is asymptotically quasi-stable on $\mathcal{A}$. Additionally if* (B.8) *holds with $c(t)$ satisfying $c_\infty = \sup_{t\in\mathbb{R}^+} c(t) < \infty$, then any full trajectory*

$$\Gamma = \{(u(t), u_t(t), \zeta(t));\ t \in \mathbb{R}\} \subset \mathcal{A}$$

enjoys the following regularity properties

$$u_t \in L^\infty(\mathbb{R}; X) \cap C(\mathbb{R}, Y), \quad u_{tt} \in L^\infty(\mathbb{R}; Y), \quad \zeta_t \in L^\infty(\mathbb{R}; Z).$$

Besides, there exists a constant $R > 0$ such that

$$\sup_{\Gamma\subset\mathcal{A}}\sup_{t\in\mathbb{R}}\left(\|u_t(t)\|_X^2 + \|u_{tt}(t)\|_Y^2 + \|\zeta_t(t)\|_Z^2\right) \leq R^2,$$

where R depends on the constant c_∞, on the seminorm $n_X(\cdot)$, and on the embedding $X \hookrightarrow Y$.

Generalized fractal exponential attractors can be also reached for quasi-stable and asymptotic quasi-stable systems as follows. The next version can be found in [18, Theorem 7.9.9]. See also [12, Theorem 3.4.7].

Theorem B.14. *Let us assume that the dynamical system (H, S_t) is dissipative and asymptotically quasi-stable on some bounded absorbing set $\mathcal{B}$. In addition, let us suppose that there exists a space $\widetilde{H} \supseteq H$ such that mapping $t \mapsto S_t z$ is Hölder continuous in $\widetilde{H}$ for each $z \in \mathcal{B}$, that is, there exist $0 < \sigma \leq 1$ and $C_{\mathcal{B},T} > 0$ ($T > 0$ given) such that*

$$\|S_{t_2}z - S_{t_1}z\|_{\widetilde{H}} \leq C_{\mathcal{B},T}|t_2 - t_1|^\sigma, \quad t_1, t_2 \in [0, T], \ z \in \mathcal{B}. \tag{B.9}$$

Then, (H, S_t) possesses a generalized fractal exponential attractor $\mathcal{A}_{\exp}$ whose dimension is finite in the space $\widetilde{H}$ ($dim^f_{\widetilde{H}}\mathcal{A}_{\exp} < \infty$).

In Theorem B.14, unless $\widetilde{H} = H$, we can only guarantee the finite fractal dimension in an extended phase space $\widetilde{H}$. Thus, to achieve such finiteness of fractal dimensional in H, one must prove (B.9) in H, which sometimes seems to be a hard task. Therefore, in order to present a tangible result with exponential attractor whose fractal dimensional is finite in H ($\dim^f_H\mathcal{A}_{\exp} < \infty$), we finally remind an useful result by following [7, 8], which relies on the construction of time-dependent exponential attractors $\mathcal{A}_{\exp} = \{\mathcal{A}_{\exp}(t);\, t \in \mathbb{R}\}$ for (continuous) dynamical systems (H, S_t) under suitable decomposition and Lipschitz properties. For previous results on the subject, we also refer to [21, 22, 23, 39, 47]. The next concept and result are based on the more recent construction developed in [7, Section 4].

• The family $\mathcal{A}_{\exp} = \left\{\mathcal{A}_{\exp}(t);\, t \in \mathbb{R}\right\}$ is called a *time-dependent exponential attractor* for (H, S_t) if there exists $0 < a < \infty$ such that $\mathcal{A}_{\exp}(t) = \mathcal{A}_{\exp}(a + t)$ for all $t \in \mathbb{R}$, and

(i) the subsets $\mathcal{A}_{\exp}(t) \subset H$ are non-empty and compact in H for all $t \in \mathbb{R}$;

(ii) the family is positively semi-invariant, that is

$$S_t\mathcal{A}_{\exp}(s) \subset \mathcal{A}_{\exp}(t + s), \quad \forall\, t \geq 0, s \in \mathbb{R};$$

(iii) the fractal dimension of the sets $\mathcal{A}_{\exp}(t)$, $t \in \mathbb{R}$, is uniformly bounded;

(iv) the family attracts exponentially all bounded subsets of H uniformly, that is, there exists a positive constant $\omega > 0$ such that for any bounded subset $\mathcal{B} \subset H$

$$\lim_{\tau\to\infty} \sup_{t\in[0,a]} e^{\omega\tau} \operatorname{dist}_{\mathrm{H}}(S_\tau \mathcal{B}, \mathcal{A}_{\exp}(t)) = 0.$$

Theorem B.15 ([7, Theorem 4.4]). *Let us assume that the dynamical system (H, S_t) can be split into $S_t = S_t^1 + S_t^2 : H \to H$ and let W be another normed space with compact embedding $(H, \|\cdot\|_H) \hookrightarrow \big(W, \|\cdot\|_W\big)$. Let us additionally suppose the following conditions:*

(S1) *(H, S_t) is dissipative, that is, it has a bounded absorbing set $\mathcal{B} \subset H$;*
(S2) *there exist a constant $0 \le c_1 < \frac{1}{2}$ and a time $T > 0$ such that S_t^1 satisfies the contraction property on $\mathcal{B}$*

$$\|S_T^1 z_1 - S_T^1 z_2\|_H \le c_1 \|z_1 - z_2\|_H, \quad z_1, z_2 \in \mathcal{B};$$

(S3) *there exists a constant $c_2 > 0$ such that S_t^2 satisfies smoothing property within $\mathcal{B}$ at time $T > 0$*

$$\|S_T^2 z_1 - S_T^2 z_2\|_H \le c_2 \|z_1 - z_2\|_W, \quad z_1, z_2 \in \mathcal{B};$$

(S4) *there exists a time $T_0 \ge 0$ such that S_t is Lipschitz on $\mathcal{B}$ for $t \ge T_0$, that is, for some constant $L_t > 0$ it holds*

$$\|S_t z_1 - S_t z_2\|_H \le L_t \|z_1 - z_2\|_H, \quad z_1, z_2 \in \mathcal{B},\ t \ge T_0.$$

Then, (H, S_t) possesses a time-dependent exponential attractor

$$\mathcal{A}_{\exp} = \big\{\mathcal{A}_{\exp}(t);\, t \in \mathbb{R}\big\},$$

whose sections are compact subsets of H with finite fractal dimension in H, that is,

$$dim_H^f\big(\mathcal{A}_{\exp}(t)\big) < \infty, \quad \forall\, t \in \mathbb{R}.$$

Remark B.1. Finally, according to [7, Remark 6], under the assumptions of Theorem $B.15$ one can construct an exponential attractor for the dynamical system (H, S_t). Indeed, to this end it is enough to consider (in general) the union

$$\widetilde{\mathcal{A}_{\exp}} := \bigcup_{t\in[T,2T]} S_t \mathcal{A}_{\exp}^d,$$

where $\mathcal{A}_{\exp}^d$ denotes the exponential attractor for the corresponding discrete semigroup $\{S_{nT}\}_{n\in\mathbb{N}}$.

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