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Exponential Characterization in Linear Viscoelasticity Under Delay Perturbations

E. H. Gomes Tavares[1] · M. A. Jorge Silva[1] · T. F. Ma[2]

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Abstract
We present a complete characterization of the (uniform) exponential stabilization for a class of viscoelastic models under small delay perturbations. The main ingredient under consideration is the notion of *admissible* kernels. While in the standard literature it is mostly common to request a exponential/general kernel as a sufficient condition for the exponential/general stability of the whole viscoelastic system under study, here our objective is to employ the much more general concept of *admissible* kernels and prove that it is not only sufficient but also a necessary assumption for exponential stability in linear viscoelasticity under small delay perturbations.

Keywords Admissible kernel · Exponential stability · Linear viscoelasticity · Delay

Mathematics Subject Classification 34K99 · 35B35 · 74D99 · 93D23

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✉ E. H. Gomes Tavares
eduardogomes7107@gmail.com

M. A. Jorge Silva
marcioajs@uel.br

T. F. Ma
matofu@mat.unb.br

[1] Department of Mathematics, State University of Londrina, Londrina, PR 86057-970, Brazil

[2] Department of Mathematics, University of Brasília, Brasília, DF 70910-900, Brazil

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1 Introduction

1.1 The Model

Let $\langle H, \|\cdot\|, (\cdot,\cdot)\rangle$ be a Hilbert space and let $A : D(A) \subset H \to H$ be a strictly positive self-adjoint densely defined operator. Let us study the following second-order integro-differential problem with delay

$$\partial_{tt} z + A\left(z - \int_0^\infty g(s) z(t-s)\, ds\right) + \mu\, \partial_t z(t-\tau) = 0, \quad t > 0, \tag{1.1}$$

suplemented by the initial data

$$z(t) = z_0(t),\ t \in (-\infty, 0], \quad \partial_t z\big|_{t=0} = z_1, \quad \partial_t z(t-\tau) = z_2(t-\tau),\ t \in (0,\tau). \tag{1.2}$$

Here, $\tau > 0$ is the time lag, $\mu \in \mathbb{R}$ is the delay coefficient, and g is the so-called memory kernel.

As usual, for long-memory and delay problems it is introduced an equivalent autonomous system which is, indeed, the object of study.

Displacement history. We initially follow Dafermos [3, 4], where the idea of *displacement history* was introduced. Denoting by

$$\zeta^t(s) := z(t) - z(t-s), \quad t \geq 0, \quad s > 0,$$

it is easy to verify (formally) that

$$\begin{cases} \partial_t \zeta^t(s) = -\partial_s \zeta^t(s) + \partial_t z(t), & t, s > 0, \\ \zeta^t(0) := \lim\limits_{s\to 0} \zeta^t(s) = 0, & t > 0, \\ \zeta^0(s) = z_0(0) - z_0(-s), & s > 0. \end{cases} \tag{1.3}$$

Treating the supplementary system (1.3) as a Cauchy problem and calling $V := D(A^{1/2})$, it can be studied rigorously in the *memory space*

$$\mathcal{M} := \left\{\zeta : \mathbb{R}^+ \to V;\ \int_0^\infty g(s)\|A^{1/2}\zeta(s)\|^2\, ds < \infty\right\},$$

endowed with inner product

$$(\zeta, \xi)_{\mathcal{M}} = \int_0^\infty g(s)(A^{1/2}\zeta(s), A^{1/2}\xi(s))\, ds.$$

Indeed, Grasselli and Pata [8] obtained several useful results with respect to (1.3) by showing that $\mathbb{L} : D(\mathbb{L}) \subset \mathcal{M} \to \mathcal{M}$ defined by

$$D(\mathbb{L}) := \{\zeta \in \mathcal{M},\ \mathbb{L}\zeta \in \mathcal{M} \text{ and } \zeta(0) = 0\}, \quad \mathbb{L}\zeta := -\partial_s \zeta,$$

is the infinitesimal generator of a right-translation semigroup given by

$$[R(t)\zeta](s) := \begin{cases} \zeta(s-t), & s > t, \\ 0, & 0 < s \le t. \end{cases} \tag{1.4}$$

In addition, they used (1.4) to extract an explicit formula for ζ, namely

$$\zeta^t(s) = \begin{cases} \zeta^0(s-t) + z(t) - z_0(0), & s > t, \\ z(t) - z(t-s), & 0 < s \le t. \end{cases} \tag{1.5}$$

To do so, they assumed the following general assumption on the memory kernel g.

Assumption 1.1 The kernel $g : \mathbb{R}^+ \to [0, \infty)$ is absolutely continuous, non-increasing and summable, with total mass

$$\ell := \int_0^\infty g(s)\, ds \in (0, 1). \tag{1.6}$$

We remark that, under the scenario of Assumption 1.1, g has possibly a singularity at $s = 0$ and g' exists almost everywhere with $g'(s) \le 0$ for almost every $s > 0$.
Delay term. With respect to the delay term, we follow the lines of Nicaise and Pignotti [11], and introduce the variable

$$v(t, p) := \partial_t z(t - \tau p), \quad p \in (0, 1), \tag{1.7}$$

which formally fulfils the following advection-type equation

$$\begin{cases} \partial_t v(t, p) + \tau \partial_p v(t, p) = 0, & t > 0, \quad p \in (0, 1), \\ v(t, 0) = \partial_t z(t), & t > 0, \\ v(t, 1) = \partial_t z(t - \tau), & t > 0, \\ v(0, p) = z_2(-\tau p), & 0 < p < 1. \end{cases} \tag{1.8}$$

We still stress that, relying on the characteristic method (cf. [5, Section 3.2]), one can conclude that the unique solution of (1.8) is given by (1.7).

Therefore, from (1.3) and (1.8), in combination with (1.1)–(1.2), and setting

$$\omega := 1 - \ell > 0,$$

we are led to the following equivalent autonomous problem

$$\begin{cases} \partial_{tt} z + A\left(\omega z + \displaystyle\int_0^\infty g(s)\zeta(s)\, ds\right) + \mu v(1) = 0, \quad t > 0, \\ \partial_t \zeta = \mathbb{L}\zeta + \partial_t z, \quad t > 0, \\ \tau \partial_t v = -\partial_p v, \quad t > 0. \end{cases} \tag{1.9}$$

with corresponding initial data

$$\begin{cases} z(0) = z_0 := z_0(0), \ \partial_t z(0) = z_1, \\ \zeta^0(s) = \zeta_0(s) := z_0(0) - z_0(-s), \quad s > 0, \\ v(0, p) = v_0(p) := z_2(-\tau p), \qquad 0 < p < 1, \end{cases} \tag{1.10}$$

and compatibility conditions

$$\zeta^t(0) = 0, \quad v(t, 0) = \partial_t z(t), \quad t > 0. \tag{1.11}$$

Before describing our main result concerning the characterization of stability for (1.9)–(1.11), we emphasize the notion of admissible kernels and then summarize some recent achievements on related models.

1.2 State of the Art: Admissible Kernels

Let g be a memory kernel satisfying Assumption 1.1. The most common type of exponential kernels that can be found in the literature are those satisfying: *there exists* $\delta > 0$ *such that*

$$g'(s) \leq -\delta g(s), \tag{1.12}$$

for almost every $s > 0$. Inequality (1.12) was used in several works to control an integral term arising from the dissipation. Specifically, we have

$$\int_0^\infty g'(s) \|A^{1/2}\zeta(s)\|^2 \, ds \leq -\delta \|\zeta\|^2_{\mathcal{M}}, \quad \forall \ \zeta \in \mathcal{M}. \tag{1.13}$$

We still note that (1.12) is equivalent to

$$g(t + s) \leq e^{-\delta t} g(s),$$

for every $t > 0$ and for almost every $s > 0$. This fact motivates us to consider a large class of memory kernels as in Chepyzhov and Pata [2] by requiring that: *there exist* $\delta > 0$ *and* $c \geq 1$ *such that* g *satisfies*

$$g(t + s) \leq c e^{-\delta t} g(s), \tag{1.14}$$

for every $t > 0$ *and for almost every* $s > 0$.

Additionally, from the physical point of view, condition (1.12) is still too restrictive when compared with (1.14) for $c > 1$. Indeed, as pointed out in [6, 13–15], we quote

> "*Under the assumption (1.12), g does not have flat zones or even horizontal inflection points, when it should be conceivably true that exponential stability should be preserved if, say, we consider a kernel which is equal to a decreasing exponential, except on a small set. On the other hand, if $c > 1$, then the gap*

between the (1.14) and (1.12) is huge since every compactly supported kernel and some kernels with small flat zones, satisfy (1.14), but not (1.12)."

Now, performing a simple integration in (1.14) we obtain the following inequality

$$\int_s^\infty g(y)\,dy \le \frac{c}{\delta}\, g(s), \quad \forall\ s > 0,$$

which in turn motivates the construction of a new class of kernels as defined below.

Definition 1.1 (Admissible Kernel) A function $g : \mathbb{R}^+ \to [0, \infty)$ is an *admissible kernel* if there exists $\Theta > 0$ such that

$$\int_s^\infty g(y)\,dy \le \Theta\, g(s), \quad \forall\ s > 0. \tag{1.15}$$

Thus, by definition, condition (1.15) is more general than (1.14). However, if Assumption 1.1 holds true, then both conditions (1.15) and (1.14) are equivalent, as one can see in [6, Remark 2.3].

The previous discussion can be summarized in Figure 1 below.

1.3 A Brief Literature Overview

No delay perturbation: $\mu = 0$. In this situation, problem (1.9) falls into the purely dissipative system

$$\begin{cases} \partial_{tt} z + A\left(\omega z + \displaystyle\int_0^\infty g(s)\zeta(s)\,ds\right) = 0, \quad t > 0, \\ \partial_t \zeta = \mathbb{L}\zeta + \partial_t z, \quad t > 0, \end{cases} \tag{1.16}$$

which was studied by several authors for what concerns well-posedness and stability results. In particular, we are interested in the study of how the flatness of g influences the exponential stability of the C_0-semigroup of contractions $S_0(t)$ associated with

$$\begin{array}{rl}
\text{Differential inequality} & \boxed{g'(s) \le -\delta\, g(s),\ \delta > 0} \\
 & \Downarrow \qquad \not\Uparrow \\
\text{Pointwise inequality} & \boxed{g(t+s) \le c\, e^{-\delta t}\, g(s),\ c \ge 1,\ \delta > 0} \\
 & \Downarrow \qquad \Uparrow \quad \text{under the Assumption 1.1} \\
\text{Integral inequality} & \boxed{\displaystyle\int_s^\infty g(\tau)\, d\tau \le C\, g(s),\ C > 0}
\end{array}$$

Fig. 1 Comparison diagram for conditions (1.12), (1.14) and (1.15)

(1.16). We recall that a semigroup $T(t) : X \to X$ is exponentially stable in a Banach space X, if there exist $M \geq 1$ and $\gamma > 0$ such that

$$\|T(t)x\|_X \leq Me^{-\gamma t}\|x\|_X, \quad \forall\, x \in X. \tag{1.17}$$

For the linear viscoelastic problem (1.16) without delay ($\mu = 0$), we highlight the following useful results in (uniform) exponential stability (1.17). For the sake of reading and similarity to what we are interested in the present work, we omit results on general decays that are not uniform, although there are interesting cases in the literature.

Regarding the pioneering work by Chepyzhov and Pata [2], we summarize the following results therein:

R1. *the semigroup $R(t)$ set in (1.4) is exponentially stable on $\mathcal{M}$ iff the pointwise inequality (1.14) holds true;*

R2. *if $S_0(t)$ associated with (1.16) is exponentially stable on $V \times H \times \mathcal{M}$, then the semigroup $R(t)$ set in (1.4) is exponentially stable on $\mathcal{M}$. Consequently, (1.14) remains true by $R1$.*

Additionally, in Pata [13] it is shown that if the kernel g is not "too flat", then (1.14) implies the exponential stability of $S_0(t)$, which provides the converse of $R2$. Mathematically speaking:

R3. *If the rate of flatness (see (3.4) for its definition) of the kernel g is less than $1/2$, then (1.14) is a sufficient condition to obtain the exponential stability of $S_0(t)$.*

Concrete examples of kernels generating stability, uniform stability, and even instability for $S_0(t)$ can be found in Pata [14]. Furthermore, in Pata [15] semigroup tools are invoked to extend the result of [13] up to kernels with flatness rate higher than or equal to $1/2$, by allowing kernels almost totally flat in the study of stability in linear viscoelasticity with no delay term involved.
The role of delay perturbation: $\mu \neq 0$. Now, going back to (1.9)–(1.11), we first remark that it does not generate a dissipative semigroup $S_\mu(t)$ promptly. So, to overcome this small difficulty, we cite the work by Guesmia [9], where the well-posedness of (1.9)–(1.11) is considered for the first time. In addition, it is proved the following stability result therein:

R4. *Under the assumption (1.12), the corresponding energy functional decays exponentially provided that the delay is sufficiently small (depending on memory kernel g and structural constants).*

This result R4 corresponds to claim that the associated C_0-semigroup $S_\mu(t)$ is exponentially stable for small delays. Also, in Alabau-Boussouira et al. [1] an alternative method has been applied to conclude, roughly speaking, that the stability result R4 still holds true under the classical condition (1.12). In general, the assumption (1.12) along with the fact $g(0) > 0$ (which avoids singularity at the origin of the memory kernel) have been mostly assumed in the previous stability analysis when dealing with viscoelastic models (with or without delay), as one can see in [10, 12, 17, 18]. Nonetheless, as pointed out above, such condition is too restrictive and is usually regarded as a sufficient condition to prove the exponential stability of the corresponding (semigroup) solution, which can be summed up in the next diagram:

$$\boxed{\text{Assumption (1.12)}} \Rightarrow \boxed{\text{Exp. Stab. of } S_\mu(t)} \tag{1.18}$$

provide that $0 < |\mu| < \mu_0$ for some $\mu_0 > 0$.

1.4 Our Main Result and Organization

From the above exposition, our main goal in this work is to find out that admissible kernels are enough to ensure the converse of (1.18) for small delay perturbations. Obviously, if $g = 0$ then problem (1.9)–(1.11) is not damped and also blows up due to the presence of delay perturbations (cf. Nicaise and Pignotti [11]). Therefore, one sees the importance of the dissipativity provided by the memory kernel g and our main result (Theorem 3.1) provides the sufficient condition (Assumption (1.14)) to reach the uniform stability for (1.9)–(1.11). Though this is the relevant implication, for the sake of completeness, we still consider the converse of Theorem 3.1 which ends this work with a full characterization of the asymptotic behavior for solutions of (1.9)–(1.11) by means of admissible kernels, still including small delay perturbations (Theorem 3.8). Consequently, we prove the following equivalence:

$$\boxed{\text{Assumption (1.14)}} \Leftrightarrow \boxed{\text{Exp. Stab. of } R(t)} \Leftrightarrow \boxed{\text{Exp. Stab. of } S_\mu(t),\ \ \mu \neq 0}$$

provided that $0 < |\mu| < \mu_0$ for some $\mu_0 > 0$.

The remaining work is organized as follows. In Sect. 2 we introduce the preliminary tools in linear semigroup theory. In Sect. 3 we state and prove our main stability results.

2 Semigroup Framework

Our goal in this short section is just to prepare the semigroup framework to deal with the subsequently stability result for (1.9)–(1.11).

Denoting by

$$\mathcal{D} := L^2(0,1;H) = \left\{ v : (0,1) \to H,\ \int_0^1 \|v(p)\|^2\,dp < \infty \right\}$$

endowed with the norm

$$\|v\|_{\mathcal{D}}^2 = \int_0^1 \|v(p)\|^2\,dp,$$

we first observe that the asymptotic behavior of solutions for (1.9)–(1.11) will be done in the Hilbert space

$$\mathscr{H}_\mu := V \times H \times \mathcal{M} \times \mathcal{D}$$

equipped with the norm

$$\|(z, w, \zeta, v)\|^2_{\mathscr{H}_\mu} = \omega\|A^{1/2}z\|^2 + \|w\|^2 + \|\zeta\|^2_{\mathcal{M}} + \tau|\mu|\|v\|^2_{\mathcal{D}}.$$

The existence and uniqueness of solution for (1.9)–(1.11) is sketched as follows. Calling by $w = \partial_t z$ and $Z = (z, w, \zeta, v)^T$, for each $\mu \in \mathbb{R}$, we rewrite (1.9)–(1.11) as the abstract IVP

$$\begin{cases} \partial_t Z = \mathbb{B}_\mu Z, & t > 0, \\ Z(0) = Z_0, \end{cases} \tag{2.1}$$

where $Z_0 := (z_0, z_1, \zeta_0, v_0)^T$ and $\mathbb{B}_\mu : D(\mathbb{B}_\mu) \subset \mathscr{H}_\mu \to \mathscr{H}_\mu$ is given by

$$D(\mathbb{B}_\mu) = \Big\{(z, w, \zeta, v)^T \in V \times V \times D(\mathbb{L}) \times \mathcal{D},$$
$$\omega z + \mathbb{I}_g(\zeta) \in D(A),\ \partial_p v \in \mathcal{D},\ v(0) = w\Big\}$$

with

$$\mathbb{B}_\mu Z = \begin{bmatrix} w \\ -A\left[\omega z + \mathbb{I}_g(\zeta)\right] - \mu v(1) \\ \mathbb{L}\zeta + \partial_t z \\ -\frac{1}{\tau}\partial_p v \end{bmatrix},$$

and

$$\mathbb{I}_g(\zeta) := \int_0^\infty g(s)\zeta^t(s), \quad \zeta \in \mathcal{M}.$$

Under the Assumption 1.1 and still assuming

$$\lambda^{1/2}\|z\| \le \|A^{1/2}z\|, \quad \forall\, z \in D(A^{1/2}), \tag{2.2}$$

for some $\lambda > 0$, then proceeding similar to [9], and invoking the classical abstract results of [19], one can show that for each $\mu \in \mathbb{R}\backslash\{0\}$, the operator $\mathbb{B}_\mu$ is an infinitesimal generator of a C_0-semigroup $S_\mu(t)$ in $\mathscr{H}_\mu$. Consequently, it holds the following statements:

- If $Z_0 \in \mathscr{H}_\mu$, then (2.1) has a unique mild solution $Z(t) = S_\mu(t)Z_0$ in the class

$$Z \in C([0, +\infty); \mathscr{H}_\mu).$$

- If $Z_0 \in D(\mathbb{B}_\mu)$, then (2.1) has a unique classical solution in the class

$$Z \in C([0, +\infty); D(\mathbb{B}_\mu)) \cap C^1([0, +\infty); \mathscr{H}_\mu).$$

- The solution fulfills the energy inequality

$$\frac{d}{dt}\|Z(t)\|^2_{\mathscr{H}_\mu} \leq \int_0^\infty g'(s)\|A^{1/2}\zeta^t(s)\|^2\,ds + 2|\mu|\|\partial_t z(t)\|^2, \quad t > 0. \quad (2.3)$$

From now on, the semigroup generated by (2.1), and hence related to problem (1.9)–(1.11), is always regarded as

$$S_\mu(t) = e^{\mathbb{B}_\mu t}, \quad t \geq 0, \quad (2.4)$$

whose characterization of the exponential behavior will be concluded in the next section.

Remark 2.1 Actually, the study of existence and long-time behaviour of the semigroup $S_\mu(t)$ can be done considering a class of admissible kernels containing a finite number of jumps or an infinite increasing sequence of jumps [6, Remark 2.1].

3 Main Stability Result

We are in position to state the main result of the paper.

Theorem 3.1 *Let us assume Assumption 1.1 and that $R_g < \frac{1}{2}$. If g is an admissible kernel in the sense of Definition 1.1, then there exists a constant $\mu_0 > 0$, independent of μ, such that $S_\mu(t) = e^{\mathbb{B}_\mu t}$ given in (2.4) is exponentially stable on $\mathscr{H}_\mu$, for every $0 < |\mu| < \mu_0$.*

To prove Theorem 3.1, we are going to introduce some auxiliary tools and technical results to be developed in the next two subsections. Then, the completion of the proof is done right after, as well as its converse which gives the promised exponential characterization result.

3.1 Functional Notations

In what follows, we provide the necessary notations to set the functionals over the solutions of (1.9)–(1.11), namely, those coming from the semigroup solution (2.4) for any $\mu \in \mathbb{R}\backslash\{0\}$. The next concepts were firstly introduced by Pata [13] to deal with case $\mu = 0$ in (1.9).

Let $\varepsilon > 0$ be given. To simplify the notations, we shall always denote by C_ε all (different) positive constants depending on ε as well as by C all (different) other generic positive constants.

Let E be a mensurable set of $\mathbb{R}^+$. The *probability measure* of E associated to g is defined by

$$\hat{g}(E) := \frac{1}{\ell}\int_E g(s)\,ds.$$

For any $\kappa > 0$, we consider the disjoint decomposition $\mathbb{R}^+ = M_\kappa \cup P_\kappa$ where

$$M_\kappa = \left\{s \in \mathbb{R}^+,\ \kappa g'(s) + g(s) \leq 0\right\}, \quad P_\kappa = \left\{s \in \mathbb{R}^+,\ \kappa g'(s) + g(s) > 0\right\} \cup \mathcal{O}$$

and $\mathcal{O}$ is the null set where g' is not defined. Then, we may write

$$\mathbb{I}_g = \mathbb{I}_g^{P_\kappa} + \mathbb{I}_g^{M_\kappa} \tag{3.1}$$

where

$$\mathbb{I}_g^{P_\kappa}(\zeta) := \int_{P_\kappa} g(s)\zeta^t(s)\,ds, \quad \mathbb{I}_g^{M_\kappa}(\zeta) := \int_{M_\kappa} g(s)\zeta^t(s)\,ds,$$

and, for every $\zeta \in \mathcal{M}$, we have

$$\|\zeta\|^2_{\mathcal{M}} = \int_{P_\kappa} g(s)\|A^{1/2}\zeta(s)\|^2\,ds + \int_{M_\kappa} g(s)\|A^{1/2}\zeta(s)\|^2\,ds =: \mathbb{P}_\kappa(\zeta) + \mathbb{M}_\kappa(\zeta). \tag{3.2}$$

A straightforward application of Hölder's inequality yields the estimates

$$\|\mathbb{I}_g^{P_\kappa}(A^{1/2}\zeta)\|^2 \leq \ell\hat{g}(P_\kappa)\mathbb{P}_\kappa(\zeta), \quad \|\mathbb{I}_g^{M_\kappa}(A^{1/2}\zeta)\|^2 \leq \ell\hat{g}(M_\kappa)\mathbb{M}_\kappa(\zeta), \quad \forall\ \zeta \in \mathcal{M}. \tag{3.3}$$

Now, we define the *flatness set* of g as

$$\mathcal{F}_g := \{s \in \mathbb{R}^+,\ g(s) > 0 \text{ and } g'(s) = 0\},$$

and the *flatness rate* of g as

$$R_g := \hat{g}(\mathcal{F}_g). \tag{3.4}$$

As pointed in [15], the sets P_κ are decreasingly nested with

$$\bigcap_{\kappa>0} P_\kappa = \mathcal{F}_g \cup \mathcal{O}$$

and, consequently,

$$\lim_{\kappa\to+\infty} \hat{g}(P_\kappa) = R_g. \tag{3.5}$$

To deal with the possible singularity at $s = 0$, we observe that for any $\varepsilon \in (0, 1)$, there exist $s_\varepsilon > 0$ such that

$$\int_0^{s_\varepsilon} g(s)\,ds \leq \frac{\ell\varepsilon}{2}. \tag{3.6}$$

Then, we set the truncated kernel $g_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$g_\varepsilon(s) := g(s_\varepsilon)\chi_{(0,s_\varepsilon]}(s) + g(s)\chi_{(s_\varepsilon,\infty)}(s),$$

which satisfies

$$g_\varepsilon(s) = [g(s_\varepsilon) - g(s)]\chi_{(0,s_\varepsilon]} + g(s) \leq g(s), \quad s > 0. \tag{3.7}$$

Under the above statements, we set the following helpful functionals $\Psi_i : \mathscr{H}_\mu \to \mathbb{R}$, $i = 1, 2, 3, 4$, as

$$\begin{aligned}
&\Psi_1(z, w, \zeta, v) := (w, z),\\
&\Psi_2(z, w, \zeta, v) := -\frac{1}{\ell}\left(w, \mathbb{I}_{g_\varepsilon}(\zeta)\right),\\
&\Psi_3(z, w, \zeta, v) := \int_0^\infty \left(\int_s^\infty g(y)\chi_{P_\kappa}(y)\,dy\right)\|A^{1/2}\zeta(s) - A^{1/2}z\|^2\,ds,\\
&\Psi_4(z, w, \zeta, v) := \tau\int_0^1 e^{-2\tau p}\|v(t, p)\|^2\,dp,
\end{aligned}$$

which are object of study in the sequel.

3.2 Technical Estimates

We start by exploring the strength of admissible kernels.

Lemma 3.2 *If g is an admissible kernel, then there exists $a_0 > 0$, independent of μ, such that*

$$\sum_{i=1}^{3}|\Psi_i(z, w, \zeta, v)| \leq a_0\|(z, w, \zeta, v)\|^2_{\mathscr{H}_\mu}, \tag{3.8}$$

for every $(z, w, \zeta, v) \in \mathscr{H}_\mu$.

Proof It is a direct consequence of Cauchy-Schwarz and Young's inequalities combined with (1.15). □

Let $Z_0 = (z_0, w_0, \zeta_0, v_0) \in D(\mathbb{B}_\mu)$ and $Z(t) = S_\mu(t)Z_0 = (z(t), \partial_t z(t), \zeta^t, v(t)) \in D(\mathbb{B}_\mu)$. Since $D(\mathbb{B}_\mu)$ is dense in $\mathscr{H}_\mu$, we emphasize that all results below remain valid for $Z_0 \in \mathscr{H}_\mu$.

Lemma 3.3 *Along the solutions, the functional Ψ_1 satisfies the following inequality*

$$\begin{aligned}
\frac{d}{dt}\Psi_1(Z(t)) \leq \|\partial_t z(t)\|^2 - \omega(1-\varepsilon)\|A^{1/2}z(t)\|^2 + C_\varepsilon\mathbb{M}_\kappa(\zeta^t)\\
-(A^{1/2}z(t), \mathbb{I}_g^{P_\kappa}(A^{1/2}\zeta^t)) - \mu(v(t,1), z(t)),
\end{aligned}$$

for every $\varepsilon > 0$.

Proof Using $(1.9)_1$, we have

$$\frac{d}{dt}\Psi_1(Z(t)) = \|\partial_t z(t)\|^2 - \omega\|A^{1/2}z(t)\|^2 - \left(A^{1/2}z(t), \mathbb{I}_g(A^{1/2}\zeta^t)\right) - \mu(z(t), v(t,1)).$$

On the other hand, from (3.1), (3.2) and (3.3), we infer

$$\begin{aligned} -\left(A^{1/2}z(t), \mathbb{I}_g(A^{1/2}\zeta^t)\right) &= -\left(A^{1/2}z(t), \mathbb{I}_g^{M_\kappa}(A^{1/2}\zeta^t)\right) - \left(A^{1/2}z(t), \mathbb{I}_g^{P_\kappa}(A^{1/2}\zeta^t)\right) \\ &\le \|A^{1/2}z(t)\|\|\mathbb{I}_g^{M_\kappa}(A^{1/2}\zeta^t)\| - \left(A^{1/2}z(t), \mathbb{I}_g^{P_\kappa}(A^{1/2}\zeta^t)\right) \\ &\le \omega\varepsilon\|A^{1/2}z(t)\|^2 + C_\varepsilon\mathbb{M}_\kappa(\zeta^t) - \left(A^{1/2}z(t), \mathbb{I}_g^{P_\kappa}(A^{1/2}\zeta^t)\right). \end{aligned}$$

Combining the above estimates, the result follows. □

Lemma 3.4 *For every $\varepsilon > 0$, the functional Ψ_2 fulfills the inequality*

$$\begin{aligned} \frac{d}{dt}\Psi_2(Z(t)) \le &-(1-\varepsilon)\|\partial_t z(t)\|^2 + 2\omega\varepsilon^{1/2}\|A^{1/2}z(t)\|^2 \\ &+\left[(1+\varepsilon^{1/2})\hat{g}(P_\kappa) + \omega\varepsilon^{1/2}\right]\mathbb{P}_\kappa(\zeta^t) \\ &+C_\varepsilon\mathbb{M}_\kappa(\zeta^t) + \frac{g(s_\varepsilon)}{2\varepsilon\lambda\ell^2}\left(\int_0^\infty -g'(s)\|A^{1/2}\zeta^t(s)\|^2\,ds\right) \\ &+\frac{\omega}{\ell}(A^{1/2}z(t), \mathbb{I}_g^{P_\kappa}(A^{1/2}\zeta^t)) + \frac{\mu}{\ell}(v(t,1), \mathbb{I}_g(\zeta^t)) \end{aligned}$$

Proof Taking the derivative of $\Psi_2(Z(t))$ and using $(1.9)_1$, we get

$$\frac{d}{dt}\Psi_2(Z(t)) = \sum_{i=1}^{4}\Phi_i(Z(t)) + \frac{\mu}{\ell}(v(t,1), \mathbb{I}_g(\zeta^t)), \tag{3.9}$$

where,

$$\begin{aligned} \Phi_1(Z(t)) &:= -\frac{1}{\ell}\left(\int_0^\infty g_\varepsilon(s)\,ds\right)\|\partial_t z(t)\|^2, \\ \Phi_2(Z(t)) &:= -\frac{1}{\ell}\left(\partial_t z(t), \mathbb{I}_{g_\varepsilon}(\mathbb{L}\zeta^t)\right), \\ \Phi_3(Z(t)) &:= \frac{\omega}{\ell}\left(A^{1/2}z(t), \mathbb{I}_{g_\varepsilon}(A^{1/2}\zeta^t)\right), \\ \Phi_4(Z(t)) &:= \frac{1}{\ell}\left(\mathbb{I}_g(A^{1/2}\zeta^t), \mathbb{I}_{g_\varepsilon}(A^{1/2}\zeta^t)\right). \end{aligned}$$

Let us give a proper estimate for the right side of (3.9). Indeed, we can write

$$\Phi_1(Z(t)) = -\frac{s_\varepsilon}{\ell}g(s_\varepsilon)\|\partial_t z(t)\|^2 + \frac{1}{\ell}\left(\int_0^{s_\varepsilon} g(s)\,ds\right)\|\partial_t z(t)\|^2 - \|\partial_t z(t)\|^2.$$

From (3.6), we obtain

$$\Phi_1(Z(t)) \leq -\left(1-\frac{\varepsilon}{2}\right)\|\partial_t z(t)\|^2.$$

Now, by definition of g_ε we infer

$$\Phi_2(Z(t)) = -\frac{g(s_\varepsilon)}{\ell}\int_0^{s_\varepsilon}\left(\partial_t z(t), \mathbb{L}\zeta^t(s)\right)ds - \frac{1}{\ell}\int_{s_\varepsilon}^{\infty} g(s)\left(\partial_t z(t), \mathbb{L}\zeta^t(s)\right)ds.$$

Since $\zeta^t \in D(\mathbb{L})$, we can use the same arguments from [8] to get

$$\Phi_2(Z(t)) = \frac{1}{\ell}\int_{s_\varepsilon}^{\infty} -g'(s)\left(\partial_t z(t), \zeta^t(s)\right)ds.$$

Thus, from Hölder's inequality and (2.2) we have

$$\begin{aligned}\Phi_2(Z(t)) &\leq \frac{1}{\ell}\|\partial_t z(t)\|\left(\int_{s_\varepsilon}^{\infty} -g'(s)\|\zeta^t(s)\|\,ds\right)\\ &\leq \frac{\varepsilon}{2}\|\partial_t z(t)\|^2 + \frac{1}{2\varepsilon\lambda\ell^2}\left(\int_{s_\varepsilon}^{\infty} -g'(s)\|A^{1/2}\zeta^t(s)\|\,ds\right)^2\\ &= \frac{\varepsilon}{2}\|\partial_t z(t)\|^2 + \frac{g(s_\varepsilon)}{2\varepsilon\lambda\ell^2}\left(\int_0^{\infty} -g'(s)\|A^{1/2}\zeta^t(s)\|^2\,ds\right).\end{aligned}$$

To estimate $\Phi_3(Z(t))$, we use (3.7) to write

$$\Phi_3(Z(t)) = \Phi_{3,1}(Z(t)) + \Phi_{3,2}(Z(t)) + \frac{\omega}{\ell}(A^{1/2}z(t), \mathbb{I}_g^{P_\kappa}(A^{1/2}\zeta^t)), \tag{3.10}$$

where,

$$\begin{aligned}\Phi_{3,1}(Z(t)) &= -\frac{\omega}{\ell}\int_0^{s_\varepsilon}[g(s)-g(s_\varepsilon)](A^{1/2}z(t), A^{1/2}\zeta^t(s))\,ds,\\ \Phi_{3,2}(Z(t)) &= \frac{\omega}{\ell}(A^{1/2}z(t), \mathbb{I}_g^{M_\kappa}(A^{1/2}\zeta^t)).\end{aligned}$$

Using again (3.6) and taking into account that g is nondecreasing, we obtain

$$\begin{aligned}\left|\Phi_{3,1}(Z(t))\right| &\leq \frac{\omega}{\ell}\left(\int_0^{s_\varepsilon} g(s)\|A^{1/2}\zeta^t(s)\|\,ds\right)\|A^{1/2}z(t)\|\\ &\leq \frac{\omega}{\ell}\left(\int_0^{s_\varepsilon} g(s)\,ds\right)^{1/2}\|\zeta^t\|_{\mathcal{M}}\|A^{1/2}z(t)\|\\ &\leq \omega\varepsilon^{1/2}\|A^{1/2}z(t)\|^2 + \frac{\omega\varepsilon^{1/2}}{8\ell}\mathbb{P}_\kappa(\zeta^t) + \frac{\omega\varepsilon^{1/2}}{8\ell}\mathbb{M}_\kappa(\zeta^t).\end{aligned}$$

From (3.3), we can estimate $\Phi_{3,2}(Z(t))$ by

$$\begin{aligned}\left|\Phi_{3,2}(Z(t))\right| &\leq \frac{\omega}{\ell}\|A^{1/2}z(t)\|\|\mathbb{I}_g^{M_\kappa}(A^{1/2}\zeta^t)\| \\ &\leq \frac{\omega}{\ell}\ell^{1/2}\|A^{1/2}z(t)\|[\mathbb{M}_\kappa(\zeta^t)]^{1/2} \\ &\leq \omega\varepsilon\|A^{1/2}z(t)\|^2 + \frac{\omega}{4\ell\varepsilon}\mathbb{M}_\kappa(\zeta^t).\end{aligned}$$

Replacing the above estimates in (3.10) we arrive at

$$\begin{aligned}\Phi_3(Z(t)) \leq 2\omega\varepsilon^{1/2}\|A^{1/2}z(t)\|^2 + \frac{\omega\varepsilon^{1/2}}{8\ell}\mathbb{P}_\kappa(\zeta^t) + C_\varepsilon\mathbb{M}_\kappa(\zeta^t) \\ +\frac{\omega}{\ell}(A^{1/2}z(t), \mathbb{I}_g^{P_\kappa}(A^{1/2}\zeta^t)).\end{aligned}$$

To estimate the term $\Phi_4(Z(t))$, we use (3.1), (3.3) and (3.7) to get

$$\begin{aligned}\Phi_4(Z(t)) &\leq \frac{1}{\ell}\left(\int_0^\infty g(s)\|A^{1/2}\zeta^t(s)\|\,ds\right)^2 \\ &\leq \hat{g}(P_\kappa)\mathbb{P}_\kappa(\zeta^t) + 2[\hat{g}(P_\kappa)\mathbb{P}_\kappa(\zeta^t)\mathbb{M}_\kappa(\zeta^t)]^{1/2} + \mathbb{M}_\kappa(\zeta^t) \\ &\leq (1+\varepsilon^{1/2})\hat{g}(P_\kappa)\mathbb{P}_\kappa(\zeta^t) + \left(\frac{1+\varepsilon^{1/2}}{\varepsilon^{1/2}}\right)\mathbb{M}_\kappa(\zeta^t).\end{aligned}$$

Plugging the above estimates in (3.9), we conclude the desire estimate. □

Lemma 3.5 *Along the solutions, the functional Ψ_3 satisfies the following equality*

$$\frac{d}{dt}\Psi_3(Z(t)) = -\mathbb{P}_\kappa(\zeta^t) + 2(A^{1/2}z(t), \mathbb{I}_g^{P_\kappa}(A^{1/2}\zeta^t)).$$

Proof Differentiating $\Psi_3(Z(t))$ and using $(1.9)_2$, we get

$$\begin{aligned}\frac{d}{dt}\Psi_3(Z(t)) = -\int_0^\infty \left(\int_s^\infty g(y)\chi_{P_\kappa}(y)\,dy\right)\frac{d}{ds}\|A^{1/2}\zeta^t(s)\|^2\,ds \\ +2\int_0^\infty\left(\int_s^\infty g(y)\chi_{P_\kappa}(y)\,dy\right)\frac{d}{ds}\left(A^{1/2}\zeta^t(s), A^{1/2}z(t)\right)ds.\end{aligned}$$

Hence, integrating by parts and taking into account that $\zeta^t \in D(\mathbb{L})$, we obtain the desire result. □

Lemma 3.6 *Along the solutions, the functional Ψ_4 fulfills the inequality*

$$\frac{d}{dt}\Psi_4(Z(t)) \leq -2\tau e^{-2\tau}\|v(t)\|^2_{\mathcal{D}} + \|\partial_t z(t)\|^2 - e^{-2\tau}\|v(t,1)\|^2.$$

Proof Taking the derivative of $\Psi_4(Z(t))$ and using $(1.9)_3$, we get

$$\begin{aligned}\frac{d}{dt}\Psi_4(Z(t)) &= -2\tau\int_0^1 e^{-2\tau p}\|v(t,p)\|^2\,dp + \|\partial_t z(t)\|^2 - e^{-2\tau}\|v(t,1)\|^2\\ &\le -2\tau e^{-2\tau}\|v(t)\|^2_{\mathcal{D}} + \|\partial_t z(t)\|^2 - e^{-2\tau}\|v(t,1)\|^2.\end{aligned}$$

□

For the next lemma, we previously stress that if g is an admissible kernel such that $R_g < \frac{1}{2}$, then from (3.5) we can pick $\kappa_0 > 0$ large enough such that $\hat{g}(P_{\kappa_0}) < \frac{1}{2}$.

Lemma 3.7 *Let g be an admissible kernel such that $R_g < \frac{1}{2}$ and, for every $n > 0$, we set*

$$\begin{aligned}\Upsilon_n(Z(t)) = n\|Z(t)\|^2_{\mathscr{H}_\mu} &+ \alpha_0\Psi_1(Z(t)) + \Psi_2(Z(t)) + \frac{1}{2}\left(\frac{\omega}{\ell}+\alpha_0\right)\Psi_3(Z(t))\\ &+\frac{\beta_0\, e^{-2\tau}}{8}\Psi_4(Z(t)),\end{aligned}$$

where we denote

$$\alpha_0 := \frac{1}{2} + \hat{g}(P_{\kappa_0}) \in (1/2, 1), \quad \beta_0 := 1-\alpha_0 \in (0, 1/2).$$

Then, there exist $b_0 > 0$ and $\varepsilon_0 > 0$, independent of $|\mu|$, such that

$$\left|\Upsilon_n(Z(t)) - n\|Z(t)\|^2_{\mathscr{H}_\mu}\right| \le \left(\frac{\beta_0\, e^{-2\tau}}{8|\mu|} + b_0\right)\|Z(t)\|^2_{\mathscr{H}_\mu}, \tag{3.11}$$

and

$$\begin{aligned}\frac{d}{dt}\Upsilon_n(Z(t)) \le &-\left[\frac{5}{8}\beta_0\, e^{-2\tau} - n|\mu|\right]\|\partial_t z(t)\|^2 - \frac{\beta_0\, e^{-2\tau}}{4}\tau\|v(t)\|^2_{\mathcal{D}}\\ &-\left[\frac{\beta_0\, e^{-4\tau}}{8} - \frac{|\mu|}{\omega\ell\lambda^{1/2}}\right]\left(\omega\|A^{1/2}z(t)\|^2 + \mathbb{P}_{\kappa_0}(\zeta^t) + \|v(t,1)\|^2\right)\\ &+\frac{|\mu|}{\omega\ell\lambda^{1/2}}\mathbb{M}_{\kappa_0}(\zeta^t) - \left[n - C_{\varepsilon_0}\right]\left(\int_0^\infty -g'(s)\|A^{1/2}\zeta^t(s)\|^2\,ds\right).\end{aligned} \tag{3.12}$$

Proof First, we observe that (3.11) holds from (3.8) with

$$b_0 = a_0 \max\left\{\frac{\omega}{\ell} + \alpha_0, 1\right\} > 0.$$

Next, combining Lemma 3.3, Lemma 3.4, Lemma 3.5, Lemma 3.6 and taking into account that

$$\mathbb{M}_{\kappa_0}(\zeta^t) \leq \kappa_0 \left(\int_0^\infty -g'(s) \| A^{1/2} \zeta^t(s) \|^2 \, ds \right), \quad t > 0, \tag{3.13}$$

we have

$$\begin{aligned} \frac{d}{dt} \Upsilon_n(Z(t)) \leq & - \left[\frac{7}{8} \beta_0 \, e^{-2\tau} - \left(\varepsilon^{1/2} + n|\mu| \right) \right] \| \partial_t z(t) \|^2 \\ & - \left[\alpha_0 - \varepsilon^{1/2} \, (3 - \beta_0) \right] \omega \| A^{1/2} z(t) \|^2 \\ & - \left[\frac{\omega}{2\ell} + \frac{\beta_0}{2} - \varepsilon^{1/2} \left(\hat{g}(P_{\kappa_0}) + \omega \right) \right] \mathbb{P}_{\kappa_0}(\zeta^t) \\ & - \frac{\beta_0 \, e^{-4\tau}}{4} \tau \| v(t) \|_{\mathcal{D}}^2 - \frac{\beta_0 \, e^{-4\tau}}{8} \| v(t, 1) \|^2 \\ & - [n - C_\varepsilon] \left(\int_0^\infty -g'(s) \| A^{1/2} \zeta^t(s) \|^2 \, ds \right) \\ & + \frac{2\omega}{\ell} (A^{1/2} z(t), \mathbb{I}_g^{P_{\kappa_0}} (A^{1/2} \zeta^t)) \\ & - \mu (v(t, 1), \alpha_0 z(t)) + \frac{\mu}{\ell} (v(t, 1), \mathbb{I}_g(\zeta^t)), \end{aligned} \tag{3.14}$$

for every $\varepsilon \in (0, 1)$. Let us estimate the two last terms of (3.14). Indeed, using (3.3) we obtain

$$\frac{2\omega}{\ell} \left| (A^{1/2} z(t), \mathbb{I}_g^{P_{\kappa_0}} (A^{1/2} \zeta^t)) \right| \leq 2\omega \hat{g}(P_{\kappa_0}) \| A^{1/2} z(t) \|^2 + \frac{\omega}{2\ell} \mathbb{P}_{\kappa_0}(\zeta^t).$$

The last term can be estimate by

$$\begin{aligned} & -\mu (v(t, 1), \alpha_0 z(t)) + \frac{\mu}{\ell} (v(t, 1), \mathbb{I}_g(\zeta^t)) \\ & \leq \frac{|\mu|}{\omega \ell \lambda^{1/2}} \left[\| v(t, 1) \|^2 + \omega \| A^{1/2} z(t) \|^2 + \| \zeta^t \|_{\mathcal{M}}^2 \right]. \end{aligned}$$

Plugging the above estimates in (3.14) and using that $e^{-4\tau} < 1$, we arrive at

$$\begin{aligned} \frac{d}{dt} \Upsilon_n(Z(t)) \leq & - \left[\frac{7}{8} \beta_0 e^{-2\tau} - \left(\varepsilon^{1/2} + n|\mu| \right) \right] \| \partial_t z(t) \|^2 \\ & - \left[\frac{\beta_0 \, e^{-4\tau}}{2} - \varepsilon^{1/2} \, (3 - \beta_0) - \frac{|\mu|}{\omega \ell \lambda^{1/2}} \right] \omega \| A^{1/2} z(t) \|^2 \\ & - \left[\frac{\beta_0 \, e^{-4\tau}}{2} - \varepsilon^{1/2} \left(\hat{g}(P_{\kappa_0}) + \omega \right) \right] \mathbb{P}_{\kappa_0}(\zeta^t) + \frac{|\mu|}{\omega \ell \lambda^{1/2}} \| \zeta^t \|_{\mathcal{M}}^2 \\ & - \frac{\beta_0 \, e^{-4\tau}}{4} \tau \| v(t) \|_{\mathcal{D}}^2 - \left[\frac{\beta_0 \, e^{-4\tau}}{8} - \frac{|\mu|}{\omega \ell \lambda^{1/2}} \right] \| v(t, 1) \|^2 \end{aligned}$$

$$-[n-C_\varepsilon]\left(\int_0^\infty -g'(s)\|A^{1/2}\zeta^t(s)\|^2\,ds\right), \tag{3.15}$$

for every $\varepsilon \in (0, 1)$. At this point, we choose

$$\varepsilon_0 := \frac{\beta_0^2\, e^{-16\tau}}{16(3-\beta_0)^2} \in (0, 1).$$

Since

$$\varepsilon_0^{1/2} < \frac{\beta_0\, e^{-4\tau}}{4(3-\beta_0)} < \frac{\beta_0\, e^{-4\tau}}{4}\min\left\{e^{2\tau}, \frac{1}{\hat{g}(P_{\kappa_0})+\omega}\right\},$$

we conclude (3.12) from (3.15). □

3.3 Proof of Theorem 3.1 (Completion)

Let $\mu_0 > 0$ defined by

$$\mu_0 := \beta_0\, e^{-2\tau}\min\left\{\frac{3}{8b_0}, \frac{\omega\ell\lambda^{1/2}e^{-2\tau}}{16}, \frac{4\kappa_0 C_{\varepsilon_0}}{\beta_0 e^{-4\tau}+8\kappa_0 C_{\varepsilon_0}}\right\},$$

where $b_0 >$ and $C_{\varepsilon_0} > 0$ are given by Lemma 3.7. From the choice of μ_0, we observe that

$$\max\left\{\frac{\beta_0\, e^{-2\tau}}{8|\mu|}+b_0, \frac{\beta_0\, e^{-4\tau}}{8\kappa_0}+\hat{C}\right\} < \frac{\beta_0\, e^{-2\tau}}{2|\mu|}.$$

Then, we can take $n_0 > 0$ satisfying

$$\max\left\{\frac{\beta_0\, e^{-2\tau}}{8|\mu|}+b_0, \frac{\beta_0\, e^{-4\tau}}{8\kappa_0}+\hat{C}\right\} < n_0 < \frac{\beta_0\, e^{-2\tau}}{2|\mu|}.$$

For this choice of n_0, we deduce from (3.11) and (3.12) that $\Upsilon_{n_0}(Z(t)) > 0$ and

$$\begin{aligned}\frac{d}{dt}\Upsilon_{n_0}(Z(t)) \le &-\left[\frac{\beta_0\, e^{-4\tau}}{8}-\frac{|\mu|}{\omega\ell\lambda^{1/2}}\right]\\ &\times\left(\|\partial_t z(t)\|^2+\omega\|A^{1/2}z(t)\|^2+\mathbb{P}_{\kappa_0}(\zeta^t)+\tau\|v(t)\|_{\mathcal{D}}^2\right)\\ &+\frac{|\mu|}{\omega\ell\lambda^{1/2}}\mathbb{M}_{\kappa_0}(\zeta^t)-\frac{\beta_0\, e^{-4\tau}}{8\kappa_0}\left(\int_0^\infty -g'(s)\|A^{1/2}\zeta^t(s)\|^2\,ds\right),\end{aligned} \tag{3.16}$$

for every $t > 0$. Combining (3.13) and (3.16) and noting that

$$\frac{\beta_0 \, e^{-4\tau}}{8} - \frac{|\mu|}{\omega \ell \lambda^{1/2}} > \frac{\beta_0 \, e^{-4\tau}}{16},$$

we arrive at

$$\frac{d}{dt}\Upsilon_{n_0}(Z(t)) \leq -\frac{\beta_0 \, e^{-4\tau}}{16} \|S_\mu(t)Z_0\|^2_{\mathscr{H}_\mu}. \tag{3.17}$$

Let $T > 0$ be fixed. Integrating (3.17) in $(0, T)$ and noting that $\Upsilon_{n_0}(Z(T)) > 0$, we have

$$0 \leq \int_0^T \|S_\mu(t)Z_0\|^2_{\mathscr{H}_\mu} \, dt \leq \frac{16 \, e^{4\tau}}{\beta_0} \Upsilon_{n_0}(Z_0) < +\infty. \tag{3.18}$$

Since the right side of (3.18) does not depend on $T > 0$ we get

$$\int_0^\infty \|S_\mu(t)Z_0\|^2_{\mathscr{H}_\mu} < +\infty, \quad \forall \; Z_0 \in \mathscr{H}_\mu.$$

Hence, applying [16, Theorem 4.1] wiht $p = 2$, we conclude that $S_\mu(t)$ is exponentially stable in $\mathscr{H}_\mu$, for $0 < |\mu| < \mu_0$. □

3.4 Final Remark: Exponential Characterization

Actually, it is possible to characterize the uniform (exponential) stability for (1.9)–(1.11) in terms of admissible kernels (see (1.15)), the solution semigroup $S_\mu(t) = e^{\mathbb{B}_\mu t}$ given in (2.4), and right-translation semigroup $R(t) = e^{\mathbb{L}t}$ set in (1.4). It reads as follows:

Theorem 3.8 *Let us assume Assumption 1.1 and that $R_g < \frac{1}{2}$. Then, the following assertions are equivalent:*

- **(I)** *g is an admissible kernel in the sense of Definition 1.1;*
- **(II)** *there exists a constant $\mu_0 > 0$, independent of μ, such that $S_\mu(t) = e^{\mathbb{B}_\mu t}$ given in (2.4) is exponentially stable on $\mathscr{H}_\mu$, for every $0 < |\mu| < \mu_0$;*
- **(III)** *the right-translation semigroup $R(t) = e^{\mathbb{L}t}$ set in (1.4) is exponentially stable on $\mathcal{M}$.*

Therefore, if it holds one of these conditions, all of them hold and the solution $Z(t) = S_\mu(t)Z_0$ of (2.1) satisfies

$$\|Z(t)\|_{\mathscr{H}_\mu} \leq C\|Z_0\|_{\mathscr{H}_\mu} e^{-\varpi t}, \;\; t \geq 0,$$

for some structural positive constants C, ϖ.

Proof Theorem 3.1 ensures that (I) $\Longrightarrow$ (II). The proof of (II) $\Longrightarrow$ (III) is similar to [2, Theorem 3.2] but using the semigroup $S_{(\mu_0/2)}(t)$ instead of $S_0(t)$. The last implication ((III) $\Longrightarrow$ (I)) follows exactly the same arguments as in [2, Theorem 3.3]. □

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Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

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