

## Decay property for a novel partially dissipative viscoelastic beam system on the real line

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**Abstract.** We address here a viscoelastic Timoshenko model on the (one-dimensional) real line with memory damping coupled on a shear force. Our main results concern a complete decay structure of the system under the so-called *equal wave speeds* assumption, as well as without this condition. This is the first result of this type for partially dissipative beam systems with memory-type damping on the shear force. Our method is based on expanded structural conditions such as the so-called SK condition. In addition, we give a characterization of the dissipative structure of the system by using a spectral analysis method, which confirms that our decay structure is optimal.

**Keywords:** Timoshenko system; Fourier transform; decay rate estimate; stability.

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### 1. Introduction

In a recent work, Alves *et al.* [1], supported by the physical studies by Boltzmann [4, 5], Drozdov–Kolmanovskii [6], Prüss [13] and Timoshenko [19, 20], have

considered the following viscoelastic Timoshenko system:

$$\begin{cases} \rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x + \kappa \int_0^t g(t-s)(\varphi_x + \psi)_x(s)ds = 0 & \text{in } (0, L) \times \mathbb{R}^+, \\ \rho_2 \psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) - \kappa \int_0^t g(t-s)(\varphi_x + \psi)(s)ds = 0 & \text{in } (0, L) \times \mathbb{R}^+, \end{cases} \quad (1.1)$$

which describes the vibrations of a thin beam with length  $L > 0$ ,  $\mathbb{R}^+ = (0, \infty)$ , by means of the rotation angle  $\varphi$  and vertical displacement  $\psi$  considered on a reference line along the beam. The physical meaning of the positive constants  $\rho_1, \rho_2, \kappa, b$  as well as the memory kernel  $g$  are described in [1, Sec. 2]. Additionally, the authors proved that the system (1.1) is partially dissipative with memory damping acting only on the shear force. More precisely, under a suitable decaying condition on the memory kernel (cf. [1, Secs. 3 and 4]) it has been shown that

$$\text{the system (1.1) is uniformly stable} \Leftrightarrow \frac{\kappa}{\rho_1} = \frac{b}{\rho_2} \text{ (EWS).}$$

The right-hand side of the above equivalence is usually called Equal Wave Speeds (EWS) assumption and, as one can see, it is very deterministic in the stability of (1.1). A drawback of the results in [1] is not to give a decay response in the case of non-EWS assumption  $\frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2}$ , not even for more regular initial data and exponential memory kernels  $g$ . Besides, it is worth mentioning that the authors work in a bounded interval  $[0, L]$  ( $L > 0$  finite), then the tools employed in [1] can count on the following technical conditions:

- (a) Estimates with bounds  $\frac{1}{L}$  everywhere;
- (b) Poincaré's inequality.

In this paper, our main goal is to address problem (1.1) in whole one-dimensional space  $\mathbb{R}$  with respect to the spatial variable, which can be seen as  $\mathbb{R} = \lim_{L \rightarrow \infty} (-L, L)$ . In this way, problem (1.1) is described by

$$\begin{cases} \rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x + \kappa(g * (\varphi_x + \psi)_x) = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ \rho_2 \psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) - \kappa(g * (\varphi_x + \psi)) = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+, \end{cases} \quad (1.2)$$

on which we add initial data

$$\begin{aligned} \varphi(x, 0) &= \varphi_0(x), & \psi(x, 0) &= \psi_0(x), \\ \varphi_t(x, 0) &= \varphi_1(x), & \psi_t(x, 0) &= \psi_1(x), & x \in \mathbb{R}. \end{aligned} \quad (1.3)$$

The notation  $*$  stands for the pattern convolution term

$$(g * (\varphi_x + \psi))(t) := \int_0^t g(t-s)(\varphi_x + \psi)(s)ds.$$

As far as we know, there is no approach on stability results in the literature concerning problem (1.2). The results provided in [1] cannot be applied here since conditions (a)–(b) aforementioned are no longer valid in the present framework. Moreover, still in this worse unboundedness scenario but for exponential memory kernel (to be set in Sec. 2), we shall analyze both EWS and non-EWS assumptions, by showing optimal decay estimates in both cases, the second one being of regularity-loss type. Therefore, to deal with these statements, instead of the tools requested in [1] (which cannot be used in our context), we rely on a recent approach concerning the so-called  $S$  &  $K$  mixed condition (M) for hyperbolic systems with non-symmetric relaxations, being developed by Mori [12], which in turn is a generalization of previous literature on the subject, cf. [18, 21–23] and references therein. We also highlight that all the latter references [12, 22] have successfully applied their theory to viscoelastic Timoshenko-type systems with *bending moment memory damping*, cf. [9, 11], but none of them have considered problems like (1.2) with *shear force memory damping*. Therefore, our main findings feature for the first time a complete decay structure for the solution of (1.2)–(1.3) with exponential kernel  $g$ , see for instance:

- Theorem 4.1, which deals with the case of non-physical EWS assumption  $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$ ;
- Theorem 4.2, which treats the physical case of non-EWS condition  $\frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2}$ .

All precise details shall be presented in Sec. 2, where we introduce the notations as well as the problem in the Fourier space; in Sec. 3, where we recall the principal ingredients coming from the theory of first-order hyperbolic systems; and in Sec. 4, where we present our main results on pointwise and decay rate estimates related to problem (1.2)–(1.3).

Finally, we would like to mention other studies on the Timoshenko system with different dissipative mechanism; see, e.g. [14, 15] for frictional dissipation case, [7, 16, 17] for thermal dissipation case, and [3, 2, 10] for memory-type dissipation case.

## 2. Abstract Setting

### 2.1. Useful notations

For reader's convenience, we initially consider the standard notations used throughout this work as follows.

- (1) For  $k \geq 0$ ,  $\partial_x^k$  denotes the totality of all  $k$ th order derivatives with respect to  $x = (x_1, \dots, x_n)$ .
- (2) For  $1 \leq p \leq \infty$ ,  $L^p = L^p(\mathbb{R}^n)$  denotes the usual Lebesgue space over  $\mathbb{R}^n$  with the norm  $\|\cdot\|_{L^p}$ .
- (3) For  $s \geq 0$ ,  $H^s = H^s(\mathbb{R}^n)$  denotes the  $s$ th order Sobolev space with the norm  $\|\cdot\|_{H^s}$ . Note that  $H^0 = L^2$ .

(4) For a given  $m \times m$  matrix  $X$

- $X^T$  denotes the transposed matrix of  $X$ .
- $X > 0$  means that for any  $\phi \in \mathbb{C}^m$  with  $\phi \neq 0$ ,  $\operatorname{Re}\langle X\phi, \phi \rangle > 0$ , where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product in  $\phi \in \mathbb{C}^m$ .

(5) For a given real  $m \times m$  matrix  $X$

- $X^{\text{sym}}$  denotes the symmetric part of  $X$ :  $X^{\text{sym}} = (X + X^T)/2$ .
- $X^{\text{asy}}$  denotes the skew-symmetric part of  $X$ :  $X^{\text{asy}} = (X - X^T)/2$ .

(6) Let  $\hat{f} = \mathcal{F}[f]$  be the Fourier transform of  $f$ :

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) := \int_{\mathbb{R}} f(x) e^{-i\xi x} dx$$

For  $1 \leq p \leq \infty$ , we denote by  $L^p = L^p(\mathbb{R})$  the usual Lebesgue space on  $\mathbb{R}$  with the norm  $\|\cdot\|_{L^p}$ .

(7) We use  $C$  and  $c$  to denote various (universal) positive constants without confusion.

## 2.2. First-order problems

In order to simplify the notations, we are going to assume without loss of generality the following measurements  $\kappa, \rho_1, \rho_2 = 1$  and  $b = a^2$  with  $a > 0$  constant. Thus, problem (1.2)–(1.3) turns into

$$\begin{cases} \varphi_{tt} - (\varphi_x + \psi)_x + g * (\varphi_x + \psi)_x = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ \psi_{tt} - a^2 \psi_{xx} + (\varphi_x + \psi) - g * (\varphi_x + \psi) = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ (\varphi, \psi, \varphi_t, \psi_t)(x, 0) = (\varphi_0, \psi_0, \varphi_1, \psi_1)(x), & x \in \mathbb{R}. \end{cases} \quad (2.1)$$

In this way, the EWS assumption of (2.1) equals to

$$a^2 = 1 \Leftrightarrow \frac{\kappa}{\rho_1} = \frac{b}{\rho_2}. \quad (2.2)$$

Additionally, since we are dealing with exponential kernels, then to do a clearer (and simpler) discussion on the analysis of stability, we assume an explicit exponential kernel by its own, namely,

$$g(t) = \gamma e^{-\gamma t}, \quad t \geq 0, \quad (2.3)$$

where  $\gamma > 0$  is a constant.

Next, we introduce the following change of variables, which was first introduced in [8],

$$v = \varphi_x + \psi, \quad u = \varphi_t, \quad y = \psi_t, \quad z = a\psi_x$$

and set

$$\tilde{v} = g * v - v, \quad U = (\tilde{v}, y, u, z)^T.$$

Therefore, problem (2.1) can be reduced to the following first-order abstract system

$$\begin{cases} A_0 U_t + AU_x + LU = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ U(x, 0) = U_0(x), & x \in \mathbb{R}, \end{cases} \quad (2.4)$$

with the following matrix coefficients:

$$A_0 = I, \quad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -a \\ 1 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} \gamma & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.5)$$

and initial data  $U_0 := (\tilde{v}_0, y_0, u_0, z_0)^T$ . We still observe that the equation in (2.4) can be rewritten in terms of its components as follows:

$$\begin{cases} \tilde{v}_t + u_x + y + \gamma \tilde{v} = 0, \\ y_t - az_x - \tilde{v} = 0, \\ u_t + \tilde{v}_x = 0, \\ z_t - ay_x = 0. \end{cases} \quad (2.6)$$

To compute the derivatives  $\tilde{v}_t$  and  $\tilde{v}_x$ , one can take the advantage of (2.3) and derive the formulas  $(g * v)_t = \gamma v - \gamma g * v$  and  $(g * v)_x = g * v_x$ .

Under the above construction, we stress that the decay property will be analyzed upon the solution  $U$  of problem (2.4), which takes a particular form of a coming symmetric hyperbolic system, see for instance (3.1). Since the technique we shall employ here relies on pointwise estimates in the Fourier space, then taking the Fourier transform of (2.4) we arrive at

$$\begin{cases} \hat{U}_t + (i\xi A + L)\hat{U} = 0, & t > 0, \\ \hat{U}(\xi, 0) = \hat{U}_0(\xi), & \xi \in \mathbb{R}, \end{cases} \quad (2.7)$$

whose solution is given by

$$\hat{U}(\xi, t) = e^{t\hat{\Phi}(i\xi)} \hat{U}_0(\xi),$$

with

$$\hat{\Phi}(\zeta) = -(L + \zeta A), \quad \zeta \in \mathbb{C}. \quad (2.8)$$

Hence, by means of the inverse Fourier transform one gets the solution  $U$  of (2.4) as well.

In what follows, our main goal is to reach the stability properties with respect to both first order problems (2.4) and (2.7). Before doing so, we first recall the abstract theory in order to make the reading more comfortable.

### 3. A Brief Review on the Theory

First, we review some structural conditions to show the stability of the initial value problem of the following first-order hyperbolic system.

$$A_0 U_t + \sum_{j=1}^n A_j U_{x_j} + LU = 0, \quad U|_{t=0} = U_0. \quad (3.1)$$

Here,  $U = U(x, t)$  is an unknown function in  $\mathbb{R}^m$  over  $x \in \mathbb{R}^n$  and  $t > 0$ ,  $U_0 = U_0(x)$  is a given function in  $\mathbb{R}^m$  over  $x \in \mathbb{R}^n$ , where the integers  $m \geq 1$  and  $n \geq 1$  denote dimensions. The coefficient matrices  $A_j$  ( $j = 0, 1, 2, \dots, n$ ) and  $L$  are  $m \times m$  real constant matrices. By taking the Fourier transform of (3.1) with respect to  $x$ , we have

$$A_0 \hat{U}_t + i|\xi| A(\omega) \hat{U} + L \hat{U} = 0, \quad \hat{U}|_{t=0} = \hat{U}_0. \quad (3.2)$$

Hereafter,  $\xi \in \mathbb{R}^n$  denotes the Fourier variable,  $\omega = \xi/|\xi| \in S^{n-1}$  (the Schwartz class) is the unit vector whenever  $\xi \neq 0$ , and  $A(\omega) = \sum_{j=1}^n A_j \omega_j$  with  $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in S^{n-1}$ . If the system (3.1) satisfies the following conditions for the matrix coefficients, named *Condition A & L*, the system (3.1) can be regarded as symmetric and dissipative.

Below we clarify the main stability issues concerning problem (3.1).

#### 3.1. The condition (M)

**Definition 3.1 (Condition (A & L)).** The  $m \times m$  coefficient matrices  $A_j$  ( $j = 0, 1, 2, \dots, n$ ) and  $L$  satisfy the following properties:

- $(A_0)^T = A_0$ ,  $A_0 > 0$  on  $\mathbb{C}^m$ .
- $(A_j)^T = A_j$  for  $j = 1, 2, \dots, n$ .
- $L \geq 0$  on  $\mathbb{C}^m$ ,  $\text{Ker } L \neq \{0\}$ .

Next, we review the most applicable stability condition for the system (3.2), named the *S & K mixed Condition (M)* in [12]. We consider the case that  $\text{Ker } L \neq \text{Ker } L^{\text{sym}}$  and the following orthogonal decompositions hold:

$$\mathbb{C}^m = (\text{Ker } L^{\text{sym}}) \oplus (\text{Ker } L^{\text{sym}})^\perp, \quad \text{Ker } L^{\text{sym}} = (\text{Ker } L) \oplus Q_2. \quad (3.3)$$

Besides, let  $P$ ,  $P_1$  and  $P_2$  be orthogonal projections onto  $\text{Ker } L$ ,  $(\text{Ker } L^{\text{sym}})^\perp$  and  $Q_2$ , respectively. Note that

$$|P\phi|^2 + |P_1\phi|^2 + |P_2\phi|^2 = |\phi|^2, \quad (3.4)$$

for  $\phi \in \mathbb{C}^m$ . Then, the *S & K mixed Condition (M)* is stated as follows (cf. [12]).

**Definition 3.2 (S & K mixed Condition (M)).** There are  $m \times m$  real compensating matrices  $K_1(\omega), K_2(\omega) \in C^\infty(S^{n-1})$  and  $m \times m$  real constant matrices  $S_1, S_2$  with the following properties: for each  $j = 1, 2$ ,  $K_j(-\omega) = -K_j(\omega)$ ,

$(K_j(\omega)A_0)^T = -K_j(\omega)A_0$  and  $(S_j A_0)^T = S_j A_0$ . Besides they satisfy either of the following conditions of two types:

Type I

$$\begin{aligned} (K_1(\omega)A(\omega))^{\text{sym}} &> 0 && \text{on } \text{Ker } L, \\ (K_2(\omega)A(\omega) + S_1 L)^{\text{sym}} &> 0 && \text{on } Q_2, \end{aligned}$$

such that

(i)  $S_1$  satisfies the Stability Condition (I) or (II).

- Stability Condition (I):  $i(S_1 A(\omega))^{\text{asy}} = 0$  on  $\mathbb{C}^m$
- Stability Condition (II):  $i(S_1 A(\omega))^{\text{asy}} = 0$  on  $\text{Ker } L^{\text{sym}}$

(ii)  $K_2$  satisfies

$$\begin{aligned} \langle (K_2(\omega)A(\omega) + S_1 L)^{\text{sym}}\phi, \phi \rangle &\geq c|(I - P)\phi|^2 - C|P_1\phi|^2 - C|\phi||P_1\phi|, \\ |\langle i(K_2(\omega)L + S_2 A(\omega))^{\text{asy}}\phi, \phi \rangle| &\leq C|(I - P)\phi||P_1\phi|, \end{aligned}$$

for  $\phi \in \mathbb{C}^m$ , where  $S_2$  satisfies  $(S_2 L)^{\text{sym}} = 0$  on  $\mathbb{C}^m$ .

Type II

$$\begin{aligned} (K_1(\omega)A(\omega))^{\text{sym}} &> 0 && \text{on } Q_2, \\ (K_2(\omega)A(\omega) + S_1 L)^{\text{sym}} &> 0 && \text{on } \text{Ker } L, \end{aligned}$$

such that

(i)  $S_1$  satisfies the Stability Condition (I) or (II).

- Stability Condition (I):  $i(S_1 A(\omega))^{\text{asy}} = 0$  on  $\mathbb{C}^m$ ,
- Stability Condition (II):  $i(S_1 A(\omega))^{\text{asy}} = 0$  on  $\text{Ker } L^{\text{sym}}$ ,

(ii)  $K_2$  satisfies

$$\begin{aligned} \langle (K_2(\omega)A(\omega) + S_1 L)^{\text{sym}}\phi, \phi \rangle &\geq c|(I - P_2)\phi|^2 - C|P_1\phi|^2 - C|\phi| \\ |P_1\phi|, |\langle i(K_2(\omega)L + S_2 A(\omega))^{\text{asy}}\phi, \phi \rangle| &\leq C|(I - P_2)\phi||P_1\phi|, \end{aligned}$$

for  $\phi \in \mathbb{C}^m$ , where  $S_2$  satisfies  $(S_2 L)^{\text{sym}} = 0$  on  $\mathbb{C}^m$ .

Under the condition (M), the following pointwise estimates of the Fourier image  $\hat{U}$  of the solution  $U$  to the Cauchy problem (3.1) and decay estimates of the solution  $U$  are obtained as follows.

**Proposition 3.3 ([12, Theorem 3.2]).** *Suppose  $\text{Ker } L \neq \text{Ker } L^{\text{sym}}$ . Then, under the structural conditions (A, L, M) and the Stability Condition (I), the Fourier*

image  $\hat{U}$  of the solution  $U$  to the Cauchy problem (3.1) satisfies the pointwise estimate

$$|\hat{U}(\xi, t)| \leq Ce^{-c\rho_{20}(\xi)t}|\hat{U}_0(\xi)|, \quad (3.5)$$

with

$$\rho_{20}(\xi) = \frac{|\xi|^4}{(1 + |\xi|^2)^2}.$$

Furthermore, let  $s \geq 0$  be an integer and  $r$  be a real number with  $1 \leq r \leq 2$ , and suppose that the initial data  $U_0$  belongs to  $H^s \cap L^r$ . Then the solution  $U$  satisfies the following decay estimate:

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{4}(\frac{1}{r}-\frac{1}{2})-\frac{k}{4}}\|U_0\|_{L^r} + Ce^{-ct}\|\partial_x^k U_0\|_{L^2}, \quad (3.6)$$

for  $k \leq s$ .

**Proposition 3.4 ([12, Theorem 3.3]).** Suppose  $\text{Ker } L \neq \text{Ker } L^{\text{sym}}$ . Then, under the structural conditions  $(A, L, M)$  and the Stability Condition (II) instead of the Stability Condition (I), the Fourier image  $\hat{U}$  of the solution  $U$  to the Cauchy problem (3.1) satisfies the pointwise estimate

$$|\hat{U}(\xi, t)| \leq Ce^{-c\rho_{21}(\xi)t}|\hat{U}_0(\xi)|, \quad (3.7)$$

with

$$\rho_{21}(\xi) = \frac{|\xi|^4}{(1 + |\xi|^2)^3}.$$

Furthermore, let  $s \geq 0$  be an integer and  $r$  be a real number with  $1 \leq r \leq 2$ , and suppose that the initial data  $U_0$  belongs to  $H^s \cap L^r$ . Then the solution  $U$  satisfies the following decay estimate:

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{4}(\frac{1}{r}-\frac{1}{2})-\frac{k}{4}}\|U_0\|_{L^r} + C(1+t)^{-\frac{\ell}{2}}\|\partial_x^{k+\ell} U_0\|_{L^2}, \quad (3.8)$$

for  $k + \ell \leq s$ .

**Remark 3.5.** In the right-hand side of the time decay estimate (3.6), the first term is the estimate in the low-frequency region ( $|\xi| \leq 1$ ), whereas the second term is the estimate in the high-frequency region ( $|\xi| \geq 1$ ). Because the high-frequency estimate decays exponentially, the time decay rate  $t^{-\frac{n}{4}(\frac{1}{r}-\frac{1}{2})-\frac{k}{4}}$  is obtained without regularity-loss. On the other hand, in the right-hand side of the time decay estimate (3.8), the first term is the estimate in the low-frequency region ( $|\xi| \leq 1$ ), whereas the second term is the estimate in the high-frequency region ( $|\xi| \geq 1$ ). The high-frequency estimate of (3.8) decays polynomially, so that we have to assume  $\ell$ th extra higher regularity on the initial data  $U_0$  in order to get the optimal time decay rate  $t^{-\frac{n}{4}(\frac{1}{r}-\frac{1}{2})-\frac{k}{4}}$ . Therefore, the decay estimate (3.8) cannot avoid regularity-loss. Consequently, the dependence of the matrix  $S_1$  with respect to the Stability Condition (I) or the Stability Condition (II) determines whether regularity-loss occurs in the decay estimate or not.

### 3.2. The condition ( $S \& K$ )

When we can find  $S_1$  satisfying the conditions in Type I of Condition ( $M$ ) even if  $K_2 = O$ , then  $S_2$  should be  $S_2 = O$  too, and Condition ( $M$ ) can be simplified as the following Condition  $S \& K$ , which is the same condition originally developed in [21], and the decay rates get faster as follows.

**Definition 3.6 (Condition ( $S \& K$ ), cf. [21]).** There are  $m \times m$  real compensating matrices  $K(\omega) \in C^\infty(S^{n-1})$  and  $m \times m$  real constant matrices  $S$  with the following properties:  $K(-\omega) = -K(\omega)$ ,  $(K(\omega)A_0)^T = -K(\omega)A_0$ ,  $(SA_0)^T = SA_0$  and

$$(K(\omega)A(\omega))^{\text{sym}} > 0 \quad \text{on } \text{Ker } L$$

$$(SL)^{\text{sym}} + L^{\text{sym}} \geq 0 \quad \text{on } \mathbb{C}^m, \quad \text{Ker } \{(SL)^{\text{sym}} + L^{\text{sym}}\} = \text{Ker } L,$$

where  $S$  satisfies either one of the following two conditions for  $\omega \in S^{n-1}$ :

- Stability Condition (I):  $i(SA(\omega))^{\text{asy}} = 0$  on  $\mathbb{C}^m$ .
- Stability Condition (II):  $i(SA(\omega))^{\text{asy}} = 0$  on  $\text{Ker } L^{\text{sym}}$ .

**Proposition 3.7 (cf. [21]).** Suppose  $\text{Ker } L \neq \text{Ker } L^{\text{sym}}$ . Then, under the structural conditions ( $A$ ,  $L$ ,  $S$  &  $K$ ) and the Stability Condition (I), the Fourier image  $\hat{U}$  of the solution  $U$  to the Cauchy problem (3.1) satisfies the pointwise estimate

$$|\hat{U}(\xi, t)| \leq Ce^{-c\rho_{10}(\xi)t}|\hat{U}_0(\xi)|, \quad (3.9)$$

with

$$\rho_{10}(\xi) = \frac{|\xi|^2}{1 + |\xi|^2}.$$

Furthermore, let  $s \geq 0$  be an integer and  $r$  be a real number with  $1 \leq r \leq 2$ , and suppose that the initial data  $U_0$  belongs to  $H^s \cap L^r$ . Then the solution  $U$  satisfies the following decay estimate:

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})-\frac{k}{2}}\|U_0\|_{L^r} + Ce^{-ct}\|\partial_x^k U_0\|_{L^2}, \quad (3.10)$$

for  $k \leq s$ .

**Proposition 3.8 (cf. [21]).** Suppose  $\text{Ker } L \neq \text{Ker } L^{\text{sym}}$ . Then, under the structural conditions ( $A$ ,  $L$ ,  $S$  &  $K$ ) and the Stability Condition (II) instead of the Stability Condition (I), the Fourier image  $\hat{U}$  of the solution  $U$  to the Cauchy problem (3.1) satisfies the pointwise estimate

$$|\hat{U}(\xi, t)| \leq Ce^{-c\rho_{11}(\xi)t}|\hat{U}_0(\xi)|, \quad (3.11)$$

with

$$\rho_{11}(\xi) = \frac{|\xi|^2}{(1 + |\xi|^2)^2}.$$

Furthermore, let  $s \geq 0$  be an integer and  $r$  be a real number with  $1 \leq r \leq 2$ , and suppose that the initial data  $U_0$  belongs to  $H^s \cap L^r$ . Then the solution  $U$  satisfies the following decay estimate:

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})-\frac{k}{2}} \|U_0\|_{L^r} + C(1+t)^{-\frac{\ell}{2}} \|\partial_x^{k+\ell} U_0\|_{L^2}, \quad (3.12)$$

for  $k + \ell \leq s$ .

Finally, when the relaxation matrix  $L$  is symmetric in the sense of  $\text{Ker } L = \text{Ker } L^{\text{sym}}$ , Condition (M) can be rewritten into the most simplified condition, which is the same condition originally developed in [18, 23]. Note that under this situation regularity-loss never occurs in the decay estimate.

**Definition 3.9 (Condition (K), cf. [18, 23]).** There is a  $m \times m$  real compensating matrix  $K(\omega) \in C^\infty(S^{n-1})$  with the following properties:  $K(-\omega) = -K(\omega)$ ,  $(K(\omega)A_0)^T = -K(\omega)A_0$  and

$$(K(\omega)A(\omega))^{\text{sym}} + L > 0 \quad \text{on } \mathbb{C}^m,$$

for  $\omega \in S^{n-1}$ .

Thus, according to [18, 23] the following stability result holds true.

**Proposition 3.10.** If  $\text{Ker } L = \text{Ker } L^{\text{sym}}$ , under the structural conditions (A, L, K), the Fourier image  $\hat{U}$  of the solution  $U$  to the Cauchy problem (3.1) satisfies the pointwise estimate (3.9). Moreover, the solution  $U$  satisfies the decay estimate (3.10).

## 4. Main Results

### 4.1. Pointwise and decay rate estimates

Let us go back to our peculiar problems (2.4) and (2.7) and then evaluate them in terms of the previous general theory and results. More precisely, we first check that (2.4) satisfies the structural condition (A & L) whereas system (2.7) complies with Condition (M).

**Condition (A & L).** The symmetric part of  $L$  is given by

$$L^{\text{sym}} = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and we see that

$$\text{Ker } L = \text{Span } \{e_3, e_4\}, \quad \text{Ker } L^{\text{sym}} = \text{Span } \{e_2, e_3, e_4\},$$

where  $e_1 = (1, 0, 0, 0)^T$ ,  $e_2 = (0, 1, 0, 0)^T$ ,  $e_3 = (0, 0, 1, 0)^T$  and  $e_4 = (0, 0, 0, 1)^T$ . Then, the matrix coefficients  $A_0, A, L$  set in (2.5) and  $L^{\text{sym}}$  satisfy:

- $(A_0)^T = A_0 = I$ , and  $\langle A_0\phi, \phi \rangle = |\phi|^2$  for  $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in \mathbb{C}^4$ , so that  $A_0 > 0$  on  $\mathbb{C}^4$ .
- $(A)^T = A$ .
- $\operatorname{Re} \langle L\phi, \phi \rangle = \gamma|\phi_1|^2 \geq 0$  for  $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in \mathbb{C}^4$ , so that  $L \geq 0$  on  $\mathbb{C}^4$ .
- $\operatorname{Ker} L = \operatorname{Span} \{e_3, e_4\} \neq \{0\}$ .
- $\operatorname{Ker} L \neq \operatorname{Ker} L^{\text{sym}}$  for the system (2.4)

Next, we try to apply Condition (M) to the system (2.7).

**Type I of Condition (M).** Let us make  $K_1$  such that

$$K_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad K_1 A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}.$$

A simple computation gives

$$\langle (K_1 A)^{\text{sym}} \phi, \phi \rangle = -|\phi_1|^2 - a|\phi_2|^2 + |\phi_3|^2 + a|\phi_4|^2.$$

for  $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in \mathbb{C}^4$ . Now suppose that  $\phi \in \operatorname{Ker} L$ , then  $\phi = (0, 0, \phi_3, \phi_4)^T$ . For this  $\phi$ , we have

$$\langle (K_1 A)^{\text{sym}} \phi, \phi \rangle = |\phi_3|^2 + a|\phi_4|^2.$$

Next, we make  $K_2$  and  $S_1$  as

$$K_2 = O, \quad S_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/a \\ 0 & 0 & -1/a & 0 \end{pmatrix}.$$

Then,

$$S_1 L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ \gamma & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, by using  $0 < \alpha < 1$  to be chosen later, for  $\phi \in \mathbb{C}^4$ ,

$$\begin{aligned} \alpha \langle (S_1 L)^{\text{sym}} \phi, \phi \rangle + \langle L^{\text{sym}} \phi, \phi \rangle &= (\gamma - \alpha)|\phi_1|^2 + \alpha|\phi_2|^2 + \alpha\gamma \operatorname{Re}(\phi_1 \bar{\phi}_2) \\ &\geq (\gamma - \alpha)|\phi_1|^2 + \alpha|\phi_2|^2 - \alpha\gamma|\phi_1||\phi_2| \\ &\geq (\gamma - \alpha)|\phi_1|^2 + \alpha|\phi_2|^2 - \alpha(C_\varepsilon|\phi_1|^2 + \varepsilon|\phi_2|^2), \end{aligned}$$

for  $0 < \varepsilon < 1$ . If we choose  $\alpha$  so small that  $\gamma - \alpha(1 + C_\varepsilon) > 0$  holds, we have

$$\alpha \langle (S_1 L)^{\text{sym}} \phi, \phi \rangle + \langle L^{\text{sym}} \phi, \phi \rangle \geq \{\gamma - \alpha(1 + C_\varepsilon)\} |\phi_1|^2 + \alpha(1 - \varepsilon) |\phi_2|^2.$$

Here we note that the total estimate up to this point is as follows: by using  $0 < \beta < 1$  to be chosen later, for  $\phi \in \mathbb{C}^4$ ,

$$\begin{aligned} & \alpha\beta \langle (K_1 A)^{\text{sym}} \phi, \phi \rangle + \alpha \langle (S_1 L)^{\text{sym}} \phi, \phi \rangle + \langle L^{\text{sym}} \phi, \phi \rangle \\ & \geq \{\gamma - \alpha(1 + C_\varepsilon + \beta)\} |\phi_1|^2 + \alpha(1 - \varepsilon - \beta a) |\phi_2|^2 + \beta |\phi_3|^2 + \beta a |\phi_4|^2. \end{aligned}$$

Therefore, for  $0 < \varepsilon < 1$ , we choose  $\beta$  such that  $(1 - \varepsilon - \beta a) > 0$ , and for this  $\beta$  choose  $\alpha$  such that  $\gamma - \alpha(1 + C_\varepsilon + \beta) > 0$ . Then, we get

$$\begin{aligned} & \alpha\beta \langle (K_1 A)^{\text{sym}} \phi, \phi \rangle + \alpha \langle (S_1 L)^{\text{sym}} \phi, \phi \rangle + \langle L^{\text{sym}} \phi, \phi \rangle \\ & \geq c(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2). \end{aligned}$$

On the other hand,  $S_1 A$  is given by

$$S_1 A = \begin{pmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1/a & 0 & 0 & 0 \end{pmatrix}, \quad (S_1 A)^{\text{asy}} = \frac{1}{2a} \begin{pmatrix} 0 & 0 & 0 & 1 - a^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a^2 - 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\langle i(S_1 A)^{\text{asy}} \phi, \phi \rangle = \frac{1}{2a} (a^2 - 1) \operatorname{Im}(\phi_1 \bar{\phi}_4).$$

for  $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in \mathbb{C}^4$ .

- (i) **EWS**  $a^2 = 1$ . If  $a^2 - 1 = 0$ , then  $(S_1 A)^{\text{asy}} = O$ .
- (ii) **Non EWS**  $a^2 \neq 1$ . If  $a^2 - 1 \neq 0$ , let  $\phi \in \operatorname{Ker} L^{\text{sym}}$ . Then  $\phi = (0, \phi_2, \phi_3, \phi_4)^T$ . For this  $\phi$ , we have  $\langle i(S_1 A)^{\text{asy}} \phi, \phi \rangle = 0$ . That is, when we restrict the linear transformation  $(S_1 A)^{\text{asy}} : \operatorname{Ker} L^{\text{sym}} \rightarrow \operatorname{Ker} L^{\text{sym}} = \operatorname{Span} \{e_2, e_3, e_4\}$ , then  $(S_1 A)^{\text{asy}} = O$ .

In conclusion, it implies system (2.7) satisfies the Stability Condition (I) or (II) depending on whether the EWS assumption  $a^2 - 1 = 0$  is regarded or not. Consequently, we conclude that Condition (M) with  $K_2 = S_2 = O$  can be applicable to our specific system (2.7).

Hence, by means of Propositions 3.7 and 3.8, we have proved the following stability results concerning pointwise estimate (for (2.7)) and decay rate estimates (for (2.4)).

**Theorem 4.1 (Main Result I – EWS assumption).** *If  $a^2 = 1$ , then we have*

- (i) **(Pointwise Estimate)** *The solution  $\hat{U}$  of the Fourier problem (2.4) satisfies the pointwise estimate*

$$|\hat{U}(\xi, t)| \leq Ce^{-c\rho_{10}(\xi)t} |\hat{U}_0(\xi)|, \quad (4.1)$$

with

$$\rho_{10}(\xi) = \frac{\xi^2}{1 + \xi^2}. \quad (4.2)$$

- (ii) **(Decay Estimate)** *For any integer  $s \geq 0$  and real number  $r$  with  $1 \leq r \leq 2$ , if we take  $U_0 \in H^s \cap L^r(\mathbb{R})$ , then the solution  $U$  of the Cauchy problem (2.4) satisfies the following decay estimate:*

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{r}-\frac{1}{2})-\frac{k}{2}} \|U_0\|_{L^r} + Ce^{-ct} \|\partial_x^k U_0\|_{L^2}, \quad (4.3)$$

for  $k \leq s$ .

**Theorem 4.2 (Main result II – non-EWS assumption).** *If  $a^2 \neq 1$ , then we have*

- (i) **(Pointwise Estimate)** *The solution  $\hat{U}$  of the Fourier problem (2.4) satisfies the pointwise estimate*

$$|\hat{U}(\xi, t)| \leq Ce^{-c\rho_{11}(\xi)t} |\hat{U}_0(\xi)|, \quad (4.4)$$

with

$$\rho_{11}(\xi) = \frac{\xi^2}{(1 + \xi^2)^2}. \quad (4.5)$$

- (ii) **(Decay Estimate)** *For any integer  $s \geq 0$  and real number  $r$  with  $1 \leq r \leq 2$ , if we take  $U_0 \in H^s \cap L^r(\mathbb{R})$ , then the solution  $U$  of the Cauchy problem (2.4) satisfies the following decay estimate of regularity-loss type:*

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{r}-\frac{1}{2})-\frac{k}{2}} \|U_0\|_{L^r} + C(1+t)^{-\frac{\ell}{2}} \|\partial_x^{k+\ell} U_0\|_{L^2}, \quad (4.6)$$

for  $k + \ell \leq s$ .

## 4.2. Dissipative structure and optimality

Now, in order to verify the optimality of the decay rate estimates (4.3) and (4.6), we proceed with an asymptotic expansion analysis of the eigenvalues corresponding to the Fourier problem (2.7).

We first observe that the eigenvalues related to problem (2.7) are given by solutions of the equation

$$\lambda\phi + (i\xi A + L)\phi = 0, \quad (4.7)$$

where  $\lambda \in \mathbb{C}$  and  $\phi \in \mathbb{C}^4$ . We denote by  $\lambda = \lambda(i\xi)$  the eigenvalue of the problem (2.7), which satisfies (4.7) for  $\phi \neq 0$ . Let  $\lambda_j(\zeta)$  be the eigenvalues of the matrix  $\hat{\Phi}(\zeta)$  set in (2.8). Thus, under this notation, we are going to investigate the asymptotic expansion of the eigenvalues  $\lambda_j(\zeta)$  for  $|\zeta| \rightarrow 0$  and  $|\zeta| \rightarrow \infty$  to show that our decay estimates (4.3) and (4.6) are optimal.

Indeed, we start by noting that the eigenvalues  $\lambda_j(\zeta)$ ,  $j = 1, 2, 3, 4$ , are the solutions of the characteristic equation

$$\det(\lambda I - \hat{\Phi}(\zeta)) = \lambda^4 + \gamma\lambda^3 + \{1 - (a^2 + 1)\zeta^2\}\lambda^2 - \gamma\zeta^2\lambda + a^2\zeta^4 = 0.$$

The analysis shall be done in low and high frequencies as follows.

**Low-frequency case.**  $|\zeta| \rightarrow 0$ . When  $|\zeta| \rightarrow 0$ ,  $\lambda_j(\zeta)$  has the following asymptotic expansion

$$\lambda_j(\zeta) = \lambda_j^{(0)} + \lambda_j^{(1)}\zeta + \lambda_j^{(2)}\zeta^2 + \dots \quad (4.8)$$

Here each coefficient  $\lambda_j^{(k)}$  is given by direct computations as

$$\begin{aligned} \lambda_j^{(0)} &= \lambda_j^{(1)} = 0, & \lambda_j^{(2)} &= \alpha_j, & \lambda_j^{(3)} &= 0 \quad \text{for } j = 1, 2, \\ \lambda_j^{(0)} &= \beta_j, & \lambda_j^{(1)} &= 0 \quad \text{for } j = 3, 4, \end{aligned}$$

where  $\alpha_j = \frac{1}{2}(\gamma \pm \sqrt{\gamma^2 - 4a^2})$  and  $\beta_j = -\frac{1}{2}(\gamma \pm \sqrt{\gamma^2 - 4})$ . Notice that  $\operatorname{Re} \alpha_j > 0$  and  $\operatorname{Re} \beta_j < 0$ . Consequently, for  $|\xi| \rightarrow 0$ , we have

$$\operatorname{Re} \lambda_j(i\xi) = \begin{cases} -(\operatorname{Re} \alpha_j) \xi^2 + O(|\xi|^4), & \text{for } j = 1, 2, \\ \operatorname{Re} \beta_j + O(|\xi|^2), & \text{for } j = 3, 4. \end{cases} \quad (4.9)$$

According to the expansion (4.9) for  $|\xi| \rightarrow 0$ , two eigenvalues behave like  $\operatorname{Re} \lambda(i\xi) \sim -\xi^2$ , while the other two ones behave like  $\operatorname{Re} \lambda(i\xi) \sim -c$ .

**High-frequency case.**  $|\zeta| \rightarrow \infty$ . When  $|\zeta| \rightarrow \infty$ ,  $\lambda_j(\zeta)$  has the following asymptotic expansion

$$\lambda_j(\zeta) = \mu_j^{(1)}\zeta + \mu_j^{(0)} + \mu_j^{(-1)}\zeta^{-1} + \mu_j^{(-2)}\zeta^{-2} + \dots$$

Each coefficient  $\mu_j^{(k)}$  is given by direct computations as follows: For  $j = 1, 2$ , we have

$$\mu_j^{(1)} = \pm 1, \quad \mu_j^{(0)} = \delta_j \quad \text{for } a = 1,$$

$$\mu_j^{(1)} = \pm 1, \quad \mu_j^{(0)} = 0, \quad \mu_j^{(-1)} = \pm \frac{1}{2Q}, \quad \mu_j^{(-2)} = \frac{\gamma}{Q^2} \quad \text{for } a \neq 1,$$

and for  $j = 3, 4$ , we have

$$\mu_j^{(1)} = \pm a, \quad \mu_j^{(0)} = -\frac{\gamma}{2}, \quad \mu_j^{(-1)} = \pm \frac{\gamma^2}{8a},$$

where we have denoted

$$\delta_j = \frac{1}{4}(-\gamma \pm \sqrt{\gamma^2 - 4}) \quad \text{and} \quad Q = a^2 - 1.$$

Note that  $\operatorname{Re} \delta_j < 0$ . Consequently, when  $a = 1$ , we have

$$\operatorname{Re} \lambda_j(i\xi) = \begin{cases} \operatorname{Re} \delta_j + O(|\xi|^{-1}), & \text{for } j = 1, 2, \\ -\frac{\gamma}{2} + O(|\xi|^{-2}), & \text{for } j = 3, 4, \end{cases} \quad (4.10)$$

for  $|\xi| \rightarrow \infty$ ; while in the case  $a \neq 1$ , we have

$$\operatorname{Re} \lambda_j(i\xi) = \begin{cases} -\frac{\gamma}{Q^2} \xi^{-2} + O(|\xi|^{-3}), & \text{for } j = 1, 2, \\ -\frac{\gamma}{2} + O(|\xi|^{-2}), & \text{for } j = 3, 4, \end{cases} \quad (4.11)$$

for  $|\xi| \rightarrow \infty$ . According to the expansion (4.11) for  $|\xi| \rightarrow \infty$ , when  $a \neq 1$ , two eigenvalues are of the standard type and satisfy  $\operatorname{Re} \lambda(i\xi) \sim -c$ , while the other two ones are not of the standard type and satisfy  $\operatorname{Re} \lambda(i\xi) \sim -c\xi^{-2}$ .

Therefore, we can conclude that the results of the asymptotic expansion of the eigenvalues correspond to our pointwise estimates (4.2) and (4.5). Moreover, based on them the decay rates (4.3) and (4.6) are shown to be optimal.

**Remark 4.3.** We characterize the dissipative structure of the system (2.1) by the straight calculation of the asymptotic expansions of the eigenvalues of the simplified system (2.7) when the memory kernel is exactly an exponential function (assumption (2.3)). However, it suggests that even when the memory kernel is not a strictly exponential function, it can be expected that the solution  $\tilde{U} = (v, y, u, z)^T$  to problem (2.1) satisfies the same time decay rates as in Theorems 4.1 and 4.2.

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