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Attractors and their properties for a class of Kirchhoff models with integro-differential damping

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ABSTRACT

In this paper, we investigate a class of Kirchhoff models with integro-differential damping given by a possibly vanishing memory term in a past history framework and a nonlinear nonlocal strong dissipation

$$u_{tt} + \alpha_\mu \Delta^2 u - \Delta_p u - \int_{-\infty}^{t} \mu(t-s)\Delta^2 u(s)\,\mathrm{d}s - N\left(\int_\Omega |\nabla u(t)|^2 \mathrm{d}x\right)\Delta u_t + f(u) = h,$$

defined in a bounded Ω of $\mathbb{R}^N$. Our main goal is to show the well-posedness and the long-time behavior through the corresponding autonomous dynamical system by regarding the relative past history. More precisely, under the assumptions that the exponent p and the growth of $f(u)$ are up to the critical range, the well-posedness and the existence of a global attractor with its geometrical structure are established. Furthermore, in the subcritical case, such a global attractor has finite fractal dimensions as well as regularity of trajectories. A result on generalized fractal exponential attractor is also proved. These results are presented for a wide class of nonlocal damping coefficient $N(\cdot)$ and possibly degenerate memory term ($\mu \equiv 0$), which deepen and extend earlier results on the subject.

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1. Introduction

In the present article, we consider the well-posedness and the long-time dynamics to the following Kirchhoff models with past history and nonlocal damping

$$u_{tt} + \alpha_\mu \Delta^2 u - \Delta_p u - \int_{-\infty}^{t} \mu(t-s)\Delta^2 u(s)\,\mathrm{d}s - N(\|\nabla u\|^2)\Delta u_t + f(u) = h, \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$, μ is the memory kernel to be stated later, α_μ is a positive constant depending on the memory kernel, $N(\|\nabla u\|^2)$ denotes the nonlocal coefficient, where $\|\cdot\|$ stands the norm in $L^2(\Omega)$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplacian operator, on which we consider the simply supported boundary condition

$$u = \Delta u = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \tag{2}$$

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and initial condition

$$u(x,t) = u_0(x,t) \quad \text{and} \quad u_t(x,t) = \partial_t u_0(x,t), \quad (x,t) \in \Omega \times (-\infty, 0], \tag{3}$$

where $u_0 : \Omega \times (-\infty, 0]$ is the given past history of u. As initially explained below, (1)–(3) is a mathematical model motivated by the so-called Kirchhoff models and extensible beams with nonlocal damping. Then, we are going to clarify that our main results extend and generalize those presented in [1,2].

We initially start by noting that the Kirchhoff model

$$u_{tt} + \triangle_x^2 u - \operatorname{div}_x(\phi(\nabla_x u)) = \mathcal{F}(x, u, u_t),$$

where $\phi(z) \approx |z|^{p-2}z$, $p \geq 2$, and $\mathcal{F}(x, u, u_t)$ represents additional damping and forcing terms, is a valid model for several applications. Indeed, in the one-dimensional case this problem is related to a model of elastoplastic microstructure of the form

$$\rho u_{tt} + \zeta u_{xxxx} - a\left(u_x^2\right)_x = 0, \quad \rho, \zeta > 0,\ a < 0,$$

which was studied by An and Peirce [3] and Yang [4,5]. In the two-dimensional case, this problem is related to a class of Kirchhoff–Boussinesq models with weak damping for plate equations

$$u_{tt} + \Delta^2 u - \operatorname{div}\left[f_0(\nabla u)\right] + k u_t = \Delta\left[f_1(u)\right] - f_2(u),$$

being considered by Chueshov and Lasiecka [6,7]. In the n-dimensional case, the strongly damped equation

$$u_{tt} + \Delta^2 u - \operatorname{div}\left(\sigma\left(|\nabla u|^2\right)\nabla u\right) - \Delta u_t = h(x, u, u_t)$$

was extensively studied by Yang [5,8–10] and Yang and Jin [11], where several results on existence of attractors and their qualitative finite-dimensional properties are provided. Additionally, in the absence of dissipation, Yang [4], Liu and Xu [12] and Esquivel-Avila [13] considered the question of blow-up in finite time and global existence with small data for related Kirchhoff models.

With respect to Kirchhoff models with memory effects, Andrade et al. [14] firstly considered the model with fading memory $\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s$. They proved the well-posedness and the asymptotic of the system. Because of the fading memory term, the system becomes nonautonomous and does not generate a continuous semigroup. However, when the forth-order memory term preserves past history, the problem can be transformed into an autonomous system, like in [15]. In fact, Jorge Silva and Ma [1] established the well-posedness and exponential stability of

$$u_{tt} + \alpha\Delta^2 u - \Delta_p u - \int_{-\infty}^{t} \mu(t-s)\Delta^2 u(s)ds - \Delta u_t + f(u) = h. \tag{4}$$

Later, the same authors proved in [2] that the Kirchhoff model with memory (4) does have a global attractor with the finite fractal dimension under suitable conditions. Nonetheless, no more results on geometrical properties, regularity and exponential attractors are presented in [2]. It is worth mentioning that (4) is a particular case of (1) with constant damping coefficient $N \equiv 1$.

On the other hand, in what concerns problems with the nonlocal damping term, the following model

$$u_{tt} + \Delta^2 u + M(\|\nabla u\|^2_{L^2(\mathbb{R}^n)})u_t = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+ \tag{5}$$

was firstly presented by Lange and Perla Menzala [16] in the context of plate models. The existence of global classical solutions and energy decay rate were proved in [16]. Subsequently, Cavalcanti et al. [17] considered Equation (5) in a viscoelastic context by adding a memory term of fourth order.

Moreover, in recent years, there are many researches on the long-time behavior of plate and extensible beam models with nonlocal damping term. Indeed, Jorge Silva and Narciso [18,19] considered the global attractor with finite fractal dimension for the model

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u + N\left(\|\nabla u\|^2\right)u_t + f(u) = h \quad \text{in } \Omega \times \mathbb{R}^+. \tag{6}$$

For the nonlocal nonlinear damping term $N(\|\nabla u\|^2)g(u_t)$, we refer to [20–22]. More specifically, Jorge Silva and Narciso [20] presented a first analysis on the long-time dynamics for the autonomous extensible beam with this nonlinear nonlocal damping term. More recently, Li et al. [21] considered the nonautonomous model. Narciso [22] also considered the attractors for a plate equation with nonlocal nonlinearities. From the mathematical viewpoint, another type of nonlocal fractional damping is given by

$$N\left(\|\nabla u\|^2\right)(-\Delta)^{\theta}u_t, \quad 0 \le \theta \le 1. \tag{7}$$

Chueshov and Kolbasin [23] firstly considered the long-time behavior for a class of abstract equations with damping given by (7) and power covering the range $0 < \theta \le 1$. Jorge Silva and Narciso [19] proved the well-posedness and the existence of finite dimensional global and exponential attractors for the extensible beams with nonlocal damping (7). For the similar nonlocal damping term (7), we also refer the readers to [24–27] and the references therein.

Motivated by the aforementioned branches of research, our main aim in this paper is to consider the model (1), where the relation between the nonlocal damping term $-N(\|\nabla u\|_2^2)\triangle u_t$ and the p-Laplacian $\triangle_p u$ is firstly provided. The main results are Theorem 2.1 which ensures the well-posedness of problem (1)–(3) and Theorem 2.2 which establishes the existence of attractors to the dynamical system associated with problem (1)–(3) as well as their qualitative properties with respect to geometrical characterization, finite dimension, regularity of trajectories and exponential attractor. Therefore, when compared to the results in [1,2], we highlight the following novelties: the linear strong damping $-\triangle u_t$ is a particular case of the nonlinear nonlocal damping term; the conditions on the source term $f(u)$ are slightly more general; the qualitative properties of the attractors are also explored here (not addressed in [2]); the memory term can be neglected in computations and, consequently, the long-time behavior can be only driven by the nonlinear nonlocal strong damping (not considered in [2]), see e.g. Remarks 2.2 and 4.2.

We end this section by introducing a new variable that transforms problem (1)–(3) into an autonomous system. By following the same argument as in [1,2] (see also e.g. [28–30]), which has its origins in Dafermos [31], we set the new variable $\eta = \eta^t(x,s)$ corresponding to relative displacement history, namely

$$\eta^t(x,s) := u(x,t) - u(x,t-s), \quad (x,t,s) \in \ \Omega \times [0,\infty) \times \mathbb{R}^+. \tag{8}$$

By formal computations, we have

$$\begin{aligned} &\eta_t(x,s) + \eta_s(x,s) = u_t \quad (x,t,s)\Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \\ &\eta^0(x,s) = u_0(x,0) - u_0(x,-s), \quad (x,s) \in \Omega \times \mathbb{R}^+, \\ &\eta^t(x,0) := \lim_{s\to 0^+} \eta^t(x,s) = 0, (x,t) \in \Omega \times [0,\infty), \end{aligned} \tag{9}$$

as well as the original memory term can be rewritten as

$$\begin{aligned} \int_{-\infty}^{t} \mu(t-s)\Delta^2 u(s)\,\mathrm{ds} &= \int_0^\infty \mu(s)\Delta^2 u(t-s)\,\mathrm{ds} \\ &= \left(\int_0^\infty \mu(s)\mathrm{ds}\right)\Delta^2 u - \int_0^\infty \mu(s)\Delta^2\eta^t(s)\,\mathrm{ds}. \end{aligned} \tag{10}$$

Therefore, assuming for simplicity that $\alpha_\mu = 1 + \int_0^\infty \mu(s)\mathrm{d}s$, the original problem (1)–(3) turns into the following equivalent autonomous system

$$u_{tt} + \Delta^2 u - \Delta_p u + \int_0^\infty \mu(s)\Delta^2\eta^t(s)\,\mathrm{d}s - N(\|\nabla u\|^2)\Delta u_t + f(u) = h(x) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{11}$$

$$\eta_t + \eta_s = u_t \quad \text{in } \Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \tag{12}$$

with boundary conditions

$$u = \Delta u = 0 \text{ on } \partial\Omega \times \mathbb{R}^+, \quad \eta = \Delta\eta = 0 \text{ on } \partial\Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \tag{13}$$

and initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \eta^0(x,s) = \eta_0(x,s), \quad \eta^t(x,0) = 0, \tag{14}$$

where

$$\begin{array}{ll} u_0(x) = u_0(x,0), & x \in \Omega, \\ u_1(x) = \partial_t u_0(x,t)|_{t=0}, & x \in \Omega, \\ \eta_0(x,s) = u_0(x,0) - u_0(x,-s), & (x,s) \in \Omega \times \mathbb{R}^+. \end{array}$$

The remaining paper is organized as follows. Section 2 provides some notations and assumptions as well as the statements of the main results (Theorems 2.1 and 2.2). Section 3 is dedicated to the proof of Theorem 2.1, that is, the well-posedness of problem (11)–(14). Section 4 is concerned with the proof of Theorem 2.2 by means of several technical results, namely, the long-time behavior of the associated dynamical system.

2. Assumptions and main results

2.1. Functional spaces and settings.

We start by introducing some notations on the functional spaces that will be used throughout this paper. Let

$$V_0 = L^2(\Omega), \quad V_1 = H_0^1(\Omega), \quad V_2 = H^2(\Omega) \cap H_0^1(\Omega),$$

and

$$V_3 = \left\{ u \in H^3(\Omega) | u = \Delta u = 0 \text{ on } \partial\Omega \right\}$$

be the Hilbert spaces endowed with their respective norms

$$\|u\|_{V_0} = \|u\|, \quad \|u\|_{V_1} = \|\nabla u\|, \quad \|u\|_{V_2} = \|\Delta u\|, \quad \text{and} \quad \|u\|_{V_3} = \|\nabla\Delta u\|,$$

corresponding to the inner products $(\cdot,\cdot)_{V_i}$, $i = 0,1,2,3$, where $\|\cdot\|$ stands the norm in $L^2(\Omega)$. Also, the notation $(\cdot,\cdot)_{V_0} = (\cdot,\cdot)$ stands for L^2-inner product and $\|\cdot\|_p$ stands for L^p-norm. We use $\langle\cdot,\cdot\rangle$ to represent the duality pairing between any Banach space V and its dual space V'. Let λ_1 be the first eigenvalue of the operator $\triangle^2$ with boundary condition (2), then

$$\lambda_1\|u\|^2 \le \|\Delta u\|^2, \quad \lambda_1^{1/2}\|\nabla u\|^2 \le \|\Delta u\|^2, \quad \forall\, u \in V_2. \tag{15}$$

With respect to the relative displacement history, we consider the L^2_μ-weighted Hilbert spaces

$$\mathcal{M}_i = L^2_\mu\left(\mathbb{R}^+, V_i\right) = \left\{\eta : \mathbb{R}^+ \to V_i; \int_0^\infty \mu(s)\|\eta(s)\|^2_{V_i}\,\mathrm{d}s < \infty\right\}, \quad i = 0, 1, 2, 3$$

endowed with inner product and norms

$$(\eta, \zeta)_{\mathcal{M}_i} = \int_0^\infty \mu(s)(\eta(s), \zeta(s))_{V_i}\,\mathrm{d}s,$$
$$\|\eta\|^2_{\mathcal{M}_i} = \int_\Omega^\infty \mu(s)\|\eta(s)\|^2_{V_i}\,\mathrm{d}s, \quad i = 0, 1, 2, 3.$$

We also consider the following Hilbert phase space

$$\mathcal{H} = V_2 \times V_0 \times \mathcal{M}_2, \quad \|(u, v, \eta)\|^2_{\mathcal{H}} = \|\Delta u\|^2 + \|v\|^2 + \|\eta\|^2_{\mathcal{M}_2},$$
$$\mathcal{H}_1 = V_3 \times V_1 \times \mathcal{M}_3, \quad \|(u, v, \eta)\|^2_{\mathcal{H}_1} = \|\nabla\Delta u\|^2 + \|\nabla v\|^2 + \|\eta\|^2_{\mathcal{M}_3},$$

where the analysis of the long-time behavior shall be done.

2.2. Assumptions

Now we give the precise hypotheses on the problem (11)–(14). For $n \in \mathbb{N}$, we assume that

$$p \ge 1 \text{if } n = 1, 2 \quad \text{and} \quad 2 \le p \le \frac{2n-2}{n-2} \text{ if } n \ge 3, \tag{16}$$

which implies that

$$H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow W^{1,2(p-1)}_0(\Omega) \hookrightarrow H^1_0(\Omega) \hookrightarrow L^2(\Omega).$$

For the forcing term, we assume that $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function with $f(0) = 0$ and

$$\left|f'(u)\right| \le k_1\left(1 + |u|^\rho\right), \quad \forall\, u \in \mathbb{R}, \tag{17}$$

where k_1 is a positive constant and

$$\rho > 0 \text{if } 1 \le n \le 4, \quad \text{and} \quad 0 < \rho \le \frac{4}{n-4} \text{ if } n \ge 5. \tag{18}$$

Moreover, we assume that

$$-\,l_0 - \frac{\alpha_1}{2}u^2 \le \hat{f}(u) := \int_0^u f(\tau)\,\mathrm{d}\tau \le f(u)u + \frac{\alpha_1}{2}u^2 + l_1, \quad \forall\, u \in \mathbb{R} \tag{19}$$

where we consider $l_0 \ge 0$, $l_1 \ge 0$ and $\alpha_1 \in [0, \lambda_1)$.

With respect to the memory kernel $\mu \ge 0$, we assume that

$$\mu \in C^1\left(\mathbb{R}^+\right), \quad \int_0^\infty \mu(s)\,\mathrm{d}s = \mu_0 > 0 \tag{20}$$

and that there exists a constant $k_2 > 0$ such that

$$\mu'(s) \le -k_2\mu(s), \quad \forall\, s \ge 0. \tag{21}$$

For the nonlocal term N, we assume that N is a C^1-functions on $[0, \infty)$ with

$$N(\tau) > 0, \quad \forall\, \tau \ge 0. \tag{22}$$

Remark 2.1: Condition (18) implies that $V_2 \hookrightarrow L^{2(\rho+1)}(\Omega)$. It follows from condition (17) and the Mean Value Theorem that there exists a constant $k_0 > 0$ such that

$$|f(u) - f(v)| \leq k_0 \left(1 + |u|^\rho + |v|^\rho\right) |u - v|, \quad \forall\, u, v \in \mathbb{R}. \tag{23}$$

Assumptions (17)–(18) include the classical nonlinear term of the form $f(u) = |u|^\rho u$. Another example is

$$f(u) \approx |u|^\rho u \pm |u|^\sigma u, \quad 0 < \sigma < \rho.$$

See more details on the conditions and examples, we refer the readers to [19,20].

2.3. Main results

The weak solution to the problem (11)–(14) is defined as follow: Given $T > 0, h \in V_0$ and $(u_0, u_1, \eta_0) \in \mathcal{H}$, we say a function $U = (u, u_t, \eta) \in C([0, T], \mathcal{H})$ is a weak solution of the problem (11)–(14) on $[0, T]$ if $U(0) = (u_0, u_1, \eta_0)$ and

$$\begin{aligned} &\langle u_{tt}, \omega\rangle + (\Delta u, \Delta\omega) + \left\langle |\nabla u|^{p-2}\nabla u, \nabla\omega\right\rangle + \int_0^\infty \mu(s)(\Delta\eta(s), \Delta\omega)\, ds \\ &\quad + N(\|\nabla u\|^2)\,(\nabla u_t, \nabla\omega) + (f(u), \omega) = (h, \omega) \end{aligned} \tag{24}$$

$$(\partial_t\eta + \partial_s\eta, \xi)_{\mathcal{M}_2} = (u_t, \xi)_{\mathcal{M}_2} \tag{25}$$

a.e. in $[0, T]$, for all $\omega \in V_2$ and $\xi \in \mathcal{M}_2$.

Now, we are in the position to present the well-posedness of the problem (11)–(14) as follow.

Theorem 2.1: *Let the assumptions* (16)–(22) *be in force, and take* $h \in V_0$.

(i) *If initial data* $(u_0, u_1, \eta_0) \in \mathcal{H}$, *then problem* (11)–(14) *has a weak solution in the class*

$$(u, u_t, \eta) \in C([0, T], \mathcal{H}), \quad \forall\, T > 0,$$

satisfying

$$u \in L^\infty(0, T; V_2), \quad u_t \in L^\infty(0, T; V_0) \cap L^2(0, T; V_1), \quad \eta \in L^\infty(0, T; \mathcal{M}_2).$$

(ii) *If initial data* $(u_0, u_1, \eta_0) \in \mathcal{H}_1$, *then the above weak solution has higher regularity*

$$u \in L^\infty(0, T; V_3), \quad u_t \in L^\infty(0, T; V_1) \cap L^2(0, T; V_2), \quad \eta \in L^\infty(0, T; \mathcal{M}_3).$$

(iii) *In both cases, the weak solutions depend continuously on the initial data in* $\mathcal{H}$. *More precisely, given any two weak solutions* $U_1 = (u, u_t, \eta)$ *and* $U_2 = (\tilde{u}, \tilde{u}_t, \tilde{\eta})$ *of problem* (11)–(14), *then*

$$\|U_1(t) - U_2(t)\|_{\mathcal{H}}^2 \leq e^{Ct}\, \|U_1(0) - U_2(0)\|_{\mathcal{H}}^2, \quad \forall\, t \in [0, T], T > 0, \tag{26}$$

for some positive constant $C = C(\|U_1(0)\|_{\mathcal{H}}, \|U_2(0)\|_{\mathcal{H}})$. *In particular, problem* (11)–(14) *has s a unique solution.*

The proof can be done by using the Faedo-Galerkin approximation method and combining similar arguments as presente e.g. in [1,19,20,32,33]. For the sake of the reader not familiar with the subject, we proved the sketch of the proof in Section 3.

The well-posedness of problem (11)–(14) given by Theorem 2.1 implies that the one-parameter family of operators $S(t) : \mathcal{H} \to \mathcal{H}$ defined by

$$S(t)\,(u_0, u_1, \eta_0) = \left(u(t), u_t(t), \eta^t\right), \quad t \geq 0, \tag{27}$$

where $(u(t), u_t(t), \eta^t)$ is the unique solution of problem (11)–(14) ensured by Theorem 2.1, satisfies the semigroup properties

$$S(0) = I, \quad \text{and} \quad S(t+s) = S(t)S(s), \quad t, s \geq 0,$$

and defines a nonlinear C_0-semigroup, which is locally Lipschitz continuous on $\mathcal{H}$. Then, the long-time dynamic of the problem (11)–(14) can be studied through the continuous dynamical system $(\mathcal{H}, S(t))$. The main result of this paper reads as follows.

Theorem 2.2: *Under the assumptions of Theorem* 2.1*, we have:*

(1) (**Global Attractor**) *The dynamical system* $(\mathcal{H}, S(t))$ *set in* (27) *possesses a unique global attractor* $\mathcal{A} \subset \mathcal{H}$*, which is compact and connected.*
(2) (**Geometrical structure**) *The global attractor* $\mathcal{A}$ *is characterized by the unstable manifold*

$$\mathcal{A} = \mathcal{M}_+(\mathcal{N}),$$

emanating from the set of stationary solutions $\mathcal{N} = \{(u, 0, 0) \in \mathcal{H} | \Delta^2 u - \Delta_p u + f(u) = h\}$*. In addition, every trajectory stabilizes to the set* $\mathcal{N}$*, more precisely, one has*

$$\lim_{t \to +\infty} \operatorname{dist}_{\mathcal{H}}(S(t)U, \mathcal{N}) = 0, \quad \forall\, U \in \mathcal{H}.$$

In particular, there exists a global minimal attractor $\mathcal{A}_{min}$ *to the dynamical system* $(\mathcal{H}, S(t))$*, which is precisely given by the set of the stationary points, that is,* $\mathcal{A}_{min} = \mathcal{N}$*.*

Additionally, if in the conditions (16) *and* (18)*, we assume subcritical exponents*

$$p < \frac{2N-2}{N-2} \text{if } N \geq 3 \quad \text{and} \quad \rho < \frac{4}{N-4} \text{if } N \geq 5, \tag{28}$$

then we also obtain:

(3) (**Finite dimensionality**) *The compact global attractor* $\mathcal{A}$ *has finite fractal (and Hausdorff) dimension* $dim^f_{\mathcal{H}}\mathcal{A}$*.*
(4) (**Regularity**) *Any full trajectory* $\gamma = \{(u(t), u_t(t), \eta^t); t \in \mathbb{R}\}$ *from the attractor* $\mathcal{A}$ *has the follow smooth property*

$$(u_t, u_{tt}, \eta_t) \in L^\infty(\mathbb{R}; \mathcal{H}). \tag{29}$$

More precisely, there exists a constant $R > 0$ *such that*

$$\sup_{\gamma \subset \mathcal{A}} \sup_{t \in \mathbb{R}} \left(\|\Delta u_t(t)\|^2 + \|u_{tt}(t)\|^2 + \left\|\eta_t^t\right\|^2_{\mathcal{M}_2} \right) \leq R^2. \tag{30}$$

(5) (**Generalized Exponential attractor**) *The dynamical system* $(\mathcal{H}, S(t))$ *has a generalized fractal exponential attractor* $\mathcal{A}_{exp}$ *with finite fractal dimension in the extended space*

$$\mathcal{H}_{-1} := V_0 \times V_2' \times \mathcal{M}_0.$$

In addition, from interpolation theorem, the fractal exponential attractor $(\mathcal{H}, S(t))$ *has finite fractal dimension in a smaller extended space* $\mathcal{H}_{-\delta}$*, where*

$$\mathcal{H} := \mathcal{H}_0 \subsetneq \mathcal{H}_{-\delta} \subseteq \mathcal{H}_{-1}, \quad 0 < \delta \leq 1.$$

The proof of Theorem 2.2 shall be done in Section 4, by several recent abstract results coming from dynamical systems which can be found e.g. in Chueshov and Lasiecka [7,34,35].

In order to give the proof of the main results stated above, we finally define the energy functional corresponding to the weak solution $(u(t), u_t(t), \eta^t) \in \mathcal{H}$ to the system (11)–(14) as

$$E(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{1}{2}\|\eta^t\|_{\mathcal{M}_2}^2 + \int_\Omega [\hat{f}(u(t)) - hu(t)]\,\mathrm{d}x. \tag{31}$$

Remark 2.2: To perform all computations hereafter without the memory component, we can simply consider $\mathcal{H} = V_2 \times V_0$ and $\mathcal{H}_1 = V_3 \times V_1$, and cancel the terms with relative displacement history η.

3. Proof of well-posedness

The proof of the existence is given by Faedo-Galerkin method and combines arguments from [1,19,20]. We shall give the sketch of the main steps.

Step 1 – Approximate problem. Let $(w_j)_{j=1}^\infty$ be an orthonormal basis in V_0 given by eigenfunctions of Δ^2 with boundary condition (2). Then it is well known that $(w_j)_{j=1}^\infty$ is smooth and can be taken orthogonal in V_1 and V_2. For the memory term, following the argument of Giorgi et al. [36,37] and Fatori et al. [32], we can choose a smooth orthonormal basis $(\xi_j)_{j=1}^\infty$ for $\mathcal{M}_2$.

Given initial data $(u_0, u_1, \eta_0) \in \mathcal{H}$, we seek for functions in the form

$$u^m(t) = \sum_{j=1}^m a_{mj}(t) w_j \quad \text{and} \quad \eta^{t,m}(s) = \sum_{j=1}^m b_{mj}(t)\xi_j(s)$$

which satisfy the approximate problem

$$\begin{aligned} &\langle u_{tt}^m(t), w_j\rangle + (\Delta u^m(t), \Delta w_j) + \left\langle \left|\nabla u^m(t)\right|^{p-2} \nabla u^m(t), \nabla w_j\right\rangle + \left(\eta^{t,m}, w_j\right)_{\mathcal{M}_2} \\ &\quad + N(\|\nabla u^m(t)\|^2)\left(\nabla u_t^m(t), \nabla w_j\right) + \left(f\left(u^m(t)\right), w_j\right) = \left(h, w_j\right), \end{aligned} \tag{32}$$

$$\left(\partial_t \eta^{t,m}, \xi_j\right)_{\mathcal{M}_2} = -\left(\partial_s \eta^{t,m}, \xi_j\right)_{\mathcal{M}_2} + \left(u_t^m(t), \xi_j\right)_{\mathcal{M}_2}, \quad j = 1, \dots, m, \tag{33}$$

with initial conditions

$$\left(u^m(0), u_t^m(0), \eta^{0,m}\right) = \left(u_0^m, u_1^m, \eta_0^m\right) \to (u_0, u_1, \eta_0) \text{ strongly in } \mathcal{H}. \tag{34}$$

We note that the approximate problem (32)–(34) can be reduced to an ordinary differential equation system and by standard existence theory for ODEs, the problem admits a local solution $(u^m(t), \eta^{t,m})$ in some interval $[0, T_m)$ with $0 < T_m \le T$. The following estimates imply that the local solutions $(u^m(t), \eta^{t,m})$ can be extended to the interval $[0, T]$ for any given $T > 0$.

Step 2 -Weak solution. Multiplying the approximate equation (32) by a_{mj} and (33) by b_{mj}, then summing up the results in j, we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}E^m(t) + N(\|\nabla u^m(t)\|^2)\left\|\nabla u_t^m(t)\right\|_2^2 = -\left(\partial_s \eta^{t,m}, \eta^{t,m}\right)_{\mathcal{M}_2}, \tag{35}$$

where $E^m(t)$ is the energy functional (30) given by approximate solutions $(u^m(t), \eta^{t,m}(s))$.

From assumptions (20) and (21) and noting $\eta^{t,m}(0) = 0$, we have

$$\left(\partial_s \eta^{t,m}, \eta^{t,m}\right)_{\mathcal{M}_2} = -\frac{1}{2}\int_0^\infty \mu'(s)\left\|\eta^{t,m}(s)\right\|_{V_2}^2 \,\mathrm{d}s \geq \frac{k_2}{2}\||\eta^{t,m}\|_{\mathcal{M}_2}^2 \geq 0. \tag{36}$$

Hence, integrating (35) from 0 to $t \leq T_m$, we get

$$E^m(t) + \int_0^t N(\|\nabla u(s)\|^2)\|\nabla u(s)\|^2 \,\mathrm{d}s \leq -\frac{k_2}{2}\||\eta^{t,m}\|_{\mathcal{M}_2}^2 + E^m(0) \leq E^m(0). \tag{37}$$

It follows from conditions (15) and (19) that

$$\int_\Omega \hat{f}(u^m(t))\,\mathrm{d}x \geq -\frac{\alpha_1}{2\lambda_1}\|\Delta u^m(t)\|^2 - l_0|\Omega|,$$

and for any $\varrho > 0$,

$$-\int_\Omega h u^m(t)\,\mathrm{d}x \geq -\varrho\|\Delta u^m(t)\|_2^2 - \frac{1}{4\lambda_1\varrho}\|h\|^2.$$

Now, taking $\alpha_0 = \frac{1}{2}(1 - \frac{\alpha_1}{\lambda_1}) > 0$ and $\varrho = \frac{\alpha_0}{2}$, we deduce that

$$\frac{1}{2}\left\|u_t^m(t)\right\|^2 + \frac{\alpha_0}{2}\|\Delta u^m(t)\|^2 + \frac{1}{p}\|\nabla u^m(t)\|_p^p + \frac{1}{2}\left\|\eta^{t,m}\right\|_{\mathcal{M}_2}^2 \leq E^m(t) + \frac{1}{2\lambda_1\alpha_0}\|h\|^2 + l_0|\Omega|. \tag{38}$$

Since $N > 0$ on $[0, \infty)$, using the convergence (34), from (37) and (38), one has

$$\left\|u_t^m(t)\right\|^2 + \|\Delta u^m(t)\|^2 + \left\|\eta^{t,m}\right\|_{\mathcal{M}_2}^2 \leq K_1$$

for all $t \in [0, T_m)$, $m \in \mathbb{N}$, where $K_1 = K_1(\|u_1\|, \|\Delta u_0\|, \|\eta_0\|_{\mathcal{M}_2}, \|h\|, |\Omega|) > 0$. This is sufficient to extend all approximate solutions on $[0, T]$. In addition, using condition (22) again, then there exists a positive constant $n_0 = n_0(\|(u_0, u_1, \eta_0)\|_{\mathcal{H}})$ such that

$$N\left(\left\|\nabla u^m(t)\right\|^2\right) \geq n_0 > 0, \quad \forall\, t \in [0, T] \tag{39}$$

Then, from (36), we have

$$E^m(t) + n_0\int_0^t \|\nabla u_t^m(s)\|^2 \,\mathrm{d}s \leq E^m(0). \tag{40}$$

Combining (38) and (40), one has

$$\left\|u_t^m(t)\right\|^2 + \|\Delta u^m(t)\|^2 + \left\|\eta^{t,m}\right\|_{\mathcal{M}_2}^2 + \int_0^t \|\nabla u_t^m(s)\|^2 \,\mathrm{d}s \leq K_1 \tag{41}$$

for any $t \in [0, T]$ and $m \in \mathbb{N}$, and some constant $K_1 > 0$ depending on initial dada in $\mathcal{H}$. From (41), passing to a subsequence if necessary, we have

$$\begin{aligned}
&u^m \to u \text{ weakly star in } L^\infty(0, T; V_2),\\
&u_t^m \to u_t \text{ weakly star in } L^\infty(0, T; V_0),\ \text{weakly in } L^2(0, T; V_1),\\
&\eta^m \to \eta \text{ weakly star in } L^\infty(0, T; \mathcal{M}_2).
\end{aligned}$$

The above limits and Aubin-Lions Lemma imply that

$$u^m \to u \quad \text{strongly in } L^2(0, T; V_1) \cap C([0, T], V_1).$$

Then, we can pass to the limit in l the approximate problem (32)–(34) and get the desired weak solution. The limit for the p-Laplacian and the nonlocal damping terms may be less standard. We refer the readers to [14,32] for details on the limit for the p-Laplacian term. Now, it only need to show that

$$\int_0^T \left(N(\|\nabla u^m(t)\|^2)\nabla u_t^m(t), \nabla w_j\right) \vartheta(t)\,\mathrm{d}t \to \int_0^T \left(N(\|\nabla u(t)\|^2)\nabla u_t(t), \nabla w_j\right) \vartheta(t)\,\mathrm{d}t, \tag{42}$$

for test function $\vartheta \in \mathcal{D}(0,T)$. In fact

$$\begin{aligned}
&\left|\int_0^T \Big(N(\|\nabla u^m(t)\|^2)\nabla u_t^m(t) - (N(\|\nabla u(t)\|^2)\nabla u_t(t), \nabla w_j\Big)\vartheta(t)\,\mathrm{d}t\right| \\
&\quad\le \left|\int_0^T \Big([N(\|\nabla u^m(t)\|^2) - (N(\|\nabla u(t)\|^2]\nabla u_t^m(t), \nabla w_j\Big)\vartheta(t)\,\mathrm{d}t\right| \\
&\quad\quad + \left|\int_0^T \Big((N(\|\nabla u(t)\|^2[\nabla u_t^m(t) - \nabla u_t(t)], \nabla w_j\Big)\vartheta(t)\,\mathrm{d}t\right| \\
&\quad = I_1 + I_2.
\end{aligned}$$

First of all, since N is C^1 function, from (41), we have that

$$\max_{\tau\in[0,K_1\lambda^{-\frac{1}{2}}]} \{N(\tau), N'(\tau)\} := C(\|(u_0,u_1,\eta_0)\|_{\mathcal{H}}) < \infty.$$

We will denote $C > 0$ to be the several constants which depend on the initial data, but not on $t > 0$. Hence, from Young's inequality and (41), one has

$$\begin{aligned}
I_1 &\le C\|\vartheta\|_\infty \int_0^T \left(\|\nabla u^m(t)\| + \|\nabla u(t)\|\right)\|\nabla u^m(t) - \nabla u(t)\|\|\nabla u_t^m(t)\|\|\nabla w_j\|\,\mathrm{d}t \\
&\le C\left(\int_0^T \|\nabla u_t^m(t)\|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T \|\nabla u^m(t) - \nabla u(t)\|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\le C\left(\int_0^T \|\nabla u^m(t) - \nabla u(t)\|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \to 0,
\end{aligned}$$

and

$$I_2 \le C\int_0^T \|\nabla u_t^m(t) - \nabla u_t(t)\|\,\mathrm{d}t \to 0.$$

The above two estimates imply that (42) hold. This completes the proof of item (**i**) of Theorem 2.1.

Step 3 -Stronger solutions. If initial data $(u_0,u_1,\eta_0) \in \mathcal{H}_1$, let us consider the approximate problem (32)–(34) with initial data

$$\left(u^m(0), u_t^m(0), \eta^{0,m}\right) = \left(u_0^m, u_1^m, \eta_0^m\right) \to (u_0,u_1,\eta_0) \text{ strongly in } \mathcal{H}_1. \tag{43}$$

By the choice of our Galerkin basis, replacing w_j and ξ_j by $-\Delta u_t^m$ and $-\Delta\eta^{t,m}$ in the approximate Equations (32) and (33), respectively, we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left\|\nabla u_t^m(t)\right\|^2 + \left\|\nabla\Delta u^m(t)\right\|^2 + \left\|\eta^{t,m}\right\|_{\mathcal{M}_3}^2\right\} + \left\langle\Delta_p\left(u^m(t)\right), \Delta u_t^m(t)\right\rangle \\
&\quad = -N(\|\nabla u^m(t)\|^2)\left\|\Delta u_t^m(t)\right\|^2 - \left(\partial_s\eta^{t,m}, \eta^{t,m}\right)_{\mathcal{M}_3} + \left(f\left(u^m(t)\right) - h, \Delta u_t^m(t)\right)
\end{aligned}$$

Noticing that

$$\left\langle\Delta_p u^m, \Delta u_t^m\right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle\Delta_p u^m, \Delta u^m\right\rangle - J$$

with

$$J=-\int_\Omega\left[(p-2)\left|\nabla u^m\right|^{p-4}\left(\nabla u^m\cdot\nabla u_t^m\right)\nabla u^m+\left|\nabla u^m\right|^{p-2}\nabla u_t^m\right\}\cdot\nabla\Delta u^m\,\mathrm{d}x,$$

we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left\|\nabla u_t^m(t)\right\|^2+\left\|\nabla\Delta u^m(t)\right\|^2+\left\|\eta^{t,m}\right\|^2_{\mathcal{M}_3}+2\left\langle\Delta_p\left(u^m(t)\right),\Delta u^m(t)\right\rangle\right\}\\&\quad=-N(\|\nabla u^m(t)\|^2)\left\|\Delta u_t^m(t)\right\|^2-\left(\partial_s\eta^{t,m},\eta^{t,m}\right)_{\mathcal{M}_3}+\left(f\left(u^m(t)\right)-h,\Delta u_t^m(t)\right)+J.\end{aligned}\tag{44}$$

We have to estimate each term of the right-hand side of (44). It follows from the argument of (39) that there exists a constant $n_0=n_0(\|(u_0,u_1,\eta^0)\|_{\mathcal{H}})>0$ such that

$$-N(\|\nabla u^m(t)\|^2)\left\|\Delta u_t^m(t)\right\|^2\le-n_0\|\Delta u_t^m(t)\|^2.$$

By a similar argument to (36), we have

$$-\left(\partial_s\eta^{t,m},\eta^{t,m}\right)_{\mathcal{M}_3}\le-\frac{k_2}{2}\||\eta^{t,m}\|^2_{\mathcal{M}_3}\le 0.$$

Using Hölder's and Young's inequalities, and the estimate (41), it is not difficult to show that

$$|J|\le\frac{n_0}{4}\left\|\Delta u_t^m(t)\right\|^2+C\left\|\nabla\Delta u^m(t)\right\|^2$$

and

$$|\left(f\left(u^m(t)\right)-h,\Delta u_t^m(t)\right)|\le\frac{n_0}{4}\left\|\Delta u_t^m(t)\right\|^2+C\left\|\nabla\Delta u^m(t)\right\|^2.$$

Therefore, inserting the above estimates into (44), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}F^m(t)+n_0\left\|\Delta u_t^m(t)\right\|_2^2\leqslant C+C\left\|\nabla\Delta u^m(t)\right\|^2,\tag{45}$$

where

$$F^m(t)=\left\|\nabla u_t^m(t)\right\|^2+\left\|\nabla\Delta u^m(t)\right\|^2+\left\|\eta^{t,m}\right\|^2_{\mathcal{M}_3}+2\left\langle\Delta_p u^m(t),\Delta u^m(t)\right\rangle.$$

As proved in [1] and [32], there exists a constant $C>0$ such that

$$2\left\langle\Delta_p u^m(t),\Delta u^m(t)\right\rangle\ge-C-\frac{1}{2}\left\|\nabla\Delta u^m(t)\right\|^2.$$

Hence, integrating (43) from 0 to $t<T$ and using the above inequality, we deduce

$$\begin{aligned}&\left\|\nabla u_t^m(t)\right\|^2+\frac{1}{2}\left\|\nabla\Delta u^m(t)\right\|^2+\left\|\eta^{t,m}\right\|^2_{\mathcal{M}_3}+n_0\int_0^t\left\|\Delta u_t^m(\tau)\right\|^2\,\mathrm{d}\tau\\&\quad\le F^m(0)+CT+\int_0^t\left\|\nabla\Delta u^m\tau)\right\|^2\,\mathrm{d}\tau.\end{aligned}$$

Using the convergence (43) and taking into account Gronwall's inequality, we arrive at

$$\left\|\nabla u_t^m(t)\right\|^2+\frac{1}{2}\left\|\nabla\Delta u^m(t)\right\|^2+\left\|\eta^{t,m}\right\|^2_{\mathcal{M}_3}+\int_0^t\left\|\Delta u_t^m(\tau)\right\|^2\,\mathrm{d}\tau\leqslant K_2,\tag{46}$$

for all $t\in[0,T]$ $m\in\mathbb{N}$, where $K_2=K_2(\|(u_0,u_1,\eta^0)\|_{\mathcal{H}_1},\|h\|,T)>0$. From this estimate we can deduce that weak solutions have stronger regularity

$$(u,u_t,\eta)\in L^\infty\left(0,T;\mathcal{H}_1\right).$$

In addition, $u_t\in L^2(0,T;V_2)$. This finishes the proof of item (ii) of Theorem 2.1.

Step 4 -Continuous dependence. Let us first suppose that $U_1 = (u, u_t, \eta)$ and $U_2 = (\tilde{u}, \tilde{u}_t, \tilde{\eta})$ are two stronger solutions of the problem (11)–(14) with initial data $U_1(0) = (u_0, u_1, \eta_0)$, $U_2(0) = (\tilde{u}_0, \tilde{u}_1, \tilde{\eta}_0)$, respectively. Denoting $w = u - \tilde{u}$ and $\zeta = \eta - \tilde{\eta}$, the function $U_1 - U_2 = (w, w_t, \zeta)$ is a stronger solution of the problem

$$w_{tt} + \Delta^2 w - \Delta_p u + \Delta_p \tilde{u} + \int_0^\infty \mu(s)\Delta^2 \zeta^t(s)\mathrm{d}s - N(\|\nabla u\|^2)\Delta w_t + f(u) - f(\tilde{u})$$
$$+ \left(N(\|\nabla u\|^2) - N(\|\nabla \tilde{u}\|^2)\right)\Delta \tilde{u}_t = 0, \tag{47}$$
$$\zeta_t = -\zeta_s + w_t \tag{48}$$

with initial data

$$U(0) = \left(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1, \eta_0 - \tilde{\eta}_0\right). \tag{49}$$

With this regularity, multiplying (47) by w_t in V_0 and (48) by ζ in $\mathcal{M}_2$, we can infer

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}W(t) + N(\|\nabla u(t)\|^2)\,\|\nabla w_t(t)\|^2 \\
&\quad = \left\langle \Delta_p u(t) - \Delta_p \tilde{u}(t), w_t(t)\right\rangle - \left(f(u(t)) - f(\tilde{u}(t)), w_t(t)\right) - \left(\zeta^t, \zeta_s^t\right)_{\mathcal{M}_2} \\
&\qquad + \left(N(\|\nabla u(t)\|^2) - N(\|\nabla \tilde{u}(t)\|^2)\right)\left(\nabla \tilde{u}_t(t), \nabla w_t(t)\right) \\
&\quad \triangleq J_1 + J_2 + J_3 + J_4,
\end{aligned} \tag{50}$$

where

$$W(t) = \|U_1 - U_2\|_{\mathcal{H}}^2 = \|\Delta w(t)\|^2 + \|w_t(t)\|^2 + \left\|\zeta^t\right\|_{\mathcal{M}_2}^2.$$

As shown before, we have that $N(\|\nabla u(t)\|^2) \geq n_0 > 0$. By the similar argument as in [1] and [32], there exists a constant $C > 0$ such that

$$|J_1| \leq C\|\Delta w(t)\|^2 + \frac{n_0}{4}\|\nabla w_t(t)\|^2,$$
$$|J_2| \leq C\|\Delta w(t)\|^2 + \frac{1}{2}\|w_t(t)\|^2,$$
$$J_3 = \frac{1}{2}\int_0^\infty \mu'(s)\left\|\Delta \zeta^t(s)\right\|^2 \mathrm{d}s \leq 0.$$

Since N is C^1 function, making use of the Mean Value Theorem and Young's inequality, estimate (41) and the embedding $V_2 \hookrightarrow V_1$, we get

$$\begin{aligned}
|J_4| &\leq C\left[\|\nabla u(t)\| + \|\nabla \tilde{u}(t)\|\right]\|\nabla w(t)\|\|\nabla \tilde{u}_t(t)\|\|\nabla w_t(t)\| \\
&\leq C\|\Delta w(t)\|\|\nabla \tilde{u}_t(t)\|\|\nabla w_t(t)\| \\
&\leq \frac{n_0}{4}\|\nabla w_t(t)\|^2 + \frac{C^2}{n_0}\|\nabla \tilde{u}_t(t)\|^2\|\Delta w(t)\|^2
\end{aligned}$$

Inserting the above estimates into (50) and neglecting all nonnegative terms, we can deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) + n_0\|\nabla w_t(t)\|^2 \leq C\left(1 + \|\nabla \tilde{u}_t(t)\|^2\right)W(t), \tag{51}$$

for any $t \in [0, T]$. It follows from estimate (41) that $\beta(t) = 1 + \|\nabla \tilde{u}_t(t)\|^2$ has the property $\beta \in L^1(0, T)$. Thus, integrating (51) from 0 to $t \leq T$ and applying Gronwall's inequality, we have

$$W(t) + n_0\int_0^t \|\nabla w(s)\|^2 \mathrm{d}s \leq e^{C\int_0^t \beta(s)\,\mathrm{d}s}W(0).$$

Keeping in mind $W(t) = \|U_1 - U_2\|^2_{\mathcal{H}}$ and $\beta \in L^1(0,T)$, the desired estimate (26) is achieved for stronger solutions for some constant $C_0 > 0$. The same conclusion holds true for weak solutions by using density arguments, see e.g. [19,36]

In particular, we have uniqueness for both strong and weak solutions. The proof of item (iii) of Theorem 2.1 is now complete.

4. Long-time dynamics

4.1. A brief review on nonlinear dynamical systems

Before giving the proof of the main result on the long-time behavior of problem (11)–(14), in this subsection, we introduce some fundamental concepts on the theory of infinite-dimensional dynamical systems that can be applied to our particular dynamical system $(\mathcal{H}, S(t))$ generated by (27). Here, just for a reason of adaptation, we prefer to recall some facts from the book by Chueshov and Lasiecka [7,35]. We also refer readers to [38–40], among others, for other approaches within the general theory in dynamical systems

A dynamical system $(H, S(t))$ is dissipative if it possesses a bounded absorbing set, that is, a bounded set $\mathcal{B} \subset H$ such that for any bounded set $B \subset H$ there exists $t_B \geq 0$ satisfying

$$S(t)B \subset \mathcal{B}, \quad \forall\, t \geq t_B.$$

A compact set $\mathcal{A} \subset H$ is a global attractor for a dynamical system $(H, S(t))$, if it is fully invariant and uniformly attracting, that is $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and for every bounded subset $B \subset H$,

$$\lim_{t\to+\infty} dist_H(S(t)B, \mathcal{A}) = 0,$$

where $dist_H$ is the Hausdorff semidistance in H. Given a compact set M in a metric space X, the fractal dimension of M is defined by

$$dim_f^X = \limsup_{\varepsilon\to 0} \frac{\ln N(M,\varepsilon)}{\ln(1/\varepsilon)},$$

where $N(M,\varepsilon)$ is the minimal number of closed balls with radius $\varepsilon > 0$ which covers M.

A global minimal attractor for$(H, S(t))$ is a bounded closed set $\mathcal{A}_{min} \subset H$ which is positively invariant $(S(t)\mathcal{A}_{min} \subseteq \mathcal{A}_{min})$ and attracts uniformly every point z from H, that is,

$$\lim_{t\to+\infty} \mathrm{dist}_H\,(S(t)z, \mathcal{A}_{\min}) = 0 \quad \text{for any } z \in H$$

and $\mathcal{A}_{min}$ has no proper subsets possessing this two properties.

Define the unstable manifold $\mathcal{M}_+(\mathcal{N})$ emanating from the set $\mathcal{N} \subset H$, such that for each $z_0 \in \mathcal{M}_+(\mathcal{N})$ there exists a full trajectory $\gamma = \{z(t) : t \in \mathbb{R}\}$ with the properties

$$z(0) = z_0 \quad \text{and} \quad \lim_{t\to-\infty} dist_H(z(t), \mathcal{N}) = 0.$$

A dynamical system $(H, S(t))$ is called a gradient system if there exists a strict Lyapunov function for $(H, S(t))$ on the whole phase space H, that is, (a) a continuous functional $\Phi(z)$ such that the function $t \to \Phi(S(t)z)$ is nonincreasing for any $z \in H$, (b) the equation $\Phi(S(t)z) = \Phi(z)$ for all $t > 0$ implies that $S(t)z = z$ for all $t > 0$.

We recall the concept of quasi-stability. Let X, Y, Z be three reflexive Banach spaces with X compactly embedded in Y and put $H = X \times Y \times Z$, considering the dynamical system $(H, S(t))$ given by an evolution operator

$$S(t)z = (u(t), u_t(t), \xi(t)), \quad z = (u_0, u_1, \xi_0) \in H, \tag{52}$$

where the functions u and ξ have the regularity

$$u \in C(\mathbb{R}^+; X) \cap C^1(\mathbb{R}^+; Y), \quad \xi \in C(\mathbb{R}^+; Z). \tag{53}$$

Then $(H, S(t))$ is called quasi-stable on a set $B \subset H$, if there exists a compact semi-norm n_X on X (i.e. if $x_j \rightharpoonup 0$ in X one has $n_X(x_j) \to 0$) and nonnegative scalar functions a, b, c, with a, c locally bounded in $[0, \infty)$ and $b \in L^1(\mathbb{R}^+)$ satisfying $\lim_{t\to+\infty} b(t) = 0$, such that

$$\left\|S(t)z^1 - S(t)z^2\right\|_H^2 \le a(t)\left\|z^1 - z^2\right\|_H^2, \quad t \ge 0 \tag{54}$$

and

$$\left\|S(t)z^1 - S(t)z^2\right\|_H^2 \le b(t)\left\|z^1 - z^2\right\|_H^2 + c(t)\sup_{0<s<t}\left[n_X(u^1(s) - u^2(s))\right]^2 \tag{55}$$

for any $z^1, z^2 \in B$.

An exponential attractor of a dynamical system $(H, S(t))$ is a compact set $\mathfrak{A}_{exp} \subset H$, that enjoys three characteristic properties: (i) it has finite fractal dimension, (ii) it is positively invariant, (iii) for any bounded set $D \subset H$, there exist positive constants t_D, C_D, and γ_D such that

$$\mathrm{dist}_H\left(S(t)D, \mathfrak{A}_{\mathrm{exp}}\right) \le C_D \exp\left(-\gamma_D\left(t - t_D\right)\right), \quad t \ge t_D.$$

If there exists an exponential attractor only having finite fractal dimension in some extended space $\tilde{H} \supseteq H$, then it is called generalized fractal exponential attractor.

4.2. Technical results

Let $E(t)$ be the energy functional corresponding to the weak solution $(u, u_t, \eta) \in \mathcal{H}$ of problem (11)–(14).

Lemma 4.1: *Under the above assumptions, there exist positive constants β_0, $K = K(|\Omega|, \|h\|)$ and $n_0 = n_0(\|(u, u_t, \eta)\|_{\mathcal{H}})$ such that the energy functional $E(t)$ satisfies*

$$E(t) \ge \beta_0\left(\|u_t(t)\|^2 + \|\Delta u(t)\|^2 + \left\|\eta^t\right\|_{\mathcal{M}_2}^2\right) - K, \quad t \ge 0, \tag{56}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le -n_0\|\nabla u_t\|^2 - \frac{k_2}{2}\|\eta^t\|_{\mathcal{M}_2}^2. \tag{57}$$

Proof: By the similar argument as (38), we have that (56) holds for

$$\beta_0 = \frac{1}{4}\left(1 - \frac{\alpha_1}{\lambda_1}\right) \quad \text{and} \quad K = l_0|\Omega| + \frac{1}{4\lambda_1\beta_0}\|h\|^2.$$

Moreover, since $(u, u_t, \eta) \in \mathcal{H}$ is the weak solution, analogously as in proof of (35), noticing (36) and (39), then estimate (57) is ensured. ■

Now, for any $\epsilon > 0$, we define the perturbed energy

$$E_\epsilon(t) = E(t) + \epsilon\Psi(t) \quad \text{with} \quad \Psi(t) = (u_t(t), u(t)).$$

In the following, we will use $C_i > 0$ $(i = 1, 2\cdots)$ to denote the several positive constants appearing in different estimates.

Lemma 4.2: *Under the assumptions of Theorem 2.2, the semigroup $S(t)$ defined by (27) admits a bounded absorbing set $\mathcal{B} \subset \mathcal{H}$, i.e. the dynamical system $(\mathcal{H}, S(t))$ is dissipative.*

Proof: Using Young's inequality and estimate (56), we have

$$|\Psi(t)| \le \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2\lambda_1}\|\Delta u(t)\|^2 \le C_1\left(E(t) + \|h\|^2 + |\Omega|\right) \tag{58}$$

for some positive constant $C_1 = C_1(\lambda_1, \alpha_1, l_0)$.

We claim that there exists positive constants C_2 and C_2 such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) \le -E(t) + C_2\|\nabla u_t(t)\|^2 + C_3\|\eta^t\|^2_{\mathcal{M}_2} + l_1|\Omega|. \tag{59}$$

Indeed, taking derivative of function $\Psi(t)$, using Equations (11) and (12), subtracting and adding $E(t)$ into the resulting expression, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) &= -E(t) + \frac{3}{2}\|u_t(t)\|^2 - \frac{1}{2}\|\Delta u(t)\|^2 - \left(1 - \frac{1}{p}\right)\|\nabla u(t)\|_p^p \\ &\quad + \frac{1}{2}\|\eta^t\|^2_{\mathcal{M}_2} + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,\end{aligned} \tag{60}$$

where

$$\begin{aligned}\mathcal{I}_1 &= \int_\Omega [\hat{f}(u(t)) - f(u(t))u(t)]\,\mathrm{d}x, \\ \mathcal{I}_2 &= -N(\|\nabla u(t)\|^2)\int_\Omega \nabla u(t)\cdot\nabla u_t(t)\,\mathrm{d}x, \\ \mathcal{I}_3 &= -\int_0^\infty \mu(s)\left(\Delta\eta^t(s), \Delta u(t)\right) ds.\end{aligned}$$

From the right-hand side of condition (19), we obtain

$$\mathcal{I}_1 \le \frac{\alpha_1}{2\lambda_1}\|\Delta u(t)\| + l_1|\Omega|.$$

Since N is uniformly bounded, using Young's inequality and embedding $V_2 \hookrightarrow V_1$, we have

$$\begin{aligned}|\mathcal{I}_2| &\le N(\|\nabla u(t)\|^2)\|\nabla u(t)\|\|\nabla u_t(t)\| \\ &\le \sigma\|\Delta u(t)\|^2 + C_\sigma\|\nabla u_t(t)\|^2\end{aligned}$$

for any $\sigma > 0$. Using Young's inequality again, we obtain

$$|\mathcal{I}_3| \le \sigma\|\Delta u(t)\|^2 + \frac{\mu_0}{4\sigma}\|\eta^t\|^2_{\mathcal{M}_2}$$

for any $\sigma > 0$. Inserting the above three estimates into (60)and neglecting the negative term $\|\nabla u\|_p^p$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) \le -E(t) + \left(\frac{3}{2}d + C_\sigma\right)\|\nabla u_t(t)\|^2 - \left(\frac{1}{2} - 2\sigma\right)\|\Delta u(t)\|^2 + \left(\frac{1}{2} + \frac{\mu_0}{4\sigma}\right)\|\eta^t\|^2_{\mathcal{M}_2} + l_1|\Omega|,$$

where d is the embedding constant for $V_1 \hookrightarrow V_0$. Hence, choosing $\sigma \in (0, \frac{1}{4}]$ sufficiently small, the inequality (59) is ensured.

Now let us fix $n_1 = \min\{\frac{n_0}{C_2}, \frac{k_2}{2C_3}\} > 0$. Then for all $\epsilon \le n_1$, follows from (57) and (59) that

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\epsilon(t) \le -\epsilon E(t) + \epsilon l_1 |\Omega|, \quad t \ge 0. \tag{61}$$

Let us choose $\epsilon_1 = \min\{\frac{1}{2C_1}, n_1\} > 0$. For any $\epsilon \in (0, \epsilon_1]$, it follows from (58) that

$$\frac{1}{2}E(t) - \frac{1}{2}\left(\|h\|_2^2 + |\Omega|\right) \le E_\epsilon(t) \le \frac{3}{2}E(t) + \frac{1}{2}\left(\|h\|_2^2 + |\Omega|\right). \tag{62}$$

Combining (61) with (62) yields

$$E(t) \le 3E(0)e^{-\frac{2}{3}\epsilon t} + C_4, \quad t \ge 0,$$

for some constant $C_4 = C_4(\|h\|, |\Omega|, l_1) > 0$. Using (56) again, we conclude that

$$\left\| \left(u(t), u_t(t), \eta^t\right) \right\|_{\mathcal{H}}^2 \le \frac{3}{\beta_0} E(0) e^{-\frac{2}{3}\epsilon t} + \frac{1}{\beta_0}\left(K + C_4\right). \tag{63}$$

Hence, taking the closed ball $\mathcal{B} = \overline{B}_{\mathcal{H}}(0, R)$ with $R = \sqrt{\frac{2}{\beta_0}(K + C_4)}$, we have that $\mathcal{B}$ is a bounded absorbing set for $S(t)$. This completes the proof of Lemma 4.2. ■

Remark 4.1: If we take the null constants l_0, l_1 and $h = 0$ in V_0, from (63) and the choices of K and C_4, we have the exponential stability for the solution of problem (11)–(14).

The existence of a bounded absorbing set implies that for initial data lying in bounded sets $B \subset \mathcal{H}$, the solutions of problem (11)–(14) are globally bounded in $\mathcal{H}$, that is, if (u, u_t, η) is a solution of (11)–(14) with initial data (u_0, u_1, η_0) in a bounded set B, then one has

$$\left\| \left(u(t), u_t(t), \eta^t\right) \right\|_{\mathcal{H}} \le C_B, \quad t \ge 0, \tag{64}$$

where $C_B > 0$ is a constant depending on the size of B. Moreover, from (41), we also infer

$$\limsup_{t \to +\infty} \left[\int_0^t \|\nabla u_t(s)\|_2^2 \, \mathrm{d}s \right] \le \tilde{C}_B, \tag{65}$$

for some constant $\tilde{C}_B > 0$.

Let us take the functional Φ as the energy E defined in (31). Then, we have the following results.

Lemma 4.3: (1) *The dynamical system* $(\mathcal{H}, S(t))$ *given in* (27) *is gradient;*
(2) *The Lyapunov functional* Φ *is bounded from above on any bounded subset of* $\mathcal{H}$*;*
(3) *The set* $\Phi_R = \{U \in \mathcal{H}; \Phi(U) \le R\}$ *is bounded in* $\mathcal{H}$ *for every* $R > 0$.

Proof: (1) For given initial data $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}$, by the similar arguments as (35) and (36), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi\left(S(t)U_0\right) = -N(\nabla u(t)) \|\nabla u_t(t)\|^2 + \frac{1}{2}\int_0^\infty \mu'(s) \|\Delta u(t)\|^2 \, \mathrm{d}s \le 0, \quad t > 0, \tag{66}$$

which implies the map $t \mapsto \Phi(S(t)U_0)$ is nonincreasing. Thus, supposing $\Phi(S(t)U_0) = \Phi(u_0)$ for all $t > 0$, yields

$$N(\nabla u(t)) \, \|\nabla u_t(t)\|_2^2 = 0 \quad \text{and} \quad \int_0^\infty \mu'(s) \left\| \Delta \eta^t(s) \right\|_2^2 \mathrm{d}s = 0, \quad t \ge 0. \tag{67}$$

The first term of (67) and (22) imply that $u(t) = u_0$ for all $t \ge 0$. On the other hand, from the second term of (67) and (21), we deduce that $\eta^t(s) = 0$ for all $t, s \ge 0$. Hence, $S(t)U_0 = (u_0, 0, 0)$ is a stationary solution, which shows that Φ is a strict Lyapunov functional. This proves that $(\mathcal{H}, S(t))$ is gradient.

(2) From (66) and the definition of Φ, it is easy to obtain that Φ is bounded from above on bounded subsets of $\mathcal{H}$.

(3) Let $(u(t), u_t(t), \eta^t) = S(t)U_0 \in \mathcal{H}$ be any weak solution of problem (11)–(14) such that $\Phi(S(t)U_0) \leq R$. Then we infer from (56) that

$$\|S(t)U_0\|_{\mathcal{H}}^2 \leq \frac{1}{\beta_0}(R+K),$$

which implies that Φ_R is bounded in $\mathcal{H}$ for every $R > 0$. ■

Lemma 4.4: *The set* $\mathcal{N} = \{(u, 0, 0) \in \mathcal{H} | \Delta^2 u - \Delta_p u + f(u) = h\}$ *of stationary solutions of the problem* (11)–(14) *is bounded in* $\mathcal{H}$.

Proof: Let $(u, 0, 0)$ be any stationary solution of problem (11)–(14). It follows from (11) that

$$\|\Delta u\|^2 + \|\nabla u\|_p^p = -\int_\Omega f(u)u \, \mathrm{d}x + \int_\Omega hu \, \mathrm{d}x.$$

It follows from the assumption (19) that

$$-\int_\Omega f(u)u \, \mathrm{d}x \leq \frac{\alpha_1}{\lambda_1}\|\Delta u\|_2^2 + (l_0 + l_1)|\Omega|.$$

Using Young's inequality with $\varrho > 0$ again and (15), we infer

$$\int_\Omega hu \, \mathrm{d}x \leq \varrho\|\Delta u\|^2 + \frac{1}{4\lambda_1\varrho}\|h\|^2.$$

Hence, we have

$$\left(1 - \frac{\alpha_1}{\lambda_1} - \varrho\right)\|\Delta u\|^2 \leq (l_0 + l_1)|\Omega| + \frac{1}{4\lambda_1\varrho}\|h\|^2$$

Choosing $\varrho > 0$ sufficiently small one concludes that $\mathcal{N}$ is bounded in $\mathcal{H}$. ■

Now we shall provide an essential inequality usually called stabilizability inequality that will be a key point for the existence of a global attractor for the dynamical system $(\mathcal{H}, S(t))$ and its properties as well.

Lemma 4.5: *Let assumptions of Theorem* 2.1 *be in force. Given a bounded set* $B \subset \mathcal{H}$*, we consider two weak solutions* $S(t)U^1 = (u(t), u_t(t), \eta^t)$ *and* $S(t)U^2 = (\tilde{u}(t), \tilde{u}_t(t), \tilde{\eta}^t)$ *of the problem* (11)–(14) *with corresponding initial data* $U^i \in B$*. Then, there exist positive constants* b_B, γ_B *and* C_B *such that*

$$\begin{aligned}\left\|S(t)U^1 - S(t)U^2\right\|_{\mathcal{H}}^2 \leq\ & b_B e^{-\gamma_B t}\left\|U^1 - U^2\right\|_{\mathcal{H}}^2 \\ & + C_B\int_0^t e^{-\gamma_B(t-s)}\left(\|\nabla w(s)\|_{2(p-1)}^2 + \|w(s)\|_{2(\rho+1)}^2\right)\mathrm{d}s \qquad (68)\end{aligned}$$

for all $t \geq 0$*, where* $w = u - \tilde{u}$.

Proof: Let us first fix a bounded set $B \subset \mathcal{H}$ and denote $\zeta = \eta - \tilde{\eta}$, then the triplet (w, w_t, ζ) is a weak solution of

$$w_{tt} + \Delta^2 w - \Delta_p u + \Delta_p \tilde{u} + \int_0^\infty \mu(s)\Delta^2\zeta^t(s)\mathrm{d}s - N(\|\nabla u\|^2)\Delta w_t + f(u) - f(\tilde{u})$$

$$+ \left(N(\|\nabla u\|^2) - N(\|\nabla \tilde{u}\|^2)\right)\Delta\tilde{u}_t = 0, \tag{69}$$

$$\zeta_t = -\zeta_s + w_t, \tag{70}$$

with initial data

$$\left(u(0), u_t(0), \eta^0\right) = U^1 - U^2. \tag{71}$$

To this problem, we consider the energy functional

$$F(t) = \frac{1}{2}\|w_t(t)\|^2 + \frac{1}{2}\|\Delta w(t)\|^2 + \frac{1}{2}\|\zeta^t\|^2_{\mathcal{M}_2}, \quad t \geq 0. \tag{72}$$

In the following, for any given $\varepsilon > 0$, C_0 and $C_{0,\varepsilon}$ will denote several positive constants depending on the size of B.

Step1. The estimate for $F'(t)$. Proceeding the same arguments as in the proof Theorem 2.1 (*iii*), we obtain the following inequality

$$F'(t) + n_B\|\nabla w_t(t)\|^2 \leq \sum_{k=1}^{4} J_k, \tag{73}$$

for some constant $n_B > 0$ by the global estimate (64) and $N(\tau) > 0$, where J_k, $k = 1, 2, 3, 4$ are the same ones given in (50). Now since there exists $M_p > 0$ such that

$$\left||x|^{p-2}x - |y|^{p-2}y\right| \leq M_p\left(|x|^{p-2} + |y|^{p-2}\right)|x - y|, \quad x, y \in \mathbb{R}^n,$$

using generalized Hölder's inequality, assumption (16) and the estimate (64), we conclude

$$\begin{aligned}
J_1 &\leq M_p\int_\Omega \left(|\nabla u(t)|^{p-2} + \left|\nabla\tilde{u}(t)\right|^{p-2}\right)|\nabla w(t)|\,|\nabla w_t(t)|\ \mathrm{d}x \\
&\leq M_p\left(\|\nabla u(t)\|^{p-2}_{2(p-1)} + \left\|\nabla\tilde{u}(t)\right\|^{p-2}_{2(p-1)}\right)\|\nabla w(t)\|_{2(p-1)}\ \|\nabla w_t(t)\|_2 \\
&\leq C_0\|\nabla w(t)\|_{2(p-1)}\ \|\nabla w_t(t)\|_2 \\
&\leq \varepsilon\|\nabla w_t(t)\|^2 + C_{0,\varepsilon}\|\nabla w(t)\|^2_{2(p-1)}
\end{aligned}$$

for any $\varepsilon > 0$. Again by generalized Hölder's inequality, assumption (17)–(18) and the estimate (64), we have

$$\begin{aligned}
J_2 &\leq k_0\int_\Omega \left(1 + |u(t)|^\rho + \left|\tilde{u}(t)\right|^\rho\right)|w(t)|\,|w_t(t)|\ \mathrm{d}x \\
&\leq k_0\big(|\Omega|^{\frac{\rho}{2\rho+1}} + \|u(t)\|^\rho_{2(\rho+1)} + \|\tilde{u}(t)\|^\rho_{2(\rho+1)}\big)\|w(t)\|_{2(\rho+1)}\|w_t(t)\|_2 \\
&\leq C_0\|w(t)\|_{2(\rho+1)}\ \|\nabla w_t(t)\|_2 \\
&\leq \varepsilon\|\nabla w_t(t)\|^2 + C_{0,\varepsilon}\|w(t)\|^2_{2(\rho+1)}
\end{aligned}$$

for any $\varepsilon > 0$. It follows assumption (21) that

$$J_3 = \frac{1}{2}\int_0^\infty \mu'(s)\left\|\Delta\zeta^t(s)\right\|^2\ \mathrm{d}s \leq -\frac{k_2}{2}\|\zeta^t\|^2_{\mathcal{M}_2}.$$

Now since $N \in C^1([0,\infty))$, then from (64) and $V_2 \hookrightarrow V_1$, we infer

$$\begin{aligned} J_4 &\le C\,[\|\nabla u(t)\| + \|\nabla \tilde{u}(t)\|]\,\|\nabla w(t)\|\|\nabla \tilde{u}_t(t)\|\|\nabla w_t(t)\| \\ &\le C_0\|\Delta w(t)\|\|\nabla \tilde{u}_t(t)\|\|\nabla w_t(t)\| \\ &\le \varepsilon\,\|\nabla w_t(t)\|^2 + C_{0,\varepsilon}\|\nabla \tilde{u}_t(t)\|^2\|\Delta w(t)\|^2 \end{aligned}$$

for any $\varepsilon > 0$. Inserting these last four estimates into (73) and choosing ε sufficiently small such that there exists positive constants C_5 and C_6 depending on the size of B, satisfy

$$F'(t) \le -\frac{n_B}{2}\,\|\nabla w_t(t)\|^2 - \frac{k_2}{2}\|\zeta^t\|^2_{\mathcal{M}_2} + C_5\|\nabla \tilde{u}_t(t)\|^2\|\Delta w(t)\|^2 + C_6\mathcal{W}(t) \tag{74}$$

where we define

$$\mathcal{W}(t) = \|\nabla w(t)\|^2_{2(p-1)} + \|w(t)\|^2_{2(\rho+1)}.$$

Step2. The estimate for $\Phi'(t)$, where we define the functional

$$\Phi(t) = \int_\Omega w_t(t)w(t)\,\mathrm{d}x.$$

We claim that there exist positive constants C_7, C_8, C_9 depending on the size of B, satisfy

$$\Phi'(t) \le -F(t) + C_7\|\nabla w_t(t)\|^2 + C_8\|\zeta^t\|^2_{\mathcal{M}_2} + C_9\mathcal{W}(t). \tag{75}$$

Indeed, taking the derivative of Φ, using (69) in the weak sense, subtracting and adding $F(t)$ in the resulting expression, we obtain

$$\Phi'(t) = -F(t) + \frac{3}{2}\,\|w_t(t)\|_2^2 - \frac{1}{2}\|\Delta w(t)\|_2^2 + \frac{1}{2}\,\left\|\zeta^t\right\|^2_{\mathcal{M}_2} + \sum_{j=1}^{5} L_j \tag{76}$$

where

$$\begin{aligned} L_1 &= -\int_0^\infty \mu(s)\left(\Delta\zeta'(s), \Delta w(t)\right)\,\mathrm{d}s, \\ L_2 &= -N(\|\nabla u(t)\|^2)\,(\nabla w_t(t), \nabla w(t))\,, \\ L_3 &= \left(N(\|\nabla u(t)\|^2) - N(\|\nabla \tilde{u}(t)\|^2)\right)\left(\nabla \tilde{u}(t), \nabla w(t)\right), \\ L_4 &= \left\langle \Delta_p u(t) - \Delta_p \tilde{u}(t), w(t)\right\rangle, \\ L_5 &= -\left(f\,(u(t)) - f\left(\tilde{u}(t)\right), w(t)\right). \end{aligned}$$

To estimate L_1, using the similar argument used to estimate $\mathcal{I}_3$ given in (60), we have

$$|L_1| \le \varepsilon\|\Delta w(t)\|^2 + \frac{\mu_0}{4\varepsilon}\|\zeta^t\|^2_{\mathcal{M}_2}$$

for any $\varepsilon > 0$. Since N is uniformly bounded, using (64) and Young's inequality with $\varepsilon > 0$ and embedding $V_2 \hookrightarrow V_1$, we arrive at

$$|L_2| \le C_0\|\nabla w(t)\|\|\nabla w_t(t)\| \le \varepsilon\|\Delta w(t)\|^2 + C_{0,\varepsilon}\|\nabla w_t(t)\|^2.$$

To estimate L_3, L_4 and L_5, modifying the arguments used to estimate J_4, J_1 and J_2 given in (50), we arrive at

$$|L_3| \le C\left[\|\nabla u(t)\| + \|\nabla \tilde{u}(t)\|\right] \|\nabla w(t)\| \|\nabla \tilde{u}(t)\| \|\nabla w(t)\| \le C_0 \|\nabla w(t)\|^2 \le C_0 \|\nabla w(t)\|^2_{2(p-1)},$$

$$|L_4| \le M_p \left(\|\nabla u(t)\|^{p-2}_{2(p-1)} + \left\|\nabla \tilde{u}(t)\right\|^{p-2}_{2(p-1)} \right) \|\nabla w(t)\|_{2(p-1)} \|\nabla w(t)\|_2 \le C_0 \|\nabla w(t)\|^2_{2(p-1)},$$

$$|L_5| \le k_0 \big(|\Omega|^{\frac{\rho}{2\rho+1}} + \|u(t)\|^{\rho}_{2(\rho+1)} + \|\tilde{u}(t)\|^{\rho}_{2(\rho+1)} \big) \|w(t)\|_{2(\rho+1)} \|w(t)\|_2 \le C_0 \|w(t)\|^2_{2(\rho+1)},$$

for any $\varepsilon > 0$ and some constant $C_0 > 0$, where we also use the embedding conditions (16) and (18). Going back (76) and inserting these last five estimates, after choosing $\varepsilon > 0$ sufficiently small and using the embedding $V_1 \hookrightarrow V_0$, we have that (75) holds true.

Step3. Conclusion of the proof. Now let us define the Lyapunov perturbed functional

$$F_\eta(t) = F(t) + \eta \Phi(t),$$

where $\eta > 0$ will be determined later. Combining (74) with (75), noticing $\|\Delta w(t)\|^2 \le 2F(t)$ and choosing $\eta_0 = \min\{\frac{n_B}{2C_7}, \frac{k_2}{2C_8}\}$, then there exists a constant $C_0 > 0$ such that

$$F'_\eta(t) \le -\eta F(t) + C_0 \left\|\nabla \tilde{u}_t(t)\right\|^2 F(t) + C_0 \mathcal{W}(t), \tag{77}$$

for all $t \ge 0$ and $\eta = \eta_B \in (0, \eta_0]$.

On the other hand, taking $C_{10} = \max\{2, \frac{2}{\lambda_1}\} > 0$, it is easy to conclude that

$$\left|F_\eta(t) - F(t)\right| \le \eta C_{10} F(t), \quad \forall\, t \ge 0,\ \forall\, \eta > 0. \tag{78}$$

Now choosing $\eta_1 = \min\{\frac{1}{2C_{10}}, \eta_0\}$ and selecting $\eta \le \eta_1$ in (78), we obtain

$$\frac{1}{2} F(t) \le F_\eta(t) \le \frac{3}{2} F(t), \quad \forall\, t \ge 0. \tag{79}$$

From (77) and (79), we have

$$F'_\eta(t) \le \phi_\eta(t) F_\eta(t) + C_0 \mathcal{W}(t), \quad t > 0,$$

where

$$\phi_\eta(t) = -\frac{2\eta}{3} + 2C_0 \left\|\nabla \tilde{u}_t(t)\right\|^2.$$

It follows from Gronwall's inequality that

$$F_\eta(t) \le e^{\int_0^t \phi_\eta(s)\,\mathrm{d}s} \left(F_\eta(0) + C_0 \int_0^t e^{-\int_0^s \phi_\eta(\xi)\,\mathrm{d}\xi} \mathcal{W}(s)\,\mathrm{d}s \right). \tag{80}$$

From estimate (65), we also obtain

$$\int_0^t \phi_\eta(s)\,\mathrm{d}s = -\frac{2\eta}{3} t + 2C_0 \int_0^t \left\|\nabla \tilde{u}_t(s)\right\|_2^2 \,\mathrm{d}s \le -\frac{2\eta}{3} t + \tilde{C}_0$$

for all $t \ge 0$ and some constant $\tilde{C}_0$. Thus (80) can be rewritten as

$$F_\eta(t) \le C_0 F_\eta(0) e^{-\delta t} + C_0 \int_0^t e^{-\delta(t-s)} W(s)\,\mathrm{d}s \tag{81}$$

for all $t > 0$, and some constants $C_0 > 0$, $\delta = \frac{2\eta}{3}$. Recalling that

$$2F(t) = \left\| \big(u(t), u_t(t), \eta^t\big) \right\|^2_{\mathcal{H}} = \left\| S(t)U^1 - S(t)U^2 \right\|^2_{\mathcal{H}}, \quad t \ge 0,$$

we deduce that the stability inequality holds true by renaming the constants in (81). This completes the proof Lemma 4.5. ■

Remark 4.2: By virtue of the above technical results, we are finally in position to prove Theorem 2.2 as follows. We stress that all proofs of such technical results, namely, from Lemma 4.1 to Lemma 4.5, can done by neglecting the memory term and following verbatim the same arguments, so that the conclusion below also holds true for the particular phase spaces given in Remark 2.2 when the memory is canceled. For the sake of brevity, we omit the details on computations here.

4.3. Proof of Theorem 2.2: completion

***Proof of Theorem 2.2 – item* (1):** Lemma 4.2 implies that the dynamical system $(\mathcal{H}, S(t))$ defined by (27) is dissipative. Thus, applying Theorem 7.2.3 in [35], it remains to show that it is asymptotically smooth, which can be deduced from Lemma 4.5 and Theorem 7.1.11 in [35].

In fact, from Lemma 4.5 and by the same argument as Lemma 4.3 in [2], we conclude that the dynamical system $(\mathcal{H}, S(t))$ is asymptotically smooth. We omit the details here. This completes the proof of Theorem 2.2- item (1). ∎

***Proof of Theorem 2.2 – item* (2):** Since $(\mathcal{H}, S(t))$ is a gradient dynamical system (see Lemma 4.3 (1)) and has a compact global attractor $\mathcal{A}$ (see Theorem 2.2 (i)), then our conclusion follows directly from Theorem 2.28 in [7], and also from Theorem 7.5.10 and Corollary 7.5.7 in [35]. ∎

***Proof of Theorem 2.2 – item* (3) *and* (4):** We show that under the additional assumption (28), the inequality (68) implies the condition (55) holds, which gives that the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable on any bounded positively invariant set $B \subset \mathcal{H}$.

Since the dynamical system $(\mathcal{H}, S(t))$ is defined through the solution of problem (11)–(14), then from Theorem 2.1 (1), we can deduce that (52) and (53) hold with $X = V_2$, $Y = V_0$ and $Z = \mathcal{M}_2$. Thus we need to verify the Lipschitz condition (54) and stabilizability inequality (55).

Indeed, we consider a bounded positively invariant set $B \subset \mathcal{H}$ with respect to $S(t)$ and take $U^1, U^2 \in B$. As above we denote $S(t)U^1 = (u(t), u_t(t), \eta^t)$ and $S(t)U^2 = (\tilde{u}(t), \tilde{u}_t(t), \tilde{\eta}^t)$.

Firstly, from Theorem 2.1 (3) we see that the Lipschitz condition (54) holds true with $a(t) = e^{Ct}$ locally bounded on $[0, \infty)$, for some constant $C > 0$ depending on the size of B.

Secondly, to verify (55), we consider the seminorm

$$n_X(u) = \|\nabla u\|_{2(p-1)} + \|u\|_{2(\rho+1)}.$$

It follows from the additional assumption (28) that the embeddings

$$V_2 \hookrightarrow W_0^{1,2(p-1)}(\Omega) \quad \text{and} \quad V_2 \hookrightarrow L^{2(\rho+1)}(\Omega)$$

are compact. Thus we infer that $n_X(\cdot)$ is a compact seminorm on $X = V_2$. Then, from (68) in Lemma 4.5 it follows that

$$\left\|S(t)U^1 - S(t)U^2\right\|_{\mathcal{H}}^2 \leq b(t) \left\|U^1 - U^2\right\|_{\mathcal{H}}^2 + c(t) \sup_{0<s<t} \left[n_X\left(u(s) - \tilde{u}(s)\right)\right]^2.$$

with

$$b(t) = b_B e^{-\gamma_B t} \quad \text{and} \quad c(t) = C_B \int_0^t e^{-\gamma_B(t-s)}\, \mathrm{d}s, \qquad t > 0.$$

Since B is bounded, it is easy to show that

$$b \in L^1\left(\mathbb{R}^+\right), \quad \lim_{t\to\infty} b(t) = 0 \quad \text{and} \quad c(t) \text{ is locally bounded.}$$

This implies the condition (55) also holds true. Hence, the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable on any bounded positively invariant set of $\mathcal{H}$. In particular, $(\mathcal{H}, S(t))$ is quasi-stable on the attractor

$\mathcal{A}$. Thus, applying Theorem 7.9.6 in [35], we conclude that $\mathcal{A}$ has finite fractal dimension. Moreover, the Hausdorff dimension of $\mathcal{A}$ is also finite since it is bounded by the fractal dimension (see, e.g. Section 7.3 in [35]).

Besides, since the condition (55) holds true with the function $c(t)$ possessing the property

$$c_\infty = \sup_{t\in\mathbb{R}^+} c(t) \le \frac{C_B}{\gamma_B} < \infty,$$

then the regularity properties (29) and (30) can be achieved as a consequence of Theorem 7.9.8 in [35]. ■

***Proof of Theorem 2.2 – item* (5):** As noticed above, our dynamical system $(\mathcal{H}, S(t))$ is dissipative and satisfies the conditions (52) and (53).

Now let us take $\mathcal{B} = \{U \in \mathcal{H} | \Phi(U) \le R\}$ for any given $R > 0$, where $\Phi = E$ is the strict Lyapunov functional considered in Lemma 4.3. Then for R sufficiently large, it is possible to conclude form Lemmas 4.2 and 4.3 (3) that $\mathcal{B}$ is a positively invariant bounded absorbing set. In particular, the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable on $\mathcal{B}$.

On the other hand, we consider the weak solution $U(t) = S(t)U_0 = (u(t), u_t(t), \eta^t) \in \mathcal{H}$ with the initial data $U_0 \in \mathcal{B}$. we infer from the system (11)–(14) that

$$(u_t, u_{tt}, \eta_t) \in L^2_{loc}\left(\mathbb{R}^+, \mathcal{H}_{-1}\right), \quad \mathcal{H}_{-1} := V_0 \times V_2' \times \mathcal{M}_0.$$

Since $U_0 \in \mathcal{B}$ and the positive invariance of $\mathcal{B}$, we infer that, for any $T > 0$,

$$\int_0^T \|U_t(s)\|^2_{\mathcal{H}-1}\, ds \le C^2_{\mathcal{B}T},$$

which implies that

$$\|S(t_1)\, U_0 - S(t_2)\, U_0\|_{\tilde{\mathcal{H}}_{-1}} \le \int_{t_1}^{t_2} \|U_t(s)\|_{\mathcal{H}_{-1}}\, ds \le C_{\mathcal{B}T}\, |t_1 - t_2|^{\frac{1}{2}},$$

where $C_{\mathcal{B}T}$ is a positive constant. Hence we have that for any initial data $U_0 \in \mathcal{H}$ the map $t \mapsto S(t)U_0$ is Hölder continuous in the extended phase space with exponent $\frac{1}{2}$. Hence, as a consequence of Theorem 7.9.9 in [35], we obtain that the dynamical system $(\mathcal{H}, S(t))$ possesses a generalized fractal exponential attractor whose fractal dimension is finite in the extended space $\mathcal{H}_{-1}$.

The rest conclusion can be obtained by using the interpolation theorem, which an be done by the same arguments as in [41] and [19] where two related systems are considered. ■

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Availability of data and materials

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Authors contributions

All authors contributed equally to this work.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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