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Unified stability analysis for a Volterra integro-differential equation under creation time perspective

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Abstract. Many real-world applications are modeled by Volterra integral–differential equations of the form

$$u_{tt} - \Delta u + \int_{\alpha}^{t} g(t-s)\Delta u(s)\,\mathrm{d}s = 0 \ \text{ in } \ \Omega \times (0,\infty),$$

where Ω is a bounded domain of $\mathbb{R}^N$ and g is a memory kernel. Our main concern is with the concept of so-called *creation time*, the time α where past history begins. Separately, the cases $\alpha = -\infty$ (history) and $\alpha = 0$ (null history) were extensively studied in the literature. However, as far as we know, there is no unified approach with respect to the intermediate case $-\infty < \alpha < 0$. Therefore we provide new stability results featuring (i) uniform and general stability when the creation time α varies over full range $(-\infty, 0)$ and (ii) connection between the history and the null history cases by means of a rigorous backward ($\alpha \to -\infty$) and forward ($\alpha \to 0^-$) limit analysis.

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Contents

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1. Introduction

In the present paper, inspired by a model that emerged at the beginning of the twentieth century with Volterra [37,38] and revisited more recently by Fabrizio, Giorgi and Pata [12], we shall deal with the following N-dimensional initial-boundary value problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_{\alpha}^{t} g(t-s)\Delta u(s)\,\mathrm{d}s = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (\alpha,\infty), \\ u(x,t) = u_0(x,t),\ (x,t) \in \Omega \times (\alpha, 0],\ u_t(x,0) = u_1(x),\ x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and g is a memory kernel. The parameter $\alpha \leq 0$ is called *creation time*, see, e.g., [12, Section 2], which means that the past history is assumed to vanish before the time α. The initial data $u_0 : \Omega \times (\alpha, 0] \to \mathbb{R}$ are the prescribed *finite past history* of the unknown displacement $u = u(x,t)$ and $u_1 = \partial_t u_0\big|_{t=0}$.

We note that the Volterra integro-differential equation in (1.1) can be obtained by means of the Volterra stress–strain constitutive relation

$$\sigma(x,t) = \mathbb{G}_0(x)\varepsilon(x,t) + \int_{\alpha}^{t} \mathbb{G}'(x,t-s)\varepsilon(x,s)\,\mathrm{d}s, \quad \alpha \leq 0, \tag{1.2}$$

where $\mathbb{G} = \mathbb{G}(\cdot, s)$, $s > 0$, stands for a symmetric tensor, commonly called nowadays by *Boltzmann function*, $\mathbb{G}_0(\cdot) = \lim_{s\to 0} \mathbb{G}(\cdot, s)$, and $\mathbb{G}'(\cdot, s)$ is the *relaxation function* given by the derivative of $\mathbb{G}(\cdot, s)$ with respect to s, see, e.g., Boltzmann [4,5]. We refer again to [12, Section 2] for more details on the derivation of viscoelastic constitutive law (1.2) as well as the physical modeling of problem (1.1).

In the history and null history scenarios, namely, formally taking the limit cases $\alpha = -\infty$ and $\alpha = 0$, particular model (1.1) has been deeply studied in literature. Indeed, we refer to [3,8,10,11,13,15–18,20,30,33,34,39] for problems involving the *backward memory limit* $\alpha = -\infty$ and [1,2,6,7,9,21,22,24,27,28,32,36,39] for models where the *forward memory limit* $\alpha = 0$ is invoked. We also quote the books [14,23,35] where a wide class of viscoelastic problems is addressed in both cases.

On the other hand, when one considers intermediate creation time $\alpha \in (-\infty, 0)$, as before, we highlight that the past history must vanish before α, giving the idea of "beginning" time, that is, the principle (alpha) of the history.

Therefore, under the above statements, our main goal is to analyze intermediate problem (1.1) with respect to the creation time $-\infty < \alpha < 0$. Our main results are concerned with existence and uniqueness of solution, general stability and limit analysis when $\alpha \to 0^-$ and $\alpha \to -\infty$. In what follows, we are going to clarify the main novelties and contributions of such results.

1.1. The intermediate stability results

In the intermediate instance $\alpha \in [-\infty, 0]$, for a given solution u of problem (1.1), we define the energy functional $E_\alpha(t) = E_\alpha(u(t), u_t(t))$, $t \geq 0$, as

$$E_\alpha(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - h_\alpha(t)\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \overset{\alpha}{\square} \nabla u)(t), \tag{1.3}$$

where, for simplicity, we have used the notation (see also Sect. 3),

$$h_\alpha(t) := \int_0^{t-\alpha} g(s)\,\mathrm{d}s, \quad (g \overset{\alpha}{\square} \nabla u)(t) = \int_0^{t-\alpha} g(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2\,\mathrm{d}s, \quad t \geq 0. \tag{1.4}$$

Our first main result (Theorem 3.1) features the exponential and general stabilities with respect to α-energy defined in (1.3). It is briefly stated as it follows.

Exponential behavior. *For any $t_0 > 0$, the energy functional $E_\alpha(t)$ satisfies the exponential-type stability*

$$E_\alpha(t) \le C_\alpha(t_0) E_\alpha(0)\, e^{-\gamma_\alpha(t_0)t}, \quad t > t_0 \tag{1.5}$$

for some structural positive constants $C_\alpha(t_0)$ and $\gamma_\alpha(t_0)$ (explicitly given in (3.5)—Sect. 3), provided that the memory kernel satisfies

$$g'(t) \le -\xi_0\, g(t), \quad t > 0, \tag{1.6}$$

for some constant $\xi_0 > 0$.

As a matter of fact, from computations point of view, exponential stability (1.5) will be a consequence of a more general stability result, which is obtained under a more generic memory kernel (in a certain way) than (1.6). More precisely, we have:

A general behavior. *Assume that*

$$K_\alpha := \sup_{\tau \in (\alpha,0)} \|\nabla u_0(\tau)\|_2^2 < \infty. \tag{1.7}$$

For any $t_0 > 0$, the energy satisfies the following decay rate

$$\begin{aligned} E_\alpha(t) \le \tilde{C}_\alpha(t_0) &\left(E_\alpha(0) + \int_0^t e^{\tilde{\gamma}_\alpha(t_0)\int_0^s \xi(\tau)\mathrm{d}\tau} [g(s) - g(s-\alpha)]\,\mathrm{d}s \right) e^{-\tilde{\gamma}_\alpha(t_0)\int_0^t \xi(s)\mathrm{d}s} \\ &+ \tilde{C}_\alpha(t_0) \int_t^{t-\alpha} g(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{1.8}$$

for all $t > t_0$, for some positive constants $\tilde{C}_\alpha(t_0) = \tilde{C}_\alpha(t_0, E_\alpha(0), K_\alpha)$ and $\tilde{\gamma}_\alpha(t_0)$ (explicitly given in (3.9)–Sect. 3), provided that, for some suitable function $\xi : [0,\infty) \to \mathbb{R}^+$, the memory kernel satisfies

$$g'(t) \le -\xi(t)\, g(t), \quad t > 0. \tag{1.9}$$

Let us give some comments on general decay (1.8) as well as on assumptions (1.7)–(1.9) at this moment. At first glance, new and explicit decay rate (1.8) may seem mildly strange, but it will be a natural extension that intermediates, with respect to the creation time α, the energy decay rates related to the history ($\alpha = -\infty$) and the null history ($\alpha = 0$) problems, both under assumption (1.9). This statement will be clearer later, after recalling the stability results to the limit problems. Before doing so, we first remark that hypothesis (1.9) generalizes, in some direction, the exponential memory kernel and yields explicit decay rates for (1.1) in the already known situations $\alpha = 0$ and $\alpha = -\infty$ according to [20,27] and still in the present intermediate case $-\infty < \alpha < 0$ as well. We also mention that there are some other works dealing with more general kernels, see, e.g., [1,2,8,9,19,24,25,29,31–34,36]. For these latter, the issue falls on the difficult in obtaining explicit stabilities for all cases with respect to the parameter α, which justifies our choice of assumption (1.9) for the memory kernel in this pioneering article handling the intermediate case. The drawback of (1.8) is that it is a non-uniform stability (nor optimal) due to the constant C_α depending on boundedness condition (1.7) and the "tail" given by $\int_t^{t-\alpha} g(s)\,\mathrm{d}s$. Nevertheless, (1.5) represents a uniform (exponential) stability that does not depend on (1.7).

1.2. Regarding the history and null history problems

Let us start by formally considering the forward case $\alpha = 0$. In this context, the energy functional $E_0(t) = E_0(u(t), u_t(t)),\ t \geq 0$, corresponding to problem $(1.1)_{\alpha=0}$, is given by

$$E_0(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g\Box\nabla u)(t), \tag{1.10}$$

where the notation $(g\Box\nabla u)$ is obtained from (1.4) in the formal limit $\alpha = 0$. In this occasion, the existing stability results are given as follows.

Exponential stability. It is well-known since near the last decade of the previous century, see, e.g., the classical references [14,23,35], that: *there exist constants* $C_0, \gamma > 0$ *such that*

$$E_0(t) \leq C_0\, E_0(0)\, e^{-\gamma t}, \quad \forall\, t > 0, \tag{1.11}$$

since assumption (1.6) *is regarded.*

More recently and generally somehow, we also have:

General Stability. As stated in [27,28], both works appearing in the year 2008, one has: *there exist constants* $C_0, \gamma > 0$ *such that*

$$E_0(t) \leq C_0\, E_0(0) e^{-\gamma \int_0^t \xi(s)\,\mathrm{d}s}, \quad \forall\, t > 0, \tag{1.12}$$

once hypothesis (1.9) *is taken into account.*

In the present paper, under assumption (1.9), we still prove that general stability (1.12) can be concluded by combining energy perturbation with a previous result proposed by Martinez in 1999 for dissipative systems, see, e.g., [26, Lemma 1].

On the other hand, in the formal backward limit $\alpha = -\infty$, problem $(1.1)_{\alpha=-\infty}$ is equivalent to a (viscoelastic) initial-boundary value problem involving the following equations

$$\begin{cases} u_{tt} - \left(1 - \displaystyle\int_0^\infty g(s)\,\mathrm{d}s\right)\Delta u - \displaystyle\int_0^\infty g(s)\Delta\,\eta^t(s)\,\mathrm{d}s = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ \eta_t^t + \eta_s^t = u_t \quad \text{in} \quad \Omega \times (0,\infty) \times (0,\infty), \end{cases} \tag{1.13}$$

where the variable η, originally introduced by Dafermos [11], is called *relative displacement history*. It is defined as

$$\eta^t(\cdot, s) := u(\cdot, t) - u(\cdot, t - s), \quad t \geq 0,\, s > 0. \tag{1.14}$$

We refer to Grasseli and Pata [18] (see Sections 3 and 4 therein) for a detailed study on the one-to-one correspondence between problems $(1.1)_{\alpha=-\infty}$ and (1.13) with its corresponding initial-boundary conditions.

In this case, the energy functional associated with (1.13) is defined as

$$E_\infty(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^\infty g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}\int_0^\infty g(s)\,\|\nabla\eta^t(s)\|_2^2\,\mathrm{d}s,$$

and the *exponential stability* result in the history case reads exactly like in (1.11), replacing $E_0(t)$ by $E_\infty(t)$. See also [13,15,30] among others. A more recent and explicit somehow *general stability* for this case is proved in [20] and reads as: *for some* $t_0 > 0$, *there exist constants* $\delta_0 \in (0,1)$ *and* $\tilde{C} > 0$ *depending on initial data such that*

$$E_\infty(t) \leq \tilde{C}\left(E_\infty(0) + \int_0^t g^{1-\delta_0}(s)\,\mathrm{d}s\right) e^{-\delta_0 \int_0^t \xi(s)\,\mathrm{d}s} + \tilde{C}\int_t^\infty g(s)\,\mathrm{d}s, \tag{1.15}$$

for all $t > t_0$, *provided that* (1.7) *holds for* $\alpha = -\infty$ *and* g *satisfies* (1.9).

As stated above, stability results (1.12) and (1.15) are already known in the literature for their respective cases. In addition, it is worth pointing out that both decay rates can be reached by formally proceeding the limits $\alpha \to 0^-$ and $\alpha \to -\infty$ in (1.8), respectively. It means that (1.8) predicts a link between the two already known results. Actually, in the present work, both stabilities (1.12) and (1.15) will be achieved as a consequence of a rigorous (forward and backward) limit analysis on new intermediate general decay (1.8) (resp. (1.5) in the exponential case), not merely replacing $\alpha = 0$ and $\alpha = -\infty$ therein roughly. This is the main topic of Sect. 4, as clarified below.

1.3. Organization of the paper and main contributions

Under the previous statements, we are now in position to highlight our main contributions.

#1. In Sect. 2, we present the existence and uniqueness of solution for (1.1) in the intermediate case $\alpha \in (-\infty, 0)$, by providing an understanding of how to work with null extension of initial data before the creation time α. Thus, we extend the methodology employed in [9, Section 4], from the limit case $\alpha = 0$ to the intermediate scenario.

#2. In Sect. 3, we state and prove Theorem 3.1. As far as we know, it is the first result that delivers explicit exponential and general decay rates for the energy corresponding to intermediate problem (1.1). To do so, we employ a key regularizing procedure as introduced by Guo et al. [21]. Therefore, we first work with regular solutions and then achieve the desired stability result for the weak solution through Proposition 3.3.

#3. In Sect. 4, by means of Theorems 4.3 and 4.6, we rigorously analyze the forward ($\alpha \to 0^-$) and backward ($\alpha \to -\infty$) limits, for every fixed time $t > 0$, of the intermediate energy $E_\alpha(t)$, by proving that it converges to $E_0(t)$ and $E_\infty(t)$ as $\alpha \to 0^-$ and $\alpha \to -\infty$, respectively. Summarizing, for any $t > 0$ given, we have the following convergence diagram:

$$\boxed{E_\infty(t) \quad \overset{\alpha\to-\infty}{\longleftarrow} \quad E_\alpha(t) \quad \overset{\alpha\to 0^-}{\longrightarrow} \quad E_0(t)}$$

As a consequence, known key results (1.12) and (1.15) can be obtained as limit cases from our unified analysis. See for instance Corollaries 4.4 and 4.7. Hence, we finally conclude that, via Theorems 3.1, 4.3 and 4.6, intermediate general stability (1.8) provides a "bridge" linking the history and null history stability cases (1.15) and (1.12), as the time creation α varies the range $(-\infty, 0)$ from backward to forward. In particular, all statements also hold true in the exponential setting.

All the precise assumptions and the (above) statements will be carefully formalized in Sects. 2, 3 and 4 in the remaining paper.

2. Existence and uniqueness

Let us start by considering the initial-boundary value problem for the intermediate system with α notation to the creation time

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_\alpha^t g(t-s)\Delta u(s)\,\mathrm{d}s = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (\alpha,\infty), \\ u(x,t) = u_0(x,t),\ (x,t) \in \Omega \times (\alpha, 0],\ u_t(x,0) = u_1(x),\ x \in \Omega. \end{cases} \tag{2.1}$$

The well-posedness for (2.1) in the limit situations $\alpha = 0$ and $\alpha = -\infty$ is very well-known, as stated in introduction. In this section, the main goal is to clarify that (2.1) is also well posed for any $-\infty < \alpha < 0$

as well. Indeed, although problem (2.1) does not generate a linear semigroup for $-\infty < \alpha \leq 0$, it is possible to use the history space setting to explore its solution by considering the null extension of initial data, namely, by setting the null history $u_0(\cdot, t) = 0$ for all $t \in (-\infty, \alpha]$. Such a methodology is already known for the limit case $\alpha = 0$ as presented in [9, Section 4]. In what follows, we are going to prove that a complementary procedure can be done for all $-\infty < \alpha < 0$.

2.1. The history setting

Let us set $u(\cdot, t) = u_0(\cdot, t) = 0$ for $-\infty < t < \alpha$. Then, problem (2.1) can be rewritten with α replaced by $-\infty$, and considering the auxiliary history variable η as defined in (1.14), we can convert problem (2.1) into the next one:

$$u_{tt} - \ell \Delta u - \int_0^\infty g(s) \Delta \eta^t(s) \, \mathrm{d}s = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{2.2}$$

$$\eta_t^t + \eta_s^t = u_t \quad \text{in} \quad \Omega \times (0, \infty) \times (0, \infty), \tag{2.3}$$

with boundary conditions

$$u = 0 \ \text{ on } \ \partial\Omega \times [0, \infty), \quad \eta = 0 \ \text{ on } \ \partial\Omega \times [0, \infty) \times (0, \infty), \tag{2.4}$$

and initial data

$$u(x, 0) = \tilde{u}_0(x), \quad u_t(x, 0) = u_1(x), \quad \eta^0(x, s) = \eta_0(x, s), \ x \in \Omega, \ s > 0, \tag{2.5}$$

where we have denoted

$$\begin{cases} \ell := 1 - \displaystyle\int_0^\infty g(s) \, \mathrm{d}s, \\ \tilde{u}_0 \text{ is the null extension of } u_0 \text{ in } (-\infty, \alpha), \\ \eta_0(\cdot, s) := \begin{cases} u_0(\cdot), & s > -\alpha, \\ u_0(\cdot) - u_0(\cdot, -s), & 0 < s < -\alpha, \end{cases} \\ \eta^t(x, 0) := \displaystyle\lim_{s \to 0^+} \eta^t(x, s) = 0, \quad (x, t) \in \Omega \times [0, \infty). \end{cases} \tag{2.6}$$

Now, in order to solve problem (2.2)–(2.5), we consider the basic assumption on the memory kernel.

Assumption 2.1. Let $g \in L^1(\mathbb{R}^+) \cap C^1(\mathbb{R}^+)$ be a non-increasing positive function such that

$$g(0) > 0, \quad \ell = 1 - \int_0^\infty g(s) \, \mathrm{d}s > 0. \tag{2.7}$$

We define the standard Hilbert spaces

$$(L^2(\Omega), \| \cdot \|_2, (\cdot, \cdot)) \quad \text{and} \quad (H_0^1(\Omega), \|\nabla \cdot \|_2, (\nabla \cdot, \nabla \cdot)),$$

the usual weighted L_g^2-space

$$L_g^2(\mathbb{R}^+; H_0^1(\Omega)) = \left\{ \eta : (0, \infty) \to H_0^1(\Omega); \int_0^\infty g(s) \|\nabla \eta(s)\|_2^2 \, \mathrm{d}s < \infty \right\} := \mathcal{W}_g,$$

with inner product and norm

$$(\eta, \xi)_{\mathcal{W}_g} = \int_0^\infty g(s) (\nabla \eta(s), \nabla \xi(s)) \, \mathrm{d}s \quad \text{and} \quad \|\eta\|_{\mathcal{W}_g}^2 = (\eta, \eta)_{\mathcal{W}_g}, \quad \forall \, \eta, \xi \in \mathcal{W}_g,$$

and the Hilbert phase space

$$\mathcal{H} := H_0^1(\Omega) \times L^2(\Omega) \times \mathcal{W}_g, \quad \|(u,v,\eta)\|_{\mathcal{H}}^2 = \ell\|\nabla u\|_2^2 + \|v\|_2^2 + \|\eta\|_{\mathcal{W}_g}^2,$$

for all $(u,v,\eta) \in \mathcal{H}$. We also denote by λ_1 the constant associated to the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, that is,

$$\lambda_1\|u\|_2 \leq \|\nabla u\|_2, \quad u \in H_0^1(\Omega).$$

Under the above notations and setting $U = (u,v,\eta)$ with $v = u_t$, history system (2.2)–(2.5) is equivalent to the following abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}U = \mathcal{A}U, \quad t > 0, \\ U(0) = (\tilde{u}_0, u_1, \eta_0) := U_0, \end{cases} \tag{2.8}$$

where the linear differential operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined as

$$\mathcal{A}(u,v,\eta) := \left(v, \ell\Delta u + \int_0^\infty g(s)\Delta\,\eta(s)\,\mathrm{d}s, -\eta_s + v \right),$$

with domain

$$D(\mathcal{A}) = \left\{ (u,v,\eta) \in \mathcal{H};\ v \in H_0^1(\Omega), \ell u + \int_0^\infty g(s)\eta(s)\,\mathrm{d}s \in H^2(\Omega), \eta_s \in \mathcal{W}_g,\ \eta(\cdot,0) = 0 \right\}.$$

Theorem 2.1. [16, Thm. 2.2] *Under Assumption 2.1, if $U_0 \in \mathcal{H}$, then there exists a unique mild solution $U = (u,v,\eta) \in C([0,\infty),\mathcal{H})$ for (2.8) given by $U(t) = e^{\mathcal{A}t}U_0$. Moreover, if $U_0 \in D(\mathcal{A})$, the mild solution is the regular one $U \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty),\mathcal{H})$.*

In other words, problem (2.8) is Hadamard well-posed with respect to mild and strong solutions and, consequently, system (2.2)–(2.5) so is it.

2.2. Back to the original problem

For $U_0 = (\tilde{u}_0, u_1, \eta_0) \in \mathcal{H}$, we know that $U = (u,v,\eta)$ is a mild solution of (2.8) if and only if (u,v) satisfies the variational problem related to (2.2)–(2.5), see for instance [18, Sect. 4]. In particular, if we take initial data in the following subspace of $\mathcal{H}$

$$(\tilde{u}_0, u_1, \eta_0) \in \{(u,v,\eta) \in \mathcal{H},\ \eta(s-\alpha) = u,\ s > 0\} := \mathcal{V},$$

then we have a one-to-one correspondence between problems (2.8) and (2.1) and, therefore, the first component u of the vector solution (u,v,η) solves problem (2.1). Indeed, through the notations in (2.6) and since η can be explicitly expressed by the formula

$$\eta^t(s) = \begin{cases} u(t) - u(t-s), & t > s, \\ u(t) - \tilde{u}_0(t-s), & t \leq s, \end{cases} \tag{2.9}$$

see again [18, Sect. 4], then

$$\tilde{u}_0 = \eta_0(s-\alpha) = \tilde{u}_0 - \tilde{u}_0(\alpha - s), \quad \forall\, s > 0.$$

From this we have $\tilde{u}_0 = 0$ in $(-\infty, \alpha)$. Additionally, from (2.2)–(2.5) along with (2.6) and (2.9), it is not so difficult to prove that u is a variational solution of problem (2.1).

Besides, we still observe that the third condition in (2.6) allows us to conclude that $u_0(\cdot,s)$, $\alpha<s<0$, belongs to the space

$$\mathcal{W}_g(\alpha,0):=\left\{u:(\alpha,0)\to H_0^1(\Omega);\ \int_0^{-\alpha} g(s)\|\nabla u(-s)\|_2^2\,\mathrm{d}s<\infty\right\},$$

whenever it is taken $\tilde{u}_0=u_0\in H_0^1(\Omega)$ and $\eta_0\in\mathcal{W}_g=L_g^2(\mathbb{R}^+;H_0^1(\Omega))$. Consequently, using again the explicit formula for η in (2.9), then u belongs to the space

$$\mathcal{W}_g(\alpha,T):=\left\{u:(\alpha,T)\to H_0^1(\Omega);\ \int_0^{T-\alpha} g(s)\|\nabla u(T-s)\|_2^2\,\mathrm{d}s<\infty\right\},$$

for any time $T>0$. Hence, we conclude that problem (2.1) has a unique solution (in the weak variational sense) in the class

$$(u,u_t)\in C([0,\infty);H_0^1(\Omega)\times L^2(\Omega)),\ \text{ with }\ u\in\mathcal{W}_g(\alpha,T),\quad \forall\, T>0. \tag{2.10}$$

Finally, in what concerns regular solutions for (2.1), we notice that the above procedure cannot be done for any general regular initial data. Indeed, under the requirements in the definitions of the subspace $\mathcal{V}$ and the domain $D(\mathcal{A})$, the one-to-one correspondence between problems (2.1) and (2.8) is only possible if we consider the null initial position $\tilde{u}_0=0$, which leads to a particular case of regular solutions to problem (2.1) related to suitable initial data.

Remark 2.2. Under the above statements, one concludes that problem (2.1) is only well-posed with respect to variational (weak) solutions in the general setting of initial data. However, regularity of solution is a very useful attribute to deal with the stability results, which are usually achieved for weak solutions by density arguments. Thus, some regularizing process is necessary to work with a general regular solution of (2.1). For instance, one can define a special type of cutoff function as done in [9] (see on pages 33-34 therein) and then consider initial data in a proper subspace. Also, a regularizing procedure as presented by Guo et al. [21] can be used to approach the stability of variational solutions of (2.1).

3. Stability result

Let us start by recalling that the energy $E_\alpha(t)$ associated with problem (2.1) is given by

$$E_\alpha(t)=\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\left(1-h_\alpha(t)\right)\|\nabla u(t)\|_2^2+\frac{1}{2}(g\overset{\alpha}{\square}\nabla u)(t),\quad t\geq 0, \tag{3.1}$$

for any $\alpha\in(-\infty,0)$. In order to prove a general stability result for $E_\alpha(t)$, let us consider an additional assumption on the memory kernel.

Assumption 3.1. The function g fulfills the following linear differential inequality

$$g'(t)\leq -\xi(t)g(t),\quad t>0, \tag{3.2}$$

for some function $\xi:[0,\infty)\to\mathbb{R}^+$ satisfying either:

(**i**) ξ is a constant function, namely, $\xi\equiv\xi_0>0$;
(**ii**) ξ is a non-constant, non-increasing and differentiable function.

The first main stability result of this paper reads as follows:

Theorem 3.1. *Let us take on Assumptions 2.1 and 3.1. For any $\alpha\in(-\infty,0)$ we have:*

I. Uniform (Exponential) Stability. *Let*

$$c_\alpha(t) := \frac{\lambda_1^2}{8g(0)\ell}\left[8\ell h_\alpha(t) + 48 + 3[h_\alpha(t)]^2\right], \tag{3.3}$$

*where $h_\alpha(t)$ is set in (1.4). If we assume condition (3.2)-(**i**), then for a fixed $t_0 > 0$ the energy functional $E_\alpha(t)$ decays exponentially*

$$E_\alpha(t) \le C_\alpha(t_0)E_\alpha(0)\, e^{-\gamma_\alpha(t_0)t}, \quad t > t_0, \tag{3.4}$$

where

$$\begin{cases} C_\alpha(t_0) := 1 + \dfrac{\xi_0}{2c_\alpha(t_0)}\left(\dfrac{1}{2} + \dfrac{\lambda_1}{\ell g(0)}\right), \\ \gamma_\alpha(t_0) := \dfrac{\xi_0\lambda_1^2\ell}{4\ell g(0)c_\alpha(t_0) + \xi_0(\ell g(0) + 2\lambda_1)}[h_\alpha(t_0)]^2. \end{cases} \tag{3.5}$$

II. A Non-uniform General Stability. *Let us assume condition (3.2)-(**ii**) and suppose that*

$$K_\alpha := \sup_{\tau\in(\alpha,0)} \|\nabla u_0(\tau)\|_2^2 < \infty. \tag{3.6}$$

Let

$$\kappa_\alpha(t_0) := c_\alpha(t_0)\left(\frac{4}{\ell}E_\alpha(0) + 2K_\alpha\right). \tag{3.7}$$

Then, for a fixed $t_0 > 0$, the energy functional $E_\alpha(t)$ decays as

$$\begin{aligned} E_\alpha(t) \le \tilde{C}_\alpha(t_0)&\left(E_\alpha(0) + \int_0^t e^{\tilde{\gamma}_\alpha(t_0)\int_0^s \xi(\tau)\mathrm{d}\tau}[g(s) - g(s-\alpha)]\,\mathrm{d}s\right) e^{-\tilde{\gamma}_\alpha(t_0)\int_0^t \xi(s)\mathrm{d}s} \\ &+ \tilde{C}_\alpha(t_0)\int_t^{t-\alpha} g(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{3.8}$$

for all $t > t_0$, where

$$\begin{cases} \tilde{C}_\alpha(t_0) := \max\left\{\dfrac{\kappa_\alpha(t_0)}{\tilde{\gamma}_\alpha(t_0)}, 2c_\alpha(t_0) + \xi(0)\left(\dfrac{1}{2} + \dfrac{\lambda_1}{\ell g(0)}\right)\right\}, \\ \tilde{\gamma}_\alpha(t_0) := \dfrac{\xi(0)\lambda_1^2\ell}{4\ell g(0)c_\alpha(t_0) + \xi(0)(\ell g(0) + 2\lambda_1)}[h_\alpha(t_0)]^2. \end{cases} \tag{3.9}$$

Before proving Theorem 3.1, let us first introduce some notations and technical results.

3.1. Functional notations

In order to provide a pattern functional notation to the subsequently technical results, we introduce some notations and a useful result on the convolution setting.

For any $\alpha < 0$ and $t \ge 0$, we define:

$$(g \overset{\alpha}{*} z)(t) := \int_\alpha^t g(t-s)z(s)\,\mathrm{d}s = \int_0^{t-\alpha} g(s)z(t-s)\,\mathrm{d}s, \tag{3.10}$$

$$(g \overset{\alpha}{\diamond} z)(t) := \int_\alpha^t g(t-s)(z(t)-z(s))\,\mathrm{d}s = \int_0^{t-\alpha} g(s)(z(t)-z(t-s))\,\mathrm{d}s, \tag{3.11}$$

$$(g \overset{\alpha}{\square} z)(t) := \int_{\alpha}^{t} g(t-s)\|z(t)-z(s)\|_2^2 \, \mathrm{d}s = \int_{0}^{t-\alpha} g(s)\|z(t)-z(t-s)\|_2^2 \, \mathrm{d}s. \tag{3.12}$$

Under the above notations, we have:

Lemma 3.2. *Let us take* $z \in \left\{ u : (\alpha,t) \to L^2(\Omega); \; \int_0^{t-\alpha} g(s)\|u(t-s)\|_2^2 \, \mathrm{d}s < \infty \right\}$.

(a) *If* $g \in L^1(\mathbb{R}^+)$, *then* $(g \overset{\alpha}{\diamond} z)(t) \in L^2(\Omega)$ *and*

$$\|(g \overset{\alpha}{\diamond} z)(t)\|_2^2 \leq \|g\|_{L^1(\mathbb{R}^+)} (g \overset{\alpha}{\square} z)(t), \quad t \geq 0. \tag{3.13}$$

(b) *If* $g \in C^1(\mathbb{R}^+)$ *and* $z_t(t) \in L^2(\Omega)$, $t \geq 0$, *then*

$$\begin{aligned} ((g \overset{\alpha}{*} z)(t), z_t(t)) = & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{ h_\alpha(t)\|z(t)\|_2^2 \, \mathrm{d}s - (g \overset{\alpha}{\square} z)(t) \right\} \\ & + \frac{1}{2} g(t-\alpha)\|z(t)\|_2^2 - \frac{1}{2}(g' \overset{\alpha}{\square} z)(t). \end{aligned} \tag{3.14}$$

Proof. The proofs of both items are standard. For the sake of the readers we give the main idea to reach them. Indeed, using (3.11), (3.12) and Hölder's inequality, we obtain

$$\begin{aligned} \|(g \overset{\alpha}{\diamond} z)(t)\|_2^2 & \leq \left(\int_0^{t-\alpha} (g(s))^{\frac{1}{2}} (g(s)\|z(t)-z(t-s)\|_2^2)^{\frac{1}{2}} \, \mathrm{d}s \right)^2 \\ & \leq h_\alpha(t)(g \overset{\alpha}{\square} z)(t), \end{aligned}$$

from where (a) follows. To arrive at (b), we take the time distributional derivative in the first expression of $(g \overset{\alpha}{\square} z)(t)$ given in (3.12) and then use (3.10). □

3.2. Auxiliary technical results

Due to the weak regularity of the solution (u, u_t) in (2.10), we cannot apply multipliers directly in (2.1). Thus, in the next computations, we are going to work initially with a smooth sequence defined through the resolvent operator as regarded in [21, Sect. 3] for a uniqueness purpose. Here, this procedure will lead to the proof of (3.8) in Theorem 3.1 for such a regularizing sequence and, consequently, to the solution of (2.1) by means of the important limit properties provided by [21, Prop. 5.1].

Let (u, u_t) be the solution of (2.1) given by (2.10), and let us consider the regularizing resolvent operator $R_n := (I - n^{-1}\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$, $n \in \mathbb{N}$. We also set the smooth sequence $u^n := R_n u$ and the corresponding energy functional

$$E_\alpha^n(t) \; = \; \frac{1}{2}\|u_t^n(t)\|_2^2 + \frac{1}{2}\left(1 - h_\alpha(t)\right)\|\nabla u^n(t)\|_2^2 + \frac{1}{2}(g \overset{\alpha}{\square} \nabla u^n)(t), \quad t \geq 0, \tag{3.15}$$

where we have used notation (3.12). In this direction, the following result holds true.

Proposition 3.3. *Under the above notations, the following statements hold.*

- *If* $u \in L^2(\Omega)$, *then* $\|u^n\|_2 \leq \|u\|_2$, *and* $u^n \to u$ *in* $L^2(\Omega)$, *as* $n \to \infty$.
- *If* $u \in H_0^1(\Omega)$, *then* $\|u^n\|_{H_0^1} \leq \|u\|_{H_0^1}$, *and* $u^n \to u$ *in* $H_0^1(\Omega)$, *as* $n \to \infty$.

Proof. Both properties are direct consequences of the definition $u^n = R_n u$ and Proposition 5.1 in [21] with $\epsilon = 1/n$. □

Moreover, we have the following energy identity.

Lemma 3.4. *Under the above notations and Assumption 2.1, then the regularized energy $E_\alpha^n(t)$ satisfies the following identity:*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha^n(t) = -\frac{1}{2}\,g(t-\alpha)\|\nabla u^n(t)\|_2^2 + \frac{1}{2}(g' \overset{\alpha}{\square} \nabla u^n)(t), \quad t > 0. \tag{3.16}$$

Proof. Firstly, we apply the regularizing resolvent operator R_n on every term of problem (2.1). Then, taking the multiplier u_t^n and using identity (3.14), energy identity (3.16) follows readily. □

Let $t_0 > 0$ be fixed. To the next lemma we also set the functionals for $t > t_0$

$$\Phi_1^n(t) := (u_t^n(t), u^n(t)), \tag{3.17}$$

$$\Phi_2^n(t) := -(u_t^n(t), (g \overset{\alpha}{\diamond} u^n)(t)), \tag{3.18}$$

and the perturbed energy

$$F_\alpha^n(t) := \frac{1}{2}\left[1 + \frac{\lambda_1}{2\ell g(0)}\right] E_\alpha^n(t) + \frac{\lambda_1^2}{4g(0)}[h_\alpha(t_0)]^2\,\Phi_1^n(t) + \frac{\lambda_1^2}{g(0)}h_\alpha(t_0)\,\Phi_2^n(t). \tag{3.19}$$

At a first moment, we observe that it is easy to verify the equivalence

$$\frac{1}{2}E_\alpha^n(t) \le F_\alpha^n(t) \le \left[\frac{1}{2} + \frac{\lambda_1}{\ell g(0)}\right] E_\alpha^n(t), \quad t \ge t_0. \tag{3.20}$$

Lemma 3.5. *Under the above notations and Assumption 2.1, then the regularized perturbed energy $F_\alpha^n(t)$ satisfies, for a fixed $t_0 > 0$, the following estimate:*

$$\frac{\mathrm{d}}{\mathrm{d}t}F_\alpha^n(t) \le -\frac{\lambda_1^2}{4g(0)}[h_\alpha(t_0)]^2 E_\alpha^n(t) + c_\alpha(t_0)\,(g \overset{\alpha}{\square} \nabla u^n)(t), \quad t > t_0, \tag{3.21}$$

where $h_\alpha(t_0)$ and $c_\alpha(t_0)$ are given in (1.4) and (3.3), respectively.

Proof. First, taking the time derivative of Φ_1^n defined in (3.17) and using (2.1) under the action of R_n, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1^n(t) = \|u_t^n(t)\|_2^2 - (1 - h_\alpha(t))\,\|\nabla u^n(t)\|_2^2 - ((g \overset{\alpha}{\diamond} \nabla u^n)(t), \nabla u^n(t)). \tag{3.22}$$

From (2.7), applying Young's inequality and using (3.13), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1^n(t) \le -E_\alpha^n(t) + \frac{3}{2}\|u_t^n(t)\|_2^2 - \frac{\ell}{4}\|\nabla u^n(t)\|_2^2 + \frac{3}{2\,\ell}(g \overset{\alpha}{\square} \nabla u^n)(t). \tag{3.23}$$

Now, deriving the functional Φ_2^n defined in (3.18) and using again (2.1), after acting R_n on it, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_2^n(t) = -h_\alpha(t)\|u_t^n(t)\|_2^2 + \sum_{j=1}^{3} J_i(t), \tag{3.24}$$

where

$$J_1(t) = (1 - h_\alpha(t))\,(\nabla u^n(t), (g \overset{\alpha}{\diamond} \nabla u^n)(t)),$$

$$J_2(t) = \|(g \overset{\alpha}{\diamond} \nabla u^n)(t)\|_2^2,$$

$$J_3(t) = -(u_t^n(t), (g' \overset{\alpha}{\diamond} u^n)(t)).$$

Fixing $t_0 > 0$, we apply Cauchy-Schwarz and Young's inequalities and use (3.13) to have

$$|J_1(t)| \le \frac{\ell}{24}h_\alpha(t_0)\,\|\nabla u^n(t)\|_2^2 + \frac{6}{\ell}[h_\alpha(t_0)]^{-1}(g \overset{\alpha}{\square} \nabla u^n)(t),$$

$$|J_2(t)| \le (g \overset{\alpha}{\square} \nabla u^n)(t).$$

Moreover, from Cauchy-Schwarz, Young and Poincaré's inequalities, we infer

$$|J_3(t)| \leq \frac{1}{2}h_\alpha(t_0)\|u_t^n(t)\|_2^2 - \frac{g(0)}{2\,\lambda_1^2}[h_\alpha(t_0)]^{-1}(g' \overset{\alpha}{\Box} \nabla u^n)(t).$$

Replacing the estimates for $J_i(t)$, $i = 1, 2, 3$, in (3.24), we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_2^n(t) \leq &-\frac{h_\alpha(t_0)}{2}\|u_t^n(t)\|_2^2 + \frac{\ell}{24}h_\alpha(t_0)\|\nabla u^n(t)\|_2^2\\ &+\left[1+\frac{6}{\ell}[h_\alpha(t_0)]^{-1}\right](g \overset{\alpha}{\Box} \nabla u^n)(t)\\ &-\frac{g(0)}{2\,\lambda_1^2}[h_\alpha(t_0)]^{-1}(g' \overset{\alpha}{\Box} \nabla u^n)(t), \quad \forall\, t \geq t_0.\end{aligned} \tag{3.25}$$

Finally, deriving F_α^n defined in (3.19) and using (3.16), (3.23) and (3.25), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F_\alpha^n(t) \leq &-\frac{\lambda_1^2}{4g(0)}[h_\alpha(t_0)]^2E_\alpha^n(t) - \frac{\ell\lambda_1^2}{48g(0)}[h_\alpha(t_0)]^2\|\nabla u^n(t)\|_2^2\\ &-\frac{\lambda_1^2}{8g(0)}[h_\alpha(t_0)]^2\|u_t^n(t)\|_2^2 + c_\alpha(t_0)(g \overset{\alpha}{\Box} \nabla u^n)(t)\\ &+\frac{\lambda_1}{8\ell g(0)}(g' \overset{\alpha}{\Box} \nabla u^n)(t).\end{aligned} \tag{3.26}$$

Then, we conclude from (3.26) that (3.21) holds true. □

Now we prove two auxiliary stability results for the regularized energy $E_\alpha^n(t)$ set in (3.15).

Proposition 3.6. *Under the above notations and Assumptions 3.1-(**i**), the regularized energy $E_\alpha^n(t)$ satisfies, for a fixed $t_0 > 0$, the following n-decay rate:*

$$E_\alpha^n(t) \leq C_\alpha(t_0)E_\alpha^n(0)\,e^{-\gamma_\alpha(t_0)t}, \quad t > t_0, \tag{3.27}$$

where $C_\alpha(t_0)$ and $\gamma_\alpha(t_0)$ are given in (3.5).

Proof. From (3.2) with $\xi \equiv \xi_0$ and taking into account (3.16) we have

$$\xi_0\,(g \overset{\alpha}{\Box} \nabla u^n)(t) \leq -(g' \overset{\alpha}{\Box} \nabla u^n)(t) \leq -2\frac{\mathrm{d}}{\mathrm{d}t}E^n(t). \tag{3.28}$$

Then, combining (3.21) with (3.28), we obtain

$$\xi_0\frac{\mathrm{d}}{\mathrm{d}t}F_\alpha^n(t) \;\leq\; -\frac{\xi_0\lambda_1^2}{4g(0)}[h_\alpha(t_0)]^2\,E_\alpha^n(t) - 2\,c_\alpha(t_0)\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha^n(t) \tag{3.29}$$

where F_α^n is given in (3.19) and $c_\alpha(t_0) > 0$ comes from Lemma 3.5. In addition, defining

$$I_\alpha^n(t) := 2c_\alpha(t_0)E_\alpha^n(t) + \xi_0\,F_\alpha^n(t), \tag{3.30}$$

we readily get

$$2c_\alpha(t_0)E_\alpha^n(t) \leq I_\alpha^n(t) \leq \left[2c_\alpha(t_0) + \xi_0\left(\frac{1}{2}+\frac{\lambda_1}{\ell g(0)}\right)\right]E_\alpha^n(t). \tag{3.31}$$

Moreover, deriving (3.30) and using (3.29) and (3.31) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}I_\alpha^n(t) \leq -\gamma_\alpha(t_0)\,I_\alpha^n(t).$$

Hence, by the differential Gronwall's inequality and (3.31), one concludes that (3.27) holds true. □

Proposition 3.7. *Let*

$$\kappa_{\alpha,n}(t_0) := c_\alpha(t_0)\left(\frac{4}{\ell}E^n_\alpha(0) + 2\sup_{\tau\in(\alpha,0)}\|\nabla u^n_0(\tau)\|_2^2\right) \tag{3.32}$$

*Under the above notations and Assumption 3.1-(**ii**), the regularized energy $E^n_\alpha(t)$ satisfies, for a fixed $t_0 > 0$, the following n-decay rate:*

$$E^n_\alpha(t) \le C_{\alpha,n}(t_0)\left(E^n_\alpha(0) + \int_0^t e^{\tilde\gamma_\alpha(t_0)\int_0^s \xi(\tau)\mathrm{d}\tau}[g(s) - g(s-\alpha)]\,\mathrm{d}s\right)e^{-\tilde\gamma_\alpha(t_0)\int_0^t \xi(s)\mathrm{d}s}$$
$$+C_{\alpha,n}(t_0)\int_t^{t-\alpha} g(\tau)\,\mathrm{d}\tau, \tag{3.33}$$

where

$$\begin{cases} C_{\alpha,n}(t_0) := \max\left\{\dfrac{\kappa_{\alpha,n}(t_0)}{\tilde\gamma_\alpha(t_0)}, 2c_\alpha(t_0) + \xi(0)\left(\dfrac{1}{2} + \dfrac{\lambda_1}{\ell g(0)}\right)\right\}, \\ \gamma_\alpha(t_0) := \dfrac{\xi(0)\lambda_1^2\ell}{4\ell g(0)c_\alpha(t_0) + \xi(0)(\ell g(0) + 2\lambda_1)}[h_\alpha(t_0)]^2. \end{cases} \tag{3.34}$$

Proof. First of all, regarding notation (3.12) and applying assumption (3.2), we observe that

$$\xi(t)\int_0^t g(s)\|\nabla u^n(t) - \nabla u^n(t-s)\|_2^2\,\mathrm{d}s \;\le\; -\int_0^t g'(s)\|\nabla u^n(t) - \nabla u^n(t-s)\|_2^2\,\mathrm{d}s$$
$$\le\; -2\frac{\mathrm{d}}{\mathrm{d}t}E^n(t). \tag{3.35}$$

Then, multiplying both sides of (3.21) by $\xi(t)$ and combining with (3.35), we obtain

$$\xi(t)\frac{\mathrm{d}}{\mathrm{d}t}F^n_\alpha(t) \;\le\; -\frac{\lambda_1^2}{4g(0)}[h_\alpha(t_0)]^2\xi(t)\,E^n_\alpha(t) - 2\,c_\alpha(t_0)\frac{\mathrm{d}}{\mathrm{d}t}E^n_\alpha(t)$$
$$+\,c_\alpha(t_0)\,\xi(t)\int_t^{t-\alpha} g(s)\|\nabla u^n(t) - \nabla u^n(t-s)\|_2^2\,\mathrm{d}s, \tag{3.36}$$

where F^n_α is given in (3.19) and $c_\alpha(t_0) > 0$ is given by (3.3). Now, defining

$$J^n_\alpha(t) := 2c_\alpha(t_0)E^n_\alpha(t) + \xi(t)\,F^n_\alpha(t), \tag{3.37}$$

we obtain the equivalence

$$2c_\alpha(t_0)E^n_\alpha(t) \le J^n_\alpha(t) \le \left[2c_\alpha(t_0) + \xi(0)\left(\frac{1}{2} + \frac{\lambda_1}{\ell g(0)}\right)\right]E^n_\alpha(t). \tag{3.38}$$

So, deriving (3.37), using (3.36) and noting that $\xi'(t)\,F^n_\alpha(t) \le 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}J^n_\alpha(t) \le -\frac{\lambda_1^2}{4g(0)}[h_\alpha(t_0)]^2\xi(t)\,E^n_\alpha(t) + c_\alpha(t_0)\xi(t)\int_t^{t-\alpha} g(s)\|\nabla u^n(t) - \nabla u^n(t-s)\|_2^2\,\mathrm{d}s. \tag{3.39}$$

Now, for $s > t$, we have

$$\|\nabla u^n(t) - \nabla u^n_0(t-s)\|_2^2 \le 2\|\nabla u^n(t)\|_2^2 + 2\|\nabla u^n_0(t-s)\|_2^2 \le \frac{4}{\ell}E^n_\alpha(0) + 2\,K_{\alpha,n},$$

where, from Proposition 3.3 and assumption (3.6), one has

$$K_{\alpha,n} = \sup_{\tau\in(\alpha,0)} \|\nabla u_0^n(\tau)\|_2^2 \le \sup_{\tau\in(\alpha,0)} \|\nabla u_0(\tau)\|_2^2 = K_\alpha < \infty, \quad \forall\, n \in \mathbb{N}. \tag{3.40}$$

This yields

$$\xi(t) \int_t^{t-\alpha} g(s)\|\nabla u^n(t) - \nabla u^n(t-s)\|_2^2 \mathrm{d}s \le \left(\frac{4}{\ell} E_\alpha^n(0) + 2\,K_{\alpha,n}\right) b_\alpha(t), \tag{3.41}$$

with $b_\alpha(t) = \xi(t) \int_t^{t-\alpha} g(s)\,\mathrm{d}s$. Then, using (3.38) and (3.41) in (3.39), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} J_\alpha^n(t) + \tilde{\gamma}_\alpha(t_0)\,\xi(t) J_\alpha^n(t) \le \kappa_{\alpha,n}(t_0) b_\alpha(t), \tag{3.42}$$

where we denote

$$\kappa_{\alpha,n}(t_0) := c_\alpha(t_0)\left(\frac{4}{\ell} E_\alpha^n(0) + 2\,K_{\alpha,n}\right) = c_\alpha(t_0)\left(\frac{4}{\ell} E_\alpha^n(0) + 2 \sup_{\tau\in[\alpha,0]} \|\nabla u_0^n(\tau)\|_2^2\right).$$

Solving differential inequality (3.42), we arrive at

$$J_\alpha^n(t) \le e^{-\tilde{\gamma}_\alpha(t_0)\int_0^t \xi(s)\,\mathrm{d}s} \left(J_\alpha^n(0) + \kappa_{\alpha,n}(t_0) \int_0^t e^{\tilde{\gamma}_\alpha(t_0)\int_0^s \xi(\tau)\mathrm{d}\tau} b_\alpha(s)\mathrm{d}s \right). \tag{3.43}$$

Also, from assumption (3.2) we get

$$\begin{aligned} \int_0^t e^{\tilde{\gamma}_\alpha(t_0)\int_0^s \xi(\tau)\mathrm{d}\tau} b_\alpha(s)\mathrm{d}s &= \frac{1}{\tilde{\gamma}_\alpha(t_0)} \int_0^t \left(e^{\tilde{\gamma}_\alpha(t_0)\int_0^s \xi(\tau)\mathrm{d}\tau}\right)' \int_s^{s-\alpha} g(\tau)\mathrm{d}\tau\,\mathrm{d}s \\ &\le \frac{1}{\tilde{\gamma}_\alpha(t_0)} \left(e^{\tilde{\gamma}_\alpha(t_0)\int_0^t \xi(\tau)\mathrm{d}\tau} \int_t^{t-\alpha} g(\tau)\,\mathrm{d}\tau \right) \\ &\quad + \frac{1}{\tilde{\gamma}_\alpha(t_0)} \left(\int_0^t e^{\tilde{\gamma}_\alpha(t_0)\int_0^s \xi(\tau)\mathrm{d}\tau} [g(s) - g(s-\alpha)]\,\mathrm{d}s \right) \end{aligned}$$

Thus, we obtain from (3.43) the inequality

$$\begin{aligned} J_\alpha^n(t) \le &\left(J_\alpha^n(0) + \frac{\kappa_{\alpha,n}(t_0)}{\tilde{\gamma}_\alpha(t_0)} \int_0^t e^{\tilde{\gamma}_\alpha(t_0)\int_0^s \xi(\tau)\mathrm{d}\tau} [g(s) - g(s-\alpha)]\,\mathrm{d}s \right) e^{-\tilde{\gamma}_\alpha(t_0))\int_0^t \xi(s)\mathrm{d}s} \\ &+ \frac{\kappa_{\alpha,n}(t_0)}{\tilde{\gamma}_\alpha(t_0)} \int_t^{t-\alpha} g(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{3.44}$$

Using again (3.38) in (3.44) we finally conclude that

$$\begin{aligned} E_\alpha^n(t) &\le \frac{1}{2c_\alpha(t_0)} J_\alpha^n(t) \\ &\le \frac{1}{2c_\alpha(t_0)} \left(J_\alpha^n(0) + \frac{\kappa_{\alpha,n}(t_0)}{\tilde{\gamma}_\alpha(t_0)} \int_0^t e^{\tilde{\gamma}_\alpha(t_0)\int_0^s \xi(\tau)\mathrm{d}\tau} [g(s) - g(s-\alpha)]\,\mathrm{d}s \right) e^{-\tilde{\gamma}_\alpha(t_0)\int_0^t \xi(s)\mathrm{d}s} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\kappa_{\alpha,n}(t_0)}{2c_\alpha(t_0)\tilde{\gamma}_\alpha(t_0)}\int\limits_t^{t-\alpha} g(\tau)\,\mathrm{d}\tau \\
&\leq C_{\alpha,n}(t_0)\left(E_\alpha^n(0)+\int\limits_0^t e^{\tilde{\gamma}_\alpha(t_0)\int\limits_0^s \xi(\tau)\mathrm{d}\tau}[g(s)-g(s-\alpha)]\,\mathrm{d}s\right)e^{-\tilde{\gamma}_\alpha(t_0)\int\limits_0^t \xi(s)\mathrm{d}s} \\
&+C_{\alpha,n}(t_0)\int\limits_t^{t-\alpha} g(\tau)\,\mathrm{d}\tau,
\end{aligned}
$$

which is precisely estimate (3.33) with $C_{\alpha,n}(t_0)$ given in (3.34). $\square$

3.3. Proof of the main result

We finally conclude decay rates (3.4) and (3.8) in Theorem 3.1 by means of Proposition 3.3. Indeed, as a direct consequence of Proposition 3.3 we obtain that the first two terms of $E_\alpha^n(t)$ set in (3.15) converge to the first two terms of $E_\alpha(t)$ defined in (3.1). Moreover, we also claim that

$$
\begin{aligned}
(g\overset{\alpha}{\square}\nabla u^n)(t) &= \int\limits_0^{t-\alpha} g(s)\|\nabla u^n(t)-\nabla u^n(t-s)\|_2^2\,\mathrm{d}s \\
&\longrightarrow \int\limits_0^{t-\alpha} g(s)\|\nabla u(t)-\nabla u(t-s)\|_2^2\,\mathrm{d}s, \quad \text{as} \quad n\to\infty,
\end{aligned}
\tag{3.45}
$$

for any given parameters $-\infty<\alpha<0$ and $t>\alpha$. In fact, defining

$$
\begin{aligned}
G_1^n(t,s) &= \|\nabla u^n(t)-\nabla u^n(t-s)\|_2-\|\nabla u(t)-\nabla u(t-s)\|_2, \\
G_2^n(t,s) &= \|\nabla u^n(t)-\nabla u^n(t-s)\|_2+\|\nabla u(t)-\nabla u(t-s)\|_2,
\end{aligned}
$$

then

$$
\|\nabla u^n(t)-\nabla u^n(t-s)\|_2^2-\|\nabla u(t)-\nabla u(t-s)\|_2^2=G_1^n(t,s)G_2^n(t,s).
$$

Again from Proposition 3.3 we have

$$
|G_1^n(t,s)|\leq\|\nabla u^n(t)-\nabla u(t)\|_2+\|\nabla u^n(t-s)-\nabla u(t-s)\|_2\to 0, \tag{3.46}
$$

as $n\to\infty$. On the other hand, using once more Proposition 3.3, we estimate $G_2^n(t,s)$ as follows

$$
|G_2^n(t,s)|\leq 2\|\nabla u(t)\|_2+2\|\nabla u(t-s)\|_2, \quad s>0. \tag{3.47}
$$

Thus, from (3.46)–(3.47) we deduce

$$
g(s)G_1^n(t,s)G_2^n(t,s)\to 0, \quad \text{as} \quad n\to\infty, \quad s>0.
$$

Moreover, there exists a constant $C>0$ (which may depend on $E_\alpha(0)$ and K_α) such that

$$
|g(s)G_1^n(t,s)G_2^n(t,s)|\leq Cg(0), \quad s>0.
$$

From these two latter, desired limit (3.45) can be obtained by the Dominated Convergence Theorem and, therefore, it holds that

$$
\lim_{n\to\infty} E_\alpha^n(t)=E_\alpha(t), \quad t\geq 0, \tag{3.48}
$$

with

$$
\kappa_{\alpha,n}(t_0)\leq\kappa_\alpha(t_0), \quad \tilde{C}_{\alpha,n}(t_0)\leq\tilde{C}_\alpha(t_0), \tag{3.49}
$$

where $\kappa_\alpha(t_0)$ and $\tilde{C}_\alpha(t_0)$ are given in (3.7) and (3.9), respectively. Hence, from (3.48)–(3.49) we can pass the limit in (3.27) and (3.33) as $n \to \infty$ to reach (3.4) and (3.8), respectively. This completes the proof of Theorem 3.1. □

4. Limit analysis

This section is devoted to the precise analysis of general decay (3.8) within the limit situations $\alpha \to 0^-$ and $\alpha \to -\infty$, for every given time. Although the stability results for (2.1) in the formal (limit) cases $\alpha = 0$ and $\alpha = -\infty$ are well-known, as mentioned in introduction, we are going to see that (3.8) provides a good formulation that allows us to reach; as a consequence, the same (already known) stability results for history and null history problems $(2.1)_{\alpha=-\infty}$ and $(2.1)_{\alpha=0}$, respectively, via limit procedures. As a consequence, the same statement holds true in particular exponential case (3.4).

To organize the notations, we first set the family of negative parameters $\Lambda = \{\alpha;\, \alpha < 0\}$ and, for each $\alpha \in \Lambda$, we denote by (u^α, u_t^α) the unique weak solution of (2.1) associated with initial data (u_0, u_1). Also, we denote the corresponding energy by $E_\alpha^u(t) = E_\alpha(u^\alpha(t), u_t^\alpha(t))$ as defined in (3.1) along with notation (3.12), that is,

$$E_\alpha^u(t) \;=\; \frac{1}{2}\|u_t^\alpha(t)\|_2^2 + \frac{1}{2}\left(1 - h_\alpha(t)\right)\|\nabla u^\alpha(t)\|_2^2 + \frac{1}{2}(g \overset{\alpha}{\square} \nabla u^\alpha)(t), \quad t \geq 0. \tag{4.1}$$

4.1. Forward memory limit: $\alpha \to 0^-$

Let us first consider the classical initial-boundary value problem for the viscoelastic wave equation without history

$$\begin{cases} v_{tt} - \Delta v + \displaystyle\int_0^t g(t-s)\Delta v(s)\,\mathrm{d}s = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ v = 0 \quad \text{on} \quad \partial\Omega \times [0,\infty), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega, \end{cases} \tag{4.2}$$

and its corresponding energy $E_0^v(t) = E_0(v(t), v_t(t))$ given by

$$E_0^v(t) = \frac{1}{2}\|v_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla v(t)\|_2^2 + \frac{1}{2}(g \,\square\, \nabla v)(t), \quad t \geq 0, \tag{4.3}$$

where the notation $(g \,\square\, \nabla v)(t)$ corresponds to (3.12) with $\alpha = 0$.

The existence and uniqueness result of weak solution to problem (4.2) is very well-known, see for instance [6,7,9–11,13,14,18]. Roughly speaking, it reads as follows: *under Assumption 2.1, if $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$, then problem (4.2) has a unique weak solution in the class*

$$(v, v_t) \in C([0,\infty); H_0^1(\Omega) \times L^2(\Omega)).$$

Moreover, under additional Assumption 3.1, one proves the next general stability for $E_0^v(t)$ set in (4.3) over weak solutions of (4.2).

Theorem 4.1. *Under Assumptions 2.1 and 3.1-(**ii**), then there exist constants $C, \gamma > 0$ such that*

$$E_0^v(t) \leq C\, E_0^v(0) e^{-\gamma \int_0^t \xi(s)\,\mathrm{d}s}, \quad \forall\, t > 0. \tag{4.4}$$

Proof. See Theorem 3.6 in [27]. See also [28, Theorem 3.5] and [20, Subsection 4.1]. □

Theorem 4.1 was first proved by Messaoudi in 2008. However, here we are going to prove that it can be reached as a consequence of a Martinez's lemma proposed in 1999 to obtain decay rate estimates for dissipative systems, as readjusted to our case as follows.

Lemma 4.2. [26, Lemma 1] *Let $\mathcal{E} : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function and let $\phi : [0, +\infty) \to [0, +\infty)$ be a strictly increasing C^1-function such that*

$$\phi(t) \to +\infty, \quad \textit{as} \quad t \to +\infty.$$

Let us also assume that there exists a constant $\omega > 0$ such that

$$\int_s^{+\infty} \mathcal{E}(t)\phi'(t)\,dt \le \frac{1}{\omega}\mathcal{E}(s), \quad \forall\, s \ge 1. \tag{4.5}$$

Then, there exists a constant $C > 0$ depending on $\mathcal{E}(1)$ such that

$$\mathcal{E}(t) \le Ce^{-\omega\phi(t)}, \quad \forall t \ge 1.$$

Sketch of the proof of Theorem 4.1. Taking a closer look at Sect. 3.2 and defining corresponding functionals (3.17)–(3.19) related to $E_0^v(t)$, then it is proved similarly to (3.20)–(3.21) that

$$\frac{1}{2}E_0^v(t) \le F_0^v(t) \le \left[\frac{1}{2} + \frac{\lambda_1}{\ell g(0)}\right] E_0^v(t), \quad t \ge 0, \tag{4.6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}F_0^v(t) \le -\frac{\lambda_1^2}{4g(0)}[h_0(1)]^2 E_0^v(t) + c_0\,(g\Box\nabla v)(t), \quad t > 1, \tag{4.7}$$

for some constant $c_0 > 0$.

Multiplying (4.7) by ξ and integrating the resulting expression on the interval (s, T), $1 \le s < T$, we get

$$\frac{\lambda_1^2}{4g(0)}[h_0(1)]^2 \int_s^T E_0^v(t)\xi(t)\,dt \le -\int_s^T \frac{\mathrm{d}}{\mathrm{d}t}F_0^v(t)\xi(t)\,dt + c_0 \int_s^T \xi(t)(g\Box\nabla v)(t)\,dt. \tag{4.8}$$

From (4.8), noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}F_0(t)\xi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(F_0\xi\right)(t) - \frac{\mathrm{d}}{\mathrm{d}t}\xi(t)F_0(t) \ge \frac{\mathrm{d}}{\mathrm{d}t}\left(F_0\xi\right)(t),$$

and regarding (3.35) for the couple (v^n, E_0^v), we deduce

$$\frac{\lambda_1^2}{4g(0)}[h_0(1)]^2 \int_s^T E_0^v(t)\xi(t)\,dt \le F_0^v(s)\xi(s) - F_0^v(T)\xi(T) + 2c_0\big(E_0^v(s)\xi(s) - E_0^v(T)\xi(T)\big). \tag{4.9}$$

Thus, using (4.6) and taking $T \to \infty$ in (4.9), we obtain

$$\int_s^{+\infty} E_0^v(t)\xi(t)\,dt \le \tilde{w}E_0^v(s), \quad \forall\, s \ge 1, \tag{4.10}$$

for some constant $\tilde{w} > 0$, which is (4.5) with $\mathcal{E}(t) = E_0^v(t)$, $\phi(t) = \int_0^t \xi(s)\,\mathrm{d}s$ and $\omega = 1/\tilde{w}$. Therefore, decay rate (4.4), for $t \ge 1$, is a direct consequence of (4.10) and Lemma 4.2. Also, it can be easily obtained on $(0, 1)$, see, e.g., [27,28]. $\square$

Now, under the above notations, we are in position to prove that general decay rate (4.4) can be reached as consequence of (3.8) in the forward limit $\alpha \to 0^-$, which will be a consequence of our next second main result.

Theorem 4.3. *Let us consider $\alpha \in \Lambda$, $(u_0, u_1) \in \mathcal{W}_g(\alpha, 0) \times L^2(\Omega)$ and suppose that Assumption 2.1 and (3.6) hold. If u^α is the weak solution of (2.1) and v is the weak solution of (4.2) with $v_0 = u_0(0)$ and $v_1 = u_1$, then for every time $t \geq 0$ it holds the following limit*

$$\lim_{\alpha \to 0^-} E^u_\alpha(t) = E^v_0(t), \tag{4.11}$$

where $E^u_\alpha(t)$ and $E^v_0(t)$ are the energies given in (4.1) and (4.3), respectively.

Proof. Let $\tilde{v}_0$ be the null extension of v_0 in $(\alpha, 0)$ and $\tilde{v}$ the unique weak solution of (2.1) associated with $\tilde{v}_0$. Due to the uniqueness, then it is easy to see that $\tilde{v}$ can be expressed by:

$$\tilde{v}(\cdot, t) = \begin{cases} v(\cdot, t), \ t \geq 0, \\ 0, \qquad \alpha < t < 0. \end{cases}$$

Setting $v^\alpha = u^\alpha - \tilde{v}$, we observe that v^α is the weak solution of (2.1) with initial conditions

$$v^\alpha(x, t) = \ u_0(x, t) - \tilde{v}_0(x, t), \quad (x, t) \in \Omega \times (\alpha, 0), \quad v^\alpha_t(x, 0) = 0, \quad x \in \Omega. \tag{4.12}$$

Integrating (3.16) from 0 to t for the $v^{\alpha,n} = R_n v^\alpha$ function, using (2.7), applying Proposition 3.3 and regarding (3.48) for $E^v_\alpha(t) = E_\alpha(v^\alpha(t), v^\alpha_t(t))$, we obtain

$$E_\alpha(v^\alpha(t), v^\alpha_t(t)) \leq E_\alpha(v^\alpha(0), v^\alpha_t(0)) = \int_0^{-\alpha} g(s) \|\nabla u^\alpha(-s) - \nabla v_0\|_2^2 \, \mathrm{d}s. \tag{4.13}$$

Now, from (3.6), we note that

$$\|\nabla u^\alpha(-s) - \nabla v_0\|_2^2 \leq 2(K_\alpha + \|\nabla v_0\|_2^2) \to 4\|\nabla v_0\|_2^2, \quad \text{as} \quad \alpha \to 0^-,$$

and from (4.13) one gets

$$0 \leq \lim_{\alpha \to 0^-} E_\alpha(v^\alpha(t), v^\alpha_t(t)) \leq \lim_{\alpha \to 0^-} \left(2(K_\alpha + \|\nabla v_0\|_2^2) \int_0^{-\alpha} g(s) \, \mathrm{d}s \right) = 0,$$

that is,

$$\lim_{\alpha \to 0^-} E_\alpha(v^\alpha(t), v^\alpha_t(t)) = 0. \tag{4.14}$$

Thus, using (4.14) and recalling that $v^\alpha = u^\alpha - \tilde{v}$, we obtain the following convergences when α goes to 0^-:

$$u^\alpha(t) \longrightarrow \tilde{v}(t) = v(t) \text{ in } H^1_0(\Omega), \quad t \geq 0, \tag{4.15}$$

$$u^\alpha_t(t) \longrightarrow \tilde{v}_t(t) = v_t(t) \text{ in } L^2(\Omega), \quad t \geq 0, \tag{4.16}$$

$$(g \overset{\alpha}{\square} \nabla u^\alpha)(t) \longrightarrow (g \overset{\alpha}{\square} \nabla \tilde{v})(t) \text{ in } \mathbb{R}, \tag{4.17}$$

We claim that

$$(g \overset{\alpha}{\square} \nabla u^\alpha)(t) \longrightarrow (g \,\square\, \nabla v)(t), \quad t \geq 0. \tag{4.18}$$

Indeed, writing

$$(g \,\square\, \nabla v)(t) = (g \overset{\alpha}{\square} \nabla \tilde{v})(t) - \left(\int_0^{-\alpha} g(t - s) \, \mathrm{d}s \right) \|\nabla v(t)\|_2^2,$$

we have from (4.17) that

$$(g \overset{\alpha}{\square} \nabla u^{\alpha})(t) - (g \square \nabla v)(t) = \left[(g \overset{\alpha}{\square} \nabla u^{\alpha})(t) - (g \overset{\alpha}{\square} \nabla \tilde{v})(t)\right] + \left(\int_0^{-\alpha} g(t-s)\,\mathrm{d}s\right) \|\nabla v(t)\|_2^2 \longrightarrow 0, \quad \text{as} \quad \alpha \to 0^-,$$

which shows (4.18). Hence, from (4.15), (4.16) and (4.18), we conclude that limit (4.11) holds true. □

As a promptly consequence of Theorems 3.1 and 4.3, we also conclude stability (4.4) for null history problem (4.2). More precisely, we have:

Corollary 4.4. *Let us take Assumptions 2.1 and 3.1 into account. If u is a weak solution of (4.2), then for any fixed $t_0 > 0$, the energy $E_0^u(t)$ satisfies (4.4) (rep. (1.11) for $\xi \equiv \xi_0 > 0$ constant) for all $t \geq t_0$.*

Proof. For every time $t \geq t_0$, stability (4.4) (rep. (1.11) for $\xi \equiv \xi_0 > 0$ constant) can be directly obtained by taking the limit when $\alpha \to 0^-$ in (3.8) (resp. (3.4)) and using (4.11). □

4.2. Backward memory limit: $\alpha \to -\infty$

Now, we are going to analyze the convergence of $E_\alpha^u(t)$ set in (4.1) when $\alpha \to -\infty$, for every fixed time. Let us consider the initial-boundary value problem for the viscoelastic wave equation with (infinity) history

$$\begin{cases} w_{tt} - \Delta w + \displaystyle\int_{-\infty}^{t} g(t-s)\Delta w(s)\,\mathrm{d}s = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ w = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}, \\ w(x,t) = w_0(x,t), \quad (x,t) \in \Omega \times (-\infty, 0], \quad w_t(x,0) = w_1(x), \quad x \in \Omega, \end{cases} \tag{4.19}$$

which is equivalent to the following autonomous problem (see, e.g., [18])

$$\begin{cases} w_{tt} - \left(1 - \displaystyle\int_0^\infty g(s)\,\mathrm{d}s\right)\Delta w - \displaystyle\int_0^\infty g(s)\Delta\,\zeta(s)\,\mathrm{d}s = 0 \ \ \text{in} \ \ \Omega \times (0,\infty), \\ \zeta_t + \zeta_s = w_t \quad \text{in} \quad \Omega \times (0,\infty) \times (0,\infty), \\ w = 0 \quad \text{on} \quad \partial\Omega \times [0,\infty), \quad \zeta = 0 \quad \text{on} \quad \partial\Omega \times [0,\infty) \times (0,\infty), \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \ \zeta^0(x,s) = \zeta_0(x,s), \ x \in \Omega, \ s > 0, \\ \zeta^t(x,0) = 0 \quad (x,t) \in \Omega \times [0,\infty), \end{cases} \tag{4.20}$$

by setting the relative displacement history $\zeta^t(\cdot,s) := w(\cdot,t) - w(\cdot,t-s)$, $t \geq 0$, $s > 0$ and taking proper initial conditions such as $\zeta_0(s) = w_0 - w_0(-s)$. In this case, the energy $E_\infty^w(t) = E_\infty(w(t), w_t(t), \zeta^t)$, $t \geq 0$, associated with problem (4.20) can be denoted by

$$E_\infty^w(t) = \frac{1}{2}\|w_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^\infty g(s)\,\mathrm{d}s\right)\|\nabla w(t)\|_2^2 + \frac{1}{2}(g \overset{\infty}{\square} \nabla w)(t), \tag{4.21}$$

where the notation $(g \overset{\infty}{\square} \nabla w)(t)$ corresponds to (3.12) with $\alpha = -\infty$, that is,

$$(g \overset{\infty}{\square} \nabla w)(t) = \int_0^\infty g(s)\|\nabla[w(t) - w(t-s)]\|_2^2\,\mathrm{d}s = \int_0^\infty g(s)\|\nabla\zeta^t(s)\|_2^2\,\mathrm{d}s = \|\zeta^t\|_{\mathcal{W}_g}^2, \tag{4.22}$$

which coincides with the notation $\|\zeta^t\|_{\mathcal{M}}$, e.g., in [16,20], with respect to ζ.

The existence and uniqueness result for (4.20) is stated as in Theorem 2.1, namely, for weak solutions it reads as follows: *under Assumption* 2.1*, if* $(w_0, w_1, \zeta_0) \in \mathcal{H}$*, then system* (4.20) *has a unique weak solution in the class*

$$(w, w_t, \zeta) \in C([0,\infty); \mathcal{H}).$$

Additionally, under Assumption 3.1 and assuming that

$$K_0 := \sup_{s<0} \|\nabla w_0(s)\|_2 < +\infty, \tag{4.23}$$

one proves the next general stability for $E_\infty^w(t)$ set in (4.21) over weak solutions of (4.20).

Theorem 4.5. *Under Assumptions* 2.1 *and* 3.1-(**ii**)*, and also condition* (4.23)*, then there exist constants* $\delta_0 \in (0,1)$ *and* $C_0 > 0$ *depending on initial data such that*

$$E_\infty^w(t) \leq C_0 \left(E_\infty^w(0) + \int_0^t g^{1-\delta_0}(s)\,\mathrm{d}s \right) e^{-\delta_0 \int_0^t \xi(s)\,\mathrm{d}s} + C_0 \int_t^\infty g(s)\,\mathrm{d}s, \tag{4.24}$$

for all $t > t_0$*, where* t_0 *is a positive fixed time.*

Proof. See Theorem 2.1 in [20]. □

Now, under the above notations for the history case, we are again in position to prove that general decay rate (4.24) can be achieved as consequence of (3.8) in the backward limit $\alpha \to -\infty$, for every given time. This statement will follow as a consequence of our next third main result.

Theorem 4.6. *Let us consider* $\alpha \in \Lambda$*,* $(w_0, w_1, \zeta_0) \in \mathcal{H}$ *and suppose that Assumption* 2.1 *and* (4.23) *hold. If* (w, ζ) *is the weak solution of* (4.20) *and* u^α *is the weak solution of* (2.1) *with* $u_0 = w_0\big|_{(\alpha,0]}$ *and* $u_1 = w_1$*, then the following limit holds for every time* $t \geq 0$

$$\lim_{\alpha \to -\infty} E_\alpha^u(t) = E_\infty^w(t), \tag{4.25}$$

where $E_\alpha^u(t)$ *and* $E_\infty^w(t)$ *are the energies given in* (4.1) *and* (4.21)*, respectively.*

Proof. We first consider $\tilde{u}_0$ the null extension of u_0 in $(-\infty, \alpha]$. Then, let $(\tilde{u}^\alpha, \eta^\alpha)$ be the unique weak solution of (4.20) associated with initial data $(\tilde{u}_0, w_1, \tilde{\zeta}_0)$, where

$$\eta^{\alpha,t}(s) = \tilde{u}^\alpha(t) - \tilde{u}^\alpha(t-s), \quad \eta^{\alpha,0}(s) = \tilde{\zeta}_0(s) = w_0 - \tilde{u}_0(-s), \quad t, s > 0.$$

Due to the uniqueness of solution for (4.20), it is easy to see that $\tilde{u}^\alpha$ is given by:

$$\tilde{u}^\alpha(\cdot, t) = \begin{cases} u^\alpha(\cdot, t), \ t > \alpha, \\ 0, \qquad\quad -\infty < t \leq \alpha. \end{cases}$$

Now, we set $(w^\alpha, \zeta^\alpha) := (\tilde{u}^\alpha - w, \eta^\alpha - \zeta)$ and observe that (w^α, ζ^α) is a weak solution of (4.20) with initial data $(0, 0, \tilde{\eta}_0)$, where

$$\tilde{\eta}_0(s) = \tilde{\zeta}_0(s) - \zeta_0(s) = \begin{cases} 0, & s \in [0, -\alpha], \\ w_0(-s), & s > -\alpha. \end{cases}$$

Now, similar to (4.13), regarding expressions (4.21)–(4.22) and the above identity for $\tilde{\eta}_0$, we note that $E_\infty^{w^\alpha}(t) = E_\infty(w^\alpha(t), w_t^\alpha(t), \zeta^{\alpha,t})$ satisfies

$$E_\infty(w^\alpha(t), w_t^\alpha(t), \zeta^{\alpha,t}) \leq E_\infty(0, 0, \tilde{\eta}_0) = \int_{-\alpha}^\infty g(s) \|\nabla w_0(-s)\|_2^2\,\mathrm{d}s. \tag{4.26}$$

Due to boundedness (4.23) we can pass the limit when $\alpha \to -\infty$ in (4.26) to get

$$\lim_{\alpha \to -\infty} E_\infty(w^\alpha(t), w_t^\alpha(t), \zeta^{\alpha,t}) = 0. \tag{4.27}$$

From (4.27) and using (4.22), we obtain the following convergences when α goes to $-\infty$:

$$\tilde{u}^\alpha(t) = u^\alpha(t) \longrightarrow w(t) \quad \text{in} \quad H_0^1(\Omega), \quad t \geq 0, \tag{4.28}$$

$$\tilde{u}_t^\alpha(t) = u_t^\alpha(t) \longrightarrow w_t(t) \quad \text{in} \quad L^2(\Omega), \quad t \geq 0, \tag{4.29}$$

$$(g \overset{\infty}{\square} \nabla\tilde{u}^\alpha)(t) \longrightarrow (g \overset{\infty}{\square} \nabla w)(t) \quad \text{in} \quad \mathbb{R}. \tag{4.30}$$

We also claim that:

$$(g \overset{\alpha}{\square} \nabla u^\alpha)(t) \longrightarrow (g \overset{\infty}{\square} \nabla w)(t), \quad t \geq 0. \tag{4.31}$$

In fact, by noting that

$$(g \overset{\alpha}{\square} \nabla u^\alpha)(t) - (g \overset{\infty}{\square} \nabla w)(t) = (g \overset{\infty}{\square} \nabla\tilde{u}^\alpha)(t) - (g \overset{\infty}{\square} \nabla w)(t) - \left(\int\limits_{t-\alpha}^{\infty} g(s)\,\mathrm{d}s \right) \|\nabla u^\alpha(t)\|_2^2,$$

then (4.31) is just a direct consequence of (4.28) and (4.30).

Therefore, from (4.28), (4.29) and (4.31), we conclude that limit (4.25) holds true. □

As an immediate consequence of Theorem 3.1 and Theorem 4.6, we are going to conclude general stability (4.24) as a limit case for the history problem.

Corollary 4.7. *Let us take Assumptions 2.1 and 3.1 into account and assume that (4.23) holds. If u is a weak solution of (4.19), then for any fixed $t_0 > 0$, the energy $E_\infty^u(t)$ set in (4.21) satisfies (4.24) (rep. (1.11) for $\xi \equiv \xi_0 > 0$ constant) for all $t > t_0$.*

Proof. We initially observe that

$$\int\limits_0^t e^{\tilde{\gamma}_\alpha(t_0)\int\limits_0^s \xi(\tau)\mathrm{d}\tau} [g(s) - g(s-\alpha)]\,\mathrm{d}s \longrightarrow \int\limits_0^t e^{\tilde{\gamma}_\alpha(t_0)\int\limits_0^s \xi(\tau)\mathrm{d}\tau} g(s)\,\mathrm{d}s, \tag{4.32}$$

as $\alpha \to -\infty$, for every fixed time. From (3.2), we have

$$e^{\int\limits_0^t \xi(\tau)\mathrm{d}\tau} g(t) \leq g(0),$$

which implies

$$\int\limits_0^t e^{\tilde{\gamma}_\alpha(t_0)\int\limits_0^s \xi(\tau)\mathrm{d}\tau} g(s)\,\mathrm{d}s \leq (g(0))^{\tilde{\gamma}_\alpha(t_0)} \int\limits_0^t (g(s))^{1-\tilde{\gamma}_\alpha(t_0)}\,\mathrm{d}s. \tag{4.33}$$

Hence, taking the limit when $\alpha \to -\infty$ in (3.8) (resp. (3.4))), and using (4.25), (4.32) and (4.33), one concludes that (4.24) (resp. (1.11) for $\xi \equiv \xi_0 > 0$ constant) is satisfied for all $t > t_0$. □

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