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Exponential stability for a thermo-viscoelastic Timoshenko system with fading memory

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A B S T R A C T

We are concerned with the characterization of the (uniform) exponential stability for a thermo-viscoelastic Timoshenko beam system under the Fourier law for heat conduction and memory in a history setting. Motivated by [5,6], we explore an intrinsic (non differentiable) assumption on the memory kernel that provides a necessary and sufficient condition for the exponential stability of the whole system. It gives a substantial generalization of the stability results obtained in [10,14,18] and surely ties up loose ends about the hypothesis equivalent to exponential stability of the problem.

Contents

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1. Introduction

In this work we shall prove a characterization of the (uniform) exponential stability result with respect to the following thermo-viscoelastic Timoshenko beam system under Fourier's law and memory in a history framework

$$\begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x + \psi)_x + \sigma\theta_x = 0 & \text{in } (0, L) \times \mathbb{R}^+, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\phi_x + \psi) + \int\limits_0^\infty g(s)\psi_{xx}(s)\, ds - \sigma\theta = 0 & \text{in } (0, L) \times \mathbb{R}^+, \\ \rho_3\theta_t - \beta\theta_{xx} + \sigma(\phi_x + \psi)_t = 0 & \text{in } (0, L) \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

subject to initial-boundary conditions

$$\begin{cases} \phi_x(0, t) = \phi_x(L, t) = \psi(0, t) = \psi(L, t) = \theta(0, t) = \theta(L, t) = 0, \quad t \geq 0, \\ (\phi(x, 0), \phi_t(x, 0), \psi_t(x, 0), \theta(x, 0)) = (\phi_0, \phi_1, \psi_1, \theta_0), \quad x \in (0, L), \\ \psi(x, t) = \psi_0(x, t), \quad (x, t) \in (0, L) \times (-\infty, 0]. \end{cases} \tag{1.2}$$

Here, the unknown functions $\phi = \phi(x, t), \psi = \psi(x, t)$, and $\theta = \theta(x, t)$ represent, respectively, the vertical displacement, the rotation angle, and the temperature deviation of a beam with length $L > 0$.

We intend to clarify what is exactly the most general assumption on the kernel g that characterizes a necessary and sufficient condition for the (uniform) exponential stability of the IBVP (1.1)-(1.2). This fact has never been approached for such a problem so far and its value judgment is clarified in forthcoming arguments. Before doing so, we launch its corresponding autonomous problem, which is indeed the object of study in the present article.

1.1. Equivalent autonomous problem

As usual, to address the IBVP (1.1)-(1.2), we set the new variable (inspired by Dafermos [7,8]) known as *relative displacement history*

$$\eta^t(s) := \psi(t) - \psi(t - s), \quad t, s > 0.$$

Thus, by following similar steps as in [13], we can rewrite problem (1.1)-(1.2) equivalently as the next system

$$\begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x + \psi)_x + \sigma\theta_x = 0 & \text{in } (0, L) \times \mathbb{R}^+, \\ \rho_2\psi_{tt} - \tilde{b}\psi_{xx} + \kappa(\phi_x + \psi) - \int\limits_0^\infty g(s)\eta_{xx}(s)\, ds - \sigma\theta = 0 & \text{in } (0, L) \times \mathbb{R}^+, \\ \rho_3\theta_t - \beta\theta_{xx} + \sigma(\phi_x + \psi)_t = 0 & \text{in } (0, L) \times \mathbb{R}^+, \\ \eta_t + \eta_s = \psi_t & \text{in } (0, L) \times \mathbb{R}^+ \times \mathbb{R}^+, \end{cases} \tag{1.3}$$

with boundary conditions

$$\begin{cases} \phi_x(0, t) = \phi_x(L, t) = 0, & t \geq 0, \\ \psi(0, t) = \psi(L, t) = 0, & t \geq 0, \\ \theta(0, t) = \theta(L, t) = 0, & t \geq 0, \\ \eta^t(0, s) = \eta^t(L, s) = 0, & t, s > 0, \\ \eta^t(x, 0) = 0, \quad x \in (0, L), & t > 0, \end{cases} \tag{1.4}$$

and initial data

$$\begin{cases}(\phi(x,0),\phi_t(x,0),\psi(x,0),\psi_t(x,0),\theta(x,0)) = (\phi_0,\phi_1,\psi_0(0),\psi_1,\theta_0), & x\in(0,L),\\ \eta^0(x,s)=\psi_0(x,0)-\psi_0(x,-s), \quad (x,s)\in(0,L)\times\mathbb{R}^+, \end{cases} \tag{1.5}$$

where

$$\tilde{b} := b - \int_0^\infty g(s)\,ds.$$

Hence, from now on, all results shall be developed for the autonomous problem (1.3)-(1.5). It is known (cf. [10,14,18]) that the existence and uniqueness result for (1.3)-(1.5) only requires the following assumption on the memory kernel.

(G) $g:\mathbb{R}^+\to[0,\infty)$ *is an absolutely continuous, integrable, and non-increasing function such that*

$$\widetilde{b} := b - \int_0^\infty g(s)\,ds > 0.$$

1.2. State of the art: previous results and main goal

Very recently, in [14, Sect. 2] the authors proved that the following assumption is a *sufficient condition* to reach the (uniform) exponential stability of problem (1.3)-(1.5):

- *there exists* $\delta>0$ *such that*

$$g'(s)+\delta g(s)\le 0, \quad s>0. \tag{1.6}$$

See, for instance, [14, Thm. 2.2]. As a matter of fact, this latter result responds precisely the question early proposed by [18, Rem. 3.8] and improves notably the stability results achieved in [10, Thms. 2.2 and 2.3], since it reveals that the (uniform) exponential stability of (1.3)-(1.5) can be concluded without any extra assumption on the coefficients nor higher regularity of initial data.

The above fact leads us to the main question of the present paper:

Q1. *Conversely, if the IBVP* (1.3)-(1.5) *is exponential stable (in the semigroup sense), then must condition* (1.6) *necessarily hold?*

In other words, the above question can be remade as: *Is also condition* (1.6) *a necessary assumption for the exponential stability of* (1.3)-(1.5)*?* In order to give a precise answer to such questions, we firstly observe that condition (1.6) is equivalent to the following one (cf. [5,6]):

- *there exists* $\delta>0$ *such that*

$$g(\tau+s)\le e^{-\delta\tau}g(s), \quad s>0,\ \tau\ge 0. \tag{1.7}$$

Secondly, we still note that assumption (1.7) is a particular case of the following one so-called *admissible memory kernel*:

- *there exist $C \geq 1$ and $\delta > 0$ such that*

$$g(\tau + s) \leq Ce^{-\delta\tau} g(s), \tag{1.8}$$

for every $\tau > 0$ and for almost every $s > 0$.

We highlight that (1.8) is (for sure) more general than (1.6) and some examples can be found in [5,6]. Moreover, in what concerns (1.8), our main goal is to prove that it is not only sufficient but also a necessary condition to the (uniform) exponential stability of problem (1.3)-(1.5) in terms of its related semigroup solution. This fact is precisely stated in Theorem 3.1 (Section 3). Summarizing, under the assumption **(G)**, we have proved the next statement:

> *The admissible memory kernel assumption* (1.8) *holds if and only if the semigroup solution corresponding to* (1.3)-(1.5) *is exponentially stable.*

Therefore, our contribution is twofold:

(1) Our main stability result (Theorem 3.1) generalizes Theorem 2.2 in [14], since (1.8) is much weaker (therefore more general) than (1.6);
(2) It also gives the answer to questions **Q1** by showing that (1.8) is precisely the equivalent condition to the exponential stability property of (1.3)-(1.5), and not assumption (1.6).

In conclusion, this work ends a cycle of studies on the system (1.3)-(1.5) (resp. (1.1)-(1.2)) by giving the characterization of exponential stability in terms of the memory kernel g, which in turn means a generalization of the stability results in [10,14,18]. Besides being more general, our arguments are different from these latter once assumption (1.8) needs extra analysis on the estimates and, for this purpose, our analysis is inspired by [5,6,12]. Moreover, here the temperature plays an important role in the stability analysis because, once coupled on the shear force component, it gives a way to reach all estimates without any additional assumptions on the coefficients, as it happens for partially damped systems with (fading) memory acting only on the bending moment and no temperature deviations being taken into account, see e.g. [6]. On the other hand, when we neglect the memory term and consider the temperature acting either on the bending moment or shear force only, then we still have a partially damped system. For instance, when omitting the viscoelastic effect ($g = 0$) in (1.1)-(1.2), it becomes to a partially damped with a solely dissipation coming from the temperature coupled only on the shear force. This scenario requires a different analysis for stability results along with the equal wave speeds assumption (say, EWS for short), cf. [1–3]. Additionally, if we consider only thermal coupling on the bending moment (still no memory term), then the stability results are well understood in the literature as one can see e.g. in [4,16] where, under the EWS assumption, different computations are considered. Moreover, in this case we can go further once there are results in a much more general situation, namely, by regarding the Gurtin-Pipkin thermal law. In this direction, we quote the recent paper by Dell'Oro [9] where the stability of Bresse and Timoshenko systems with hyperbolic heat conduction are taken into account. The author provides a complete study encompassing the stability of the pure thermoelastic Timoshenko under Gurtin-Pipkin's law. Therein, since the temperature effect acts only on the bending moment, the stability depends upon the so-called *stability number*, still noting that there is no other (extra) damping in turn nor additional coupling terms as in our case. We refer to [9, Section 6] for more details on the model and the (non-exponential) stability result.

In Section 2 we provide the necessary notations as well as the existence result in terms of the linear semigroup theory, which is fundamental for the main stability result presented in Section 3.

2. Semigroup solution

2.1. Useful notations

Let $L^2(0,L)$ be the standard L^2-space with inner product and norm

$$(u,v) = \int_0^L u(x)\overline{v(x)}\,dx, \quad \|u\| = \left(\int_0^L |u(x)|^2\,dx\right)^{1/2}.$$

The space $H_0^1(0,L)$ stands for the usual Sobolev space, and

$$L_*^2(0,L) = \left\{u \in L^2(0,L),\ \frac{1}{L}\int_0^L u(x)dx = 0\right\}, \quad H_*^1(0,L) = H^1(0,L) \cap L_*^2(0,L),$$

equipped with the norms

$$\|u\|_{L_*^2(0,L)} = \|u\|, \quad \|u\|_{H_0^1(0,L)} = \|u\|_{H_*^1(0,L)} = \|u_x\|.$$

Let $h : (0,\infty) \to \mathbb{R}^+$ be a measurable function and $p \geq 1$. To deal with the relative displacement history variable, we consider the memory spaces

$$L_h^p(\mathbb{R}^+; H_0^1(0,L)) := \left\{\eta : \mathbb{R}^+ \to H_0^1(0,L);\ \int_0^\infty h(s)\|\eta_x(s)\|^p\,ds < \infty\right\}$$

with norm

$$\|\eta\|^p_{L_h^p(\mathbb{R}^+;H_0^1(0,L))} := \int_0^\infty h(s)\|\eta_x(s)\|^p\,ds.$$

In particular, for $p = 2$, we simply denote the space

$$\mathcal{M}_h := L_h^2(\mathbb{R}^+; H_0^1(0,L)),$$

which is a Hilbert space endowed with inner product

$$(\eta,\xi)_{\mathcal{M}_h} = \int_0^\infty h(s)(\eta_x(s),\xi_x(s))\,ds.$$

Additionally, under the above notations and assumption **(G)**, we consider the phase space

$$\mathcal{H} = H_*^1(0,L) \times L_*^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times L^2(0,L) \times \mathcal{M}_g$$

equipped with inner product

$$(z_1,z_2)_{\mathcal{H}} = \rho_1(\Phi_1,\Phi_2) + \rho_2(\Psi_1,\Psi_2) + \rho_3(\theta_1,\theta_2) + \kappa(\phi_{1,x}+\psi_1,\phi_{2,x}+\psi_2) + \widetilde{b}(\psi_{1,x},\psi_{2,x}) + (\eta_1,\eta_2)_{\mathcal{M}_g}$$

and norm

$$\|z\|_{\mathcal{H}}^2 = \rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2 + \rho_3\|\theta\|^2 + \kappa\|\phi_x + \psi\|^2 + \widetilde{b}\|\psi_x\|^2 + \|\eta\|_{\mathcal{M}_g}^2,$$

where $z_i = (\phi_i, \Phi_i, \psi_i, \Psi_i, \theta_i, \eta_i)$, $z = (\phi, \Phi, \psi, \Psi, \theta, \eta) \in \mathcal{H}$, $i = 1, 2$.

2.2. Right-translation and contraction semigroups

Now we set the operator $\mathbb{L} : D(\mathbb{L}) \subset \mathcal{M}_g \to \mathcal{M}_g$ given by

$$D(\mathbb{L}) := \{\eta \in \mathcal{M}_g,\, \mathbb{L}\eta \in \mathcal{M}_g \text{ and } \eta(0) = 0\}, \quad \mathbb{L}\eta := -\partial_s\eta,$$

which is the infinitesimal generator of the right-translation semigroup $R(t) : \mathcal{M}_g \to \mathcal{M}_g$ given by

$$[R(t)\eta](s) := \begin{cases} \eta(s-t), & s > t, \\ 0, & 0 < s \le t. \end{cases}$$

Accordingly, by means of (1.3)$_4$ and following the same arguments as in [13], we can express η explicitly in terms of ψ, namely

$$\eta^t(s) = \begin{cases} \eta^0(s-t) + \psi(t) - \psi_0(0), & s > t, \\ \psi(t) - \psi(t-s), & 0 < s \le t. \end{cases} \tag{2.1}$$

Moreover, by setting $\Phi = \phi_t$, $\Psi = \psi_t$, and $z_0 = (\phi_0, \phi_1, \psi_0, \psi_1, \theta_0, \eta^0)$, we can now rewrite (1.3)-(1.5) as the following the Cauchy problem

$$\begin{cases} z_t = Az, & t > 0, \\ z(0) = z_0, \end{cases} \tag{2.2}$$

where the linear operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$A = \begin{bmatrix} 0 & \mathrm{Id} & 0 & 0 & 0 & 0 \\ \frac{\kappa}{\rho_1}\partial_{xx} & 0 & -\frac{\kappa}{\rho_1}\partial_x & 0 & \frac{\sigma}{\rho_1}\partial_x & 0 \\ 0 & 0 & 0 & \mathrm{Id} & 0 & 0 \\ -\frac{\kappa}{\rho_2}\partial_x & 0 & \frac{\widetilde{b}}{\rho_2}\partial_{xx} - \frac{\kappa}{\rho_2}\mathrm{Id} & 0 & \frac{\kappa}{\rho_2}\mathrm{Id} & \frac{1}{\rho_2}\mathbb{I}_g \circ \partial_{xx} \\ 0 & -\frac{\sigma}{\rho_3}\partial_x & 0 & -\frac{\sigma}{\rho_3}\mathrm{Id} & \frac{\beta}{\rho_3}\partial_{xx} & 0 \\ 0 & 0 & 0 & \mathrm{Id} & 0 & -\partial_s \end{bmatrix} \tag{2.3}$$

with domain

$$D(A) = \left\{ z \in \mathcal{H};\;\; \phi_x, \Psi, \theta \in H_0^1(0,L),\;\; \phi, \theta, \widetilde{b}\psi + \mathbb{I}_g(\eta) \in H^2(0,L),\;\; \eta \in D(\mathbb{L}) \right\},$$

and, for simplicity, we adopt the notation

$$\mathbb{I}_g(\eta) := \int_0^\infty g(s)\eta(s)\,ds.$$

If **(G)** holds, one can prove that $D(A)$ is dense in $\mathcal{H}$, $I - A$ is onto, and

$$(Az, z)_{\mathcal{H}} = \frac{1}{2}\int_0^\infty g'(s)\|\eta_x(s)\|^2\,ds - \beta\|\theta_x\|^2 \le 0, \tag{2.4}$$

for every $z = (\phi, \Phi, \psi, \Psi, \theta, \eta) \in D(A)$ (cf. [14, Sect. 2]). Therefore, from Lumer-Phillips Theorem (cf. [15, Thm. 1.2.4]), the operator A set in (2.3) is the infinitesimal generator of a C_0-semigroup of contractions $S(t) := e^{At}$ on $\mathcal{H}$. Consequently, it yields

Theorem 2.1 *(Existence and Un1iqueness). Let assumption* **(G)** *be in turn. Then, for every $z_0 \in \mathcal{H}$, problem (2.2) has a unique mild solution $z \in C(0, \infty; \mathcal{H})$ given by*

$$z(t) = e^{At} z_0, \quad t \geq 0. \tag{2.5}$$

If, in addition $z_0 \in D(A)$, then z is the classical solution of (2.2) with

$$z \in C^1(0, \infty; \mathcal{H}) \cap C(0, \infty; D(A)).$$

Theorem 2.1 also means that problem (1.3)-(1.5) is well-posed. Below, our main goal is to give a complete characterization of its stability result.

3. Characterization of stability

Let us recall that a semigroup $T(t) : H \to H$ is exponentially stable if and only if there exist constants $M \geq 1$ and $\gamma > 0$ such that

$$\|T(t)z\|_H \leq M e^{-\gamma t} \|z\|_H, \quad \forall\ z \in H.$$

In the present paper, our main goal is prove that the semigroup $S(t) = e^{At}$ related to the solution (2.5) of problem (2.2) (and consequently (1.3)-(1.5)) is exponentially stable iff the following additional assumption holds true:

(S) *there exist constants $C \geq 1$ and $\delta > 0$ such that*

$$g(t + s) \leq C e^{-\delta t} g(s) \tag{3.1}$$

for every $t > 0$ and for almost every $s > 0$.

More precisely, we have:

Theorem 3.1 *(Main Result – Characterization of Exponential Stability). Let assumption* **(G)** *be in turn. Then, the following statements are equivalent:*

(i) g satisfies **(S)**;
(ii) the semigroup solution z given by (2.5) is exponential stable.

In conclusion, Theorem 3.1 claims that condition **(S)** is not only sufficient but also necessary for the exponential stability of problem (2.2) (resp. (1.3)-(1.5)). Its detailed proof shall be given as follows.

3.1. Proof of the main result: part I (necessity)

The arguments in the proof of Theorem 3.1 ($(ii) \Rightarrow (i)$) are inspired by [5, Thm. 3.2]. For the sake of completeness, we are going to prove it in detail.

Let us consider $\eta_0 \in \mathcal{M}_g$ and

$$\tilde{z}(t) = S(t)(0, 0, 0, 0, 0, \eta_0) = (\tilde{\phi}, \tilde{\Phi}, \tilde{\psi}, \tilde{\Psi}, \tilde{\theta}, \tilde{\eta}).$$

By assuming that (ii) holds, namely, the semigroup $S(t)$ is exponentially stable and so we have

$$\|\tilde{z}(t)\|_{\mathcal{H}}^2 = \|S(t)(0,0,0,0,0,\eta_0)\|_{\mathcal{H}}^2 \le Me^{-\gamma t}\|\eta_0\|_{\mathcal{M}_g}^2, \tag{3.2}$$

for some $M \ge 1$, $\gamma > 0$ and for every $t > 0$. Now, by the formula (2.1) and using (3.2), we deduce

$$\begin{aligned}\int_t^\infty g(s)\|\eta_{0x}(s-t)\|^2\,ds &\le 2\|\tilde{\eta}^t\|_{\mathcal{M}_g}^2 + 2b\|\tilde{\psi}_x(t)\|^2 \\ &\le 2M(1+b_0)e^{-\gamma t}\|\eta_0\|_{\mathcal{M}_g}^2, \end{aligned}\tag{3.3}$$

where $b_0 := \frac{b}{\tilde{b}} > 1$.

On the other hand, for each $t > 0$ we define

$$\mathcal{B}_t := \{s \in \mathbb{R}^+,\ g(t+s) - 2M(1+b_0)e^{-\gamma t}g(s) > 0\}.$$

We claim that $\operatorname{meas}(\mathcal{B}_t) = 0$, for every $t > 0$. Indeed, suppose by contradiction that exists $t_0 > 0$ such that $\operatorname{meas}(\mathcal{B}_{t_0}) > 0$ (possibly infinite). Then,

$$0 < \int_{\mathcal{B}_{t_0}} [g(t_0+s) - 2M(1+b_0)e^{-\gamma t_0}g(s)]\,ds < +\infty. \tag{3.4}$$

But, from (3.3),

$$\begin{aligned} 0 &\ge \int_{t_0}^\infty g(s)\|\eta_{0x}(s-t_0)\|^2\,ds - 2M(1+b_0)e^{-\gamma t_0}\int_0^\infty g(s)\|\eta_{0x}(s)\|^2\,ds \\ &= \int_0^\infty [g(t_0+s) - 2M(1+b_0)e^{-\gamma t_0}g(s)]\|\eta_{0x}(s)\|^2\,ds. \end{aligned}$$

Now we choose $\eta_0(s) = \chi_{\mathcal{B}_{t_0}}(s)\phi^*$, for some $\phi^* \in H_0^1(0,L)$ such that $\|\phi_x^*\| = 1$. Therefore,

$$\int_{\mathcal{B}_{t_0}} [g(t_0+s) - 2M(1+b_0)e^{-\gamma t_0}g(s)]\,ds \le 0,$$

which contradicts (3.4). Hence, condition (3.1) holds as desired. $\square$

3.2. General results

Before concluding the proof of Theorem 3.1 ($(i) \Rightarrow (ii)$), we are going to clarify how important the inequality (3.1) is in the controllability of specific integral terms. More specifically, we prove the following (more) general result that shall very useful in our future computations.

Lemma 3.2. *Let $p \ge 1$ and $\eta \in L_g^p(\mathbb{R}^+; H_0^1(0,L))$. If (3.1) holds, then for every $0 \le r \le p-1$ we have the estimate*

$$\int_0^\infty g(s)\left(\int_0^s \|\eta_x(\tau)\|\,d\tau\right)^{p-r} ds \le C^{(p-r)/p}\, b^{r/p}\left(\frac{p}{\delta}\right)^{p-r}\|\eta\|_{L_g^p(\mathbb{R}^+;H_0^1(0,L))}^{p-r}, \tag{3.5}$$

where $C \geq 1$ and $\delta > 0$ are given in (3.1).

Proof. Using (3.1) and Hölder inequality with $\frac{r}{p} + \frac{p-r}{p} = 1$, we obtain

$$\begin{aligned}
&\int_0^\infty g(s) \left(\int_0^s \|\eta_x(\tau)\| \, d\tau \right)^{p-r} ds \\
&\quad = \int_0^\infty g^{r/p}(s) \left(\int_0^s g^{1/p}(s) \|\eta_x(\tau)\| \, d\tau \right)^{p-r} ds \\
&\quad \leq C^{(p-r)/p} \int_0^\infty g^{r/p}(s) \left(\int_0^s e^{-\frac{\delta}{p}(s-\tau)} g^{1/p}(\tau) \|\eta_x(\tau)\| \, d\tau \right)^{p-r} ds \\
&\quad \leq C^{(p-r)/p} \, b^{r/p} \left[\int_0^\infty \left(\int_0^s e^{-\frac{\delta}{p}(s-\tau)} g^{1/p}(\tau) \|\eta_x(\tau)\| \, d\tau \right)^p ds \right]^{(p-r)/p}.
\end{aligned}$$

Now, applying Young inequality for convolutions (see e.g. [11, Thm. 8.7]) with

$$e^{-\frac{\delta}{p}(\cdot)} \in L^1(\mathbb{R}^+), \quad g^{1/p}(\cdot)\|\eta_x(\cdot)\| \in L^p(\mathbb{R}^+),$$

we get

$$\begin{aligned}
\int_0^\infty \left(\int_0^s e^{-\frac{\delta}{p}(s-\tau)} g^{1/p}(\tau) \|\eta_x(\tau)\| \, d\tau \right)^p ds &\leq \left(\int_0^\infty e^{-\frac{\delta}{p} s} \, ds \right)^p \|\eta\|^p_{L^p_g(\mathbb{R}^+; H^1_0(0,L))} \\
&= \left(\frac{p}{\delta} \right)^p \|\eta\|^p_{L^p_g(\mathbb{R}^+; H^1_0(0,L))}.
\end{aligned}$$

Collecting the above estimates, we arrive at (3.5). $\square$

Corollary 3.3. *If $\eta \in D(\mathbb{L})$, then*

$$\|\eta\|_{\mathcal{M}_g} \leq \frac{\sqrt{4C}}{\delta} \|\eta_s\|_{\mathcal{M}_g}.$$

Proof. It is a direct consequence of the following inequality

$$\|\eta\|_{\mathcal{M}_g} \leq \left[\int_0^\infty g(s) \left(\int_0^s \|\eta_{sx}(\tau)\| \, d\tau \right)^2 ds \right]^{1/2}$$

and Lemma 3.2 with $p = 2$ and $r = 0$. $\square$

3.3. Proof of the main result: part II (sufficiency)

In addition, to complete the proof of Theorem 3.1, we appeal to the following known result.

Lemma 3.4 *([12, Lemma 1.6]). Let $T(t)$ be a C_0-semigroup of contractions on a Hilbert space H, and B its infinitesimal generator. If there exists $\varepsilon > 0$ such that*

$$\inf_{\lambda\in\mathbb{R}} \|(i\lambda - B_c)z\|_{H_c} \geq \varepsilon\|z\|_{H_c}, \quad \forall\ z \in D(B_c), \tag{3.6}$$

where H_c and B_c are the complexification of H and B, respectively, that is,

$$H_c = \{u + iv, u, v \in H\}, \quad B_c : D(B_c) \subset H_c \to H_c,\ B_c(u + iv) = Bu + iBv,$$

then $T(t)$ is exponentially stable.

To simplify the notation, we will omit the complexification index (c).

Now, to prove that '$(i) \Rightarrow (ii)$', we argue by contradiction. Let us assume condition (i), but suppose that $S(t)$ is not exponentially stable. From (3.6), one can construct sequences

$$z_n = (\phi_n, \Phi_n, \psi_n, \Psi_n, \theta_n, \eta_n) \in D(A), \quad \lambda_n \in \mathbb{R},$$

with $\|z_n\|_{\mathcal{H}} = 1$ such that $\lambda_n \to \lambda_* \in [-\infty, +\infty]$ and

$$i\lambda_n z_n - Az_n \to 0 \quad \text{in} \quad \mathcal{H}. \tag{3.7}$$

Componentwise,

$$\begin{cases} i\lambda_n\phi_n - \Phi_n \to 0 & \text{in} \quad H^1_*(0, L), \\ i\lambda_n\rho_1\Phi_n - \kappa(\phi_{nx} + \psi_n)_x + \sigma\theta_{nx} \to 0 & \text{in} \quad L^2_*(0, L) \\ i\lambda_n\psi_n - \Psi_n \to 0 & \text{in} \quad H^1_0(0, L), \\ i\lambda_n\rho_2\Psi_n + \kappa(\phi_{nx} + \psi_n) - \big(\tilde{b}\psi_n + \mathbb{I}_g(\eta_n)\big)_{xx} - \sigma\theta_n \to 0 & \text{in} \quad L^2(0, L), \\ i\lambda_n\rho_3\theta_n - \beta\theta_{nxx} + \sigma(\Phi_{nx} + \Psi_n) \to 0 & \text{in} \quad L^2(0, L), \\ i\lambda_n\eta_n + \eta_{ns} - \Psi_n \to 0 & \text{in} \quad \mathcal{M}_g. \end{cases} \tag{3.8}$$

From (2.4), (3.7) and using the boundedness of z_n in $\mathcal{H}$, we have

$$-\frac{1}{2}\int_0^\infty g'(s)\|\eta_{nx}(s)\|^2\, ds + \beta\|\theta_{nx}\|^2 \leq \mathrm{Re}\,(i\lambda_n z_n - Az_n, z_n)_{\mathcal{H}} \to 0. \tag{3.9}$$

Therefore, using the embedding $H^1_0(0, L) \hookrightarrow L^2(0, L)$ and (3.9), we can reduce (3.8) to

$$\begin{cases} f^1_n := i\lambda_n\phi_n - \Phi_n \to 0 & \text{in} \quad H^1_*(0, L), \\ f^2_n := i\lambda_n\rho_1\Phi_n - \kappa(\phi_{nx} + \psi_n)_x \to 0 & \text{in} \quad L^2_*(0, L) \\ f^3_n := i\lambda_n\psi_n - \Psi_n \to 0 & \text{in} \quad H^1_0(0, L), \\ f^4_n := i\lambda_n\rho_2\Psi_n + \kappa(\phi_{nx} + \psi_n) - \big(\tilde{b}\psi_n + \mathbb{I}_g(\eta_n)\big)_{xx} \to 0 & \text{in} \quad L^2(0, L), \\ f^5_n := i\lambda_n\rho_3\theta_n - \beta\theta_{nxx} + \sigma(\Phi_{nx} + \Psi_n) \to 0 & \text{in} \quad L^2(0, L), \\ f^6_n := i\lambda_n\eta_n + \eta_{ns} - \Psi_n \to 0 & \text{in} \quad \mathcal{M}_g. \end{cases} \tag{3.10}$$

At this moment, we are going to divide the proof in three cases.

Case 1: $\lambda_* = 0$. From $(3.10)_1$, $(3.10)_3$, $(3.10)_6$ and boundness of z_n, we have

$$\Phi_n \to 0 \text{ in } H^1_*(0,L), \quad \Psi_n \to 0 \text{ in } H^1_0(0,L), \quad \eta_{ns} \to 0 \text{ in } \mathcal{M}_g.$$

Then, using the embeddings $H^1_*(0,L) \hookrightarrow L^2_*(0,L)$, $H^1_0(0,L) \hookrightarrow L^2(0,L)$ and applying Corollary 3.3, we arrive at

$$\Phi_n \to 0 \text{ in } L^2_*(0,L), \quad \Psi_n \to 0 \text{ in } L^2(0,L), \quad \eta_n \to 0 \text{ in } \mathcal{M}_g. \tag{3.11}$$

On the other hand, taking the inner product of $(3.10)_2$ with ϕ_n in $L^2_*(0,l)$, the inner product of $(3.10)_4$ with ψ_n in $L^2(0,l)$ and adding the results, we deduce

$$\kappa\|\phi_{nx} + \psi_n\|^2 + \tilde{b}\|\psi_{nx}\|^2 = (f^2_n - i\lambda_n\rho_1\Phi_n, \phi_n) + (f^4_n - i\lambda_n\rho_2\Psi_n, \psi_n) + (\mathbb{I}_g(\eta_{nx}), \psi_{nx}). \tag{3.12}$$

But, from $(3.10)_2$, $(3.10)_4$ and (3.11), we infer

$$\begin{aligned}
|(f^2_n - i\lambda_n\rho_1\Phi_n, \phi_n)| &\le \|f^2_n - i\lambda_n\rho_1\Phi_n\|\|\phi_n\| \to 0,\\
|(f^4_n - i\lambda_n\rho_2\Psi_n, \psi_n)| &\le \|f^4_n - i\lambda_n\rho_2\Psi_n\|\|\psi_n\| \to 0,\\
|(\mathbb{I}_g(\eta_{nx}), \psi_{nx})| &\le \sqrt{b}\|\eta\|_{\mathcal{M}_g}\|\psi_{nx}\| \to 0.
\end{aligned}$$

Hence, from (3.11) and (3.12) we conclude that $\|z_n\|_{\mathcal{H}} \to 0$, which contradicts $\|z_n\|_{\mathcal{H}} = 1$. This prevents that $\lambda_* = 0$.

Remark 3.5. For the two remaining cases $\lambda_* \in \mathbb{R}\backslash\{0\}$ or $\lambda_* \in \{-\infty, +\infty\}$, we need an additional convergence. Indeed, since we can not transfer the dissipation generated by $\int_0^\infty g'(s)\|\eta_{nx}(s)\|^2\, ds$ to $\|\eta_n\|_{\mathcal{M}_g}$, we restrict the kernel g to a set where we can do it. Indeed, let $\alpha > 0$ such that the set

$$N = \{s \in \mathbb{R}^+, \alpha g'(s) + g(s) < 0\}$$

has positive Lebesgue measure. The existence of such α can be found in [17]. Calling $\tilde{g}(s) := g(s)\chi_N(s)$ and noting that $\mathcal{M}_g \subset \mathcal{M}_{\tilde{g}}$, we have

$$\frac{1}{\alpha}\|\eta_n\|^2_{\mathcal{M}_{\tilde{g}}} \le -\int_0^\infty g'(s)\|\eta_{nx}(s)\|^2\, ds. \tag{3.13}$$

Then, combining (3.9) and (3.13) one gets

$$\frac{1}{2\alpha}\|\eta_n\|^2_{\mathcal{M}_{\tilde{g}}} + \beta\|\theta_{nx}\|^2 \to 0. \tag{3.14}$$

Now we are ready to study the remaining cases. Without loss of generality, we can assume that $\lambda_n \neq 0$ for every $n \in \mathbb{N}$.

Case 2: $\lambda_* \in \mathbb{R}\backslash\{0\}$. We will split this case in several parts.

Part I: $\|\psi_{nx}\|$, $\|\Psi_{nx}\| \to 0$. First, we claim that Ψ_n is bounded in $H^1_0(0,L)$. Indeed, by the triangular inequality

$$\|\Psi_{nx}\| \le \|\Psi_{nx} - i\lambda_n\psi_{nx}\| + |\lambda_n|\|\psi_{nx}\|.$$

The desire boundedness follows from $(3.10)_3$ and from the boundedness of ψ_n in $H_0^1(0,L)$.

From $(3.10)_6$ we obtain the expression for η_n

$$\eta_n(s) = \frac{1}{i\lambda_n}\left(1 - e^{-i\lambda_n s}\right)\Psi_n + \int_0^s e^{-i\lambda_n(s-\tau)} f_n^6(\tau) d\tau. \tag{3.15}$$

Taking the inner product of (3.15) with Ψ_n in $\mathcal{M}_{\tilde{g}}$, we get

$$\left[\int_0^\infty \tilde{g}(s)\left(1 - \cos(\lambda_n s)\right) ds\right] \|\Psi_{nx}\|^2 = \mathrm{Re}\left[i\lambda_n(\eta_n, \Psi_n)_{\mathcal{M}_{\tilde{g}}} + \upsilon_n\right], \tag{3.16}$$

where

$$\upsilon_n = -i\lambda_n \int_0^\infty \tilde{g}(s) \int_0^s e^{-i\lambda_n(s-\tau)}(f_{nx}^6(\tau), \Psi_{nx}) d\tau\, ds.$$

Since $\|\Psi_{nx}\|$ is bounded, we have

$$\left|\lambda_n(\eta_n, \Psi_n)_{\mathcal{M}_{\tilde{g}}}\right| \leq |\lambda_n| \|\Psi_{nx}\| \left(\int_0^\infty \tilde{g}(s)\|\eta_{nx}(s)\| ds\right) \leq \sqrt{b}|\lambda_n| \|\Psi_{nx}\| \|\eta_n\|_{\mathcal{M}_{\tilde{g}}} \to 0.$$

Using again Lemma 3.2 with $p = 2$ and $r = 1$ and the fact $\tilde{g} \leq g$, we get

$$|\upsilon_n| \leq |\lambda_n| \|\Psi_{nx}\| \left(\int_0^\infty g(s) \int_0^s \|f_{nx}^6(\tau)\| d\tau\, ds\right) \leq \frac{\sqrt{4Cb}}{\delta} |\lambda_n| \|\Psi_{nx}\| \|f_n^6\|_{\mathcal{M}_g} \to 0.$$

Now, considering the countable set

$$P := \left\{ s \in N, s = \frac{2j\pi}{\lambda_n} \text{ or } s = \frac{2j\pi}{\lambda_*},\ j, n \in \mathbb{N} \right\}$$

we obtain

$$\int_0^\infty \tilde{g}(s)\left(1 - \cos(\lambda_n s)\right) ds \to \int_{N \setminus P} g(s)\left(1 - \cos(\lambda_* s)\right) ds > 0. \tag{3.17}$$

Here we use the fact that a countable set is a null set. Taking the limit in (3.16) and using the above convergences, we conclude that $\|\Psi_{nx}\| \to 0$. Consequently, the convergence $\|\psi_{nx}\| \to 0$ holds from $(3.10)_3$.

<u>*Part II*</u>: $\|\eta_n\|_{\mathcal{M}_g} \to 0$. From the expression (3.15), we have

$$\|\eta_{nx}(s)\|^2 = |(\eta_{nx}(s), \eta_{nx}(s))| \leq \frac{2}{|\lambda_n|} \|\eta_{nx}(s)\| \|\Psi_{nx}\| + \left(\int_0^s \|f_{nx}^6(\tau)\| d\tau\right) \|\eta_{nx}(s)\|, \tag{3.18}$$

for almost every $s > 0$. Multiplying (3.18) by $g(s)$, integrating the result in $\mathbb{R}^+$ and making use of the Hölder's inequality, we deduce

$$\|\eta_n\|_{\mathcal{M}_g}^2 \leq \frac{2\sqrt{b}}{|\lambda_n|}\|\eta_n\|_{\mathcal{M}_g}\|\Psi_{nx}\| + \left[\int_0^\infty g(s)\left(\int_0^s \|f_{nx}^6(\tau)\|\, d\tau\right)^2 ds\right]^{1/2} \|\eta_n\|_{\mathcal{M}_g}. \tag{3.19}$$

Using Lemma 3.2 with $p = 2$ and $r = 0$, we obtain

$$\int_0^\infty g(s)\left(\int_0^s \|f_{nx}^6(\tau)\|\, d\tau\right)^2 ds \leq \frac{4C}{\delta^2}\|f_n^6\|_{\mathcal{M}_g}^2.$$

Hence, from (3.19) we get

$$\|\eta_n\|_{\mathcal{M}_g}^2 \leq \frac{2\sqrt{b}}{|\lambda_n|}\|\eta_n\|_{\mathcal{M}_g}\|\Psi_{nx}\| + \frac{\sqrt{4C}}{\delta}\|f_n^6\|_{\mathcal{M}_g}\|\eta_n\|_{\mathcal{M}_g}.$$

From Part I, boundedness of η_n in $\mathcal{M}_g$ and $(3.10)_6$, we conclude the desire convergence.

Part III: $\|\phi_{nx} + \psi_n\| \to 0$. Let

$$\omega_n := \tilde{b}\psi_n + \mathbb{I}_g(\eta_n).$$

From Part I and Part II, we get

$$\|\omega_{nx}\| \leq \tilde{b}\|\psi_{nx}\| + \sqrt{b}\|\eta_n\|_{\mathcal{M}_g} \to 0. \tag{3.20}$$

Now, taking the L^2-inner product of $(3.10)_4$ with $\phi_{nx} + \psi_n$ and performing an integration by parts we have

$$i\lambda_n\rho_2(\Psi_n, \phi_{nx} + \psi_n) + \kappa\|\phi_{nx} + \psi_n\|^2 - (\omega_{nx}, (\phi_{nx} + \psi_n)_x) \to 0.$$

From Part II and boundedness of $\phi_{nx} + \psi_n$ in $L^2(0, L)$, the previous convergence reduces to

$$\kappa\|\phi_{nx} + \psi_n\|^2 - (\omega_{nxx}, \phi_{nx} + \psi_n) \to 0. \tag{3.21}$$

On the other hand, taking the L^2-inner product of $(3.10)_2$ with ω_{nx}, using the boundedness of Φ_n in $L^2(0, L)$ and taking into account (3.20), we obtain the following convergence

$$((\phi_{nx} + \psi_n)_x, \omega_{nx}) \to 0. \tag{3.22}$$

Combining (3.21) and (3.22) we conclude that the desire convergence holds.

Part IV: $\|\Phi_n\| \to 0$. Taking the L^2-inner product of $(3.10)_1$ and $(3.10)_2$ with $\rho_1\Phi_n$ and ϕ_n, respectively, and adding the obtained results, we deduce

$$2i\lambda_n\rho_1 \operatorname{Re}(\phi_n, \Phi_n) - \rho_1\|\Phi_n\|^2 - \kappa((\phi_{nx} + \psi_n)_x, \phi_n) \to 0.$$

Taking the real part and performing an integration by parts, we obtain

$$-\rho_1\|\Phi_n\|^2 + \kappa \operatorname{Re}(\phi_{nx} + \psi_n, \phi_{nx}) \to 0. \tag{3.23}$$

Now, recalling $\phi_{nx} + \psi_n$, ψ_n are bounded in $L^2(0, L)$ and noting that

$$\|\phi_{nx}\| \leq \|\phi_{nx} + \psi_n\| + \|\psi_n\|,$$

we conclude that ϕ_{nx} is bounded in $L^2(0, L)$. Then, from Part III we have

$$(\phi_{nx} + \psi_n, \phi_{nx}) \to 0$$

Hence, the desire convergence holds from (3.23).

Collecting the convergences from (3.14) and Part I - Part IV, we conclude again that $\|z_n\|_{\mathcal{H}} \to 0$, which contradicts $\|z_n\|_{\mathcal{H}} = 1$. Therefore, the case $\lambda_* \in \mathbb{R}\backslash\{0\}$ can not happen either.

Case 3: $\lambda_* \in \{-\infty, +\infty\}$. This case will also be divided into several steps.

Part I: $\lambda_n \Psi_n$ is bounded in $H^{-1}(0, L)$. Let $\|\cdot\|_{-1}$ be the usual H^{-1}-norm. From $(3.10)_4$ we have

$$\rho_2 \|\lambda_n \Psi_n\|_{-1} \leq \|f_n^4\|_{-1} + \|\kappa(\phi_{nx} + \psi_n) - \tilde{b}\psi_{nxx} - \mathbb{I}_g(\eta_{nxx})\|_{-1}. \tag{3.24}$$

The first in the right-side of (3.24) is bounded since $L^2(0, L) \hookrightarrow H^{-1}(0, L)$ and $f_n^4 \to 0$ in $L^2(0, L)$. The second one can be estimated as follows

$$\|\kappa(\phi_{nx} + \psi_n) - \tilde{b}\psi_{nxx} - \mathbb{I}_g(\eta_{nxx})\|_{-1} \leq \kappa\|\phi_{nx} + \psi_n\|_{-1} + \tilde{b}\|\psi_{nx}\| + \|\mathbb{I}_g(\eta_{nx})\| \leq d'\|z_n\| = d',$$

for some $d' > 0$. Hence, $\lambda_n \Psi_n$ is bounded in $H^{-1}(0, L)$.

Part II: $\|\Psi_n\| \to 0$ Let us consider $\vartheta_n \in H^2(0, L) \cap H_0^1(0, L)$ a solution of $-\vartheta_{nxx} = \Psi_n$. Then, from (3.15) we get

$$\left[\int_0^\infty \tilde{g}(s)\left(1 - \cos(\lambda_n s)\right)\, ds\right] \|\Psi_n\|^2 = \operatorname{Re}\left[i\lambda_n(\eta_n, \vartheta_n)_{\mathcal{M}_{\tilde{g}}} + \zeta_n\right], \tag{3.25}$$

where

$$\zeta_n := -i\lambda_n \int_0^\infty \tilde{g}(s) \int_0^s e^{-i\lambda_n(s-\tau)}(f_{nx}^6(\tau), \vartheta_{nx})d\tau\, ds.$$

Then, from (3.14) and Part I, we have

$$\left|\lambda_n(\eta_n, \vartheta_n)_{\mathcal{M}_{\tilde{g}}}\right| \leq \|\lambda_n \Psi_n\|_{-1} \left(\int_0^\infty \tilde{g}(s)\|\eta_{nx}(s)\| ds\right) \leq \sqrt{b}\|\lambda_n \Psi_n\|_{-1}\|\eta_n\|_{\mathcal{M}_{\tilde{g}}} \to 0.$$

Applying Lemma 3.2 with $p = 2$ and $r = 1$, and the fact $\tilde{g} \leq g$, we get

$$|\zeta_n| \leq \|\lambda_n \Psi_n\|_{-1} \left(\int_0^\infty g(s) \int_0^s \|f_{nx}^6(\tau)\| d\tau\, ds\right) \leq \frac{\sqrt{4Cb}}{\delta}\|\lambda_n \Psi_n\|_{-1}\|f_n^6\|_{\mathcal{M}_g} \to 0.$$

Using the last two convergences to pass the limit in (3.25), we arrive at

$$\left[\int_0^\infty \tilde{g}(s)\left(1 - \cos(\lambda_n s)\right)\, ds\right] \|\Psi_n\|^2 \to 0.$$

Since $\lambda_* \in \{-\infty, +\infty\}$, we apply the Riemann-Lebesgue Lemma (see [11], Theorem 8.22) to get

$$\int_0^\infty \tilde{g}(s)\left(1-\cos(\lambda_n s)\right)\, ds \to \int_{N/Q} g(s)\, ds > 0, \tag{3.26}$$

where Q is the null set

$$Q := \left\{ s \in N, s = \frac{2j\pi}{\lambda_n},\ j, n \in \mathbb{N} \right\}.$$

Hence, we obtain the desire convergence.

<u>*Part III*</u>: $\|\psi_{nx}\| \to 0$. First, we add and subtract the term $(1-e^{-i\lambda_n s})\psi_n$ in the right side of (3.15) to get

$$\eta_n(s) = \left(1-e^{-i\lambda_n s}\right)\psi_n + \int_0^s e^{-i\lambda_n (s-\tau)} f_n^6(\tau) d\tau - \frac{1}{i\lambda_n}\left(1-e^{-i\lambda_n s}\right) f_n^3. \tag{3.27}$$

Taking the $\mathcal{M}_{\tilde{g}}$-inner product of (3.27) with ψ_n we have

$$\left[\int_0^\infty \tilde{g}(s)\left(1-\cos(\lambda_n s)\right)\, ds\right] \|\psi_{nx}\|^2 = \operatorname{Re}\left[(\eta_n, \psi_n)_{\mathcal{M}_{\tilde{g}}} + \zeta_n^1 + \zeta_n^2\right], \tag{3.28}$$

where

$$\zeta_n^1 := -\int_0^\infty \tilde{g}(s) \int_0^s e^{-i\lambda_n (s-\tau)} (f_{nx}^6(\tau), \psi_{nx}) d\tau\, ds,$$

$$\zeta_n^2 := \frac{1}{i\lambda_n}\left[\int_0^\infty \tilde{g}(s)\left(1-e^{-i\lambda_n s}\right)\, ds\right] (f_{nx}^3, \psi_{nx}).$$

Since $\frac{1}{\lambda_n}$ and $\|\psi_{nx}\|$ are bounded, we deduce

$$\left|(\eta_n, \psi_n)_{\mathcal{M}_{\tilde{g}}}\right| \le \|\psi_{nx}\| \left(\int_0^\infty \tilde{g}(s)\|\eta_{nx}(s)\| ds\right) \le \sqrt{b}\|\psi_{nx}\|\|\eta_n\|_{\mathcal{M}_{\tilde{g}}} \to 0$$

and

$$\left|\zeta_n^2\right| \le \frac{2b}{|\lambda_n|}|(f_{nx}^3, \psi_{nx})| \le \frac{2}{|\lambda_n|}\|\psi_{nx}\|\|f_{nx}^3\| \to 0.$$

Using Lemma 3.2 with $p = 2$ and $r = 1$, and the fact $\tilde{g} \le g$, we get

$$\left|\zeta_n^1\right| \le \|\psi_{nx}\| \left(\int_0^\infty g(s) \int_0^s \|f_{nx}^6(\tau)\| d\tau\, ds\right) \le \frac{\sqrt{4Cb}}{\delta}\|\psi_{nx}\|\|f_n^6\|_{\mathcal{M}_g} \to 0.$$

Passing the limit in (3.28), taking into account the last three convergences and applying (3.26) we obtain the desire convergence.

Part IV: $\|\eta_n\|_{\mathcal{M}_g} \to 0$. From the expression (3.27), we infer

$$\|\eta_{nx}(s)\|^2 \le 2\|\eta_{nx}(s)\|\|\psi_{nx}\| + \left(\int_0^s \|f^6_{nx}(\tau)\|\,d\tau\right)\|\eta_{nx}(s)\| + \frac{2}{|\lambda_n|}\|\eta_{nx}(s)\|\|f^3_{nx}\|, \tag{3.29}$$

for almost every $s > 0$. Proceeding in the same way to obtain Part II of Case 2, we arrive at

$$\|\eta_n\|^2_{\mathcal{M}_g} \le 2\sqrt{b}\|\eta_n\|_{\mathcal{M}_g}\|\psi_{nx}\| + \frac{\sqrt{4C}}{\delta}\|f^6_n\|_{\mathcal{M}_g}\|\eta_n\|_{\mathcal{M}_g} + \frac{2\sqrt{b}}{|\lambda_n|}\|\eta_n\|_{\mathcal{M}_g}\|f^3_{nx}\|.$$

From Part III, $(3.10)_3$, $(3.10)_6$ and boundedness of η_n in $\mathcal{M}_g$, we conclude the desire convergence.

Part V: $\|\phi_{nx}+\psi_n\|, \|\Phi_n\| \to 0$. This is the precise moment where we explore the strength of $(3.10)_5$. Indeed, combining $(3.10)_1$, $(3.10)_3$ and $(3.10)_5$, we have

$$i\lambda_n\rho_3\theta_n - \beta\theta_{nxx} + i\lambda_n\sigma(\phi_{nx}+\psi_n) - \sigma(f^1_{nx} + f^3_n) \to 0. \tag{3.30}$$

Therefore, taking the L^2-inner product of the sequence in (3.30) with $\kappa(\phi_{nx}+\psi_n)$ and using that $\frac{1}{\lambda_n} \to 0$, we get

$$\sigma\kappa\|\phi_{nx}+\psi_n\|^2 + \varepsilon_n \to 0, \tag{3.31}$$

where

$$\varepsilon_n := \rho_3\kappa(\theta_n, \phi_{nx}+\psi_n) + \frac{\beta\kappa}{i\lambda_n}(\theta_{nx}, (\phi_{nx}+\psi_n)_x) - \frac{\sigma\kappa}{i\lambda_n}((f^1_{nx}+f^3_n), \phi_{nx}+\psi_n).$$

From $(3.10)_2$, we can write ε_n as follows

$$\varepsilon_n := \rho_3\kappa(\theta_n, \phi_{nx}+\psi_n) - \beta\rho_1(\theta_{nx}, \Psi_n) - \frac{\beta}{i\lambda_n}(\theta_{nx}, f^2_n) + \frac{\sigma\kappa}{i\lambda_n}((f^1_{nx}+f^3_n), \phi_{nx}+\psi_n).$$

Now, using the boundedness of $\|z_n\|_{\mathcal{H}}$, the convergences (3.14) and (3.10), we deduce

$$|\varepsilon_n| \le \left(\rho_3\kappa\|\theta_n\| + \frac{\sigma\kappa}{|\lambda_n|}\|f^1_{nx}+f^3_n\|\right)\|\phi_{nx}+\psi_n\| + \left(\beta\rho_1\|\Psi_n\| + \frac{\beta}{|\lambda_n|}\|f^2_n\|\right)\|\theta_{nx}\| \to 0.$$

Hence, from (3.31) we obtain the first convergence. The last convergence follows using exactly the same argument of Part V of Case 2.

Collecting the convergences from (3.14) and Part II - Part V, we finally conclude that $\|z_n\|_{\mathcal{H}} \to 0$, which contradicts again $\|z_n\|_{\mathcal{H}} = 1$ and implies in the impossibility of such a case $\lambda_* \in \{-\infty, +\infty\}$.

Therefore, from Cases 1-3 we conclude that (ii) holds true, that is, the semigroup $S(t)$ is exponentially stable in $\mathcal{H}$.

This finishes the proof of Theorem 3.1. □

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