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ORIGINAL PAPER

Arched beams of Bresse type: observability and application in thermoelasticity

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Abstract This is the first paper of a trilogy intended by the authors in what concerns a unified approach to the stability of thermoelastic arched beams of Bresse type under Fourier's law. Our main goal in this starting work is to develop an original observability inequality for conservative Bresse systems with non-constant coefficients. Then, as a powerful application, we prove mathematically that the stability of a partially damped model in thermoelastic Bresse beams is invariant under the boundary conditions. The exponential and optimal polynomial decay rates are addressed. This approach gives a new view on the stability of Bresse systems subject to different boundary conditions as well as it provides an accurate answer for the related issue raised by Liu and Rao (Z. Angew. Math. Phys. 60(1): 54–69, 2009) from both the physical and mathematical points of view.

Keywords Bresse system · Thermoelasticity · Stability · Optimality

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Contents

1 Introduction

In a pioneering work on asymptotic stability for beams of Bresse type, Liu and Rao [24] explored the energy decay rate to following thermoelastic Bresse system

$$\begin{array}{ll}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x & \\
-k_0 l(w_x - l\varphi) + k_2 l\eta = 0 & \text{in } (0,L)\times\mathbb{R}^+,\\
\rho_2\psi_{tt} - b\,\psi_{xx} + k(\varphi_x & \\
+\psi + lw) + k_1\vartheta_x = 0 & \text{in } (0,L)\times\mathbb{R}^+,\\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x & \\
+kl(\varphi_x + \psi + lw) + k_2\eta_x = 0 & \text{in } (0,L)\times\mathbb{R}^+,\\
\rho_3\vartheta_t - \gamma_1\vartheta_{xx} + k_1\psi_{xt} = 0 & \text{in } (0,L)\times\mathbb{R}^+,\\
\rho_4\eta_t - \gamma_2\eta_{xx} + k_2(w_x - l\varphi)_t = 0 & \text{in } (0,L)\times\mathbb{R}^+,
\end{array} \tag{1.1}$$

subject to initial conditions

$$\begin{aligned}&\varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\\&\psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x),\\&w(x,0)=w_0(x),\ w_t(x,0)=w_1(x),\ \vartheta(x,0)=\vartheta_0(x),\\&\eta(x,0)=\eta_0(x),\quad x\in(0,L),\end{aligned}\tag{1.2}$$

and either the full Dirichlet or mixed Dirichlet–Neumann boundary conditions

$$\begin{aligned}\varphi(x,t)&=\psi(x,t)=w(x,t)=\vartheta(x,t)\\&=\eta(x,t)=0,\quad x\in\{0,L\},\ t\geq 0,\end{aligned}\tag{1.3}$$

or

$$\begin{aligned}\varphi(x,t)&=\psi_x(x,t)=w_x(x,t)=\vartheta(x,t)\\&=\eta(x,t)=0,\quad x\in\{0,L\},\ t\geq 0.\end{aligned}\tag{1.4}$$

As highlighted by the authors in [24, Sect. 1], the governing model (1.1) stands for a linear planar, shearable, and flexible thermoelastic beam vibration, which is a special case of networks on flexible thermoelastic beams as structured by Lagnese, Leugering, & Schmidt [22,23]. Moreover, in their works [22,23] the authors derived more general nonlinear thermoelastic flexible beams whose model (1.1) arises as a particular prototype in the linear framework. Accordingly, the physical meaning of the whole problem is described as follows: the coefficients are given by

$$\begin{aligned}&\rho_1=\rho A,\quad \rho_2=\rho I,\quad k=GA,\\&\quad b=EI,\quad k_0=EA,\\&\rho_3,\rho_4=\frac{\rho c}{T_0},\quad \gamma_1,\gamma_2=\frac{1}{T_0},\\&\quad k_1,k_2=\alpha,\quad l=\frac{1}{R},\end{aligned}\tag{1.5}$$

where ρ is the mass density per unit of the reference area, A is the cross-sectional area, I is the second moment of area of the cross section, G is the shear modulus, E is the modulus of elasticity, c is the heat capacity, α is the coefficient of thermal expansion, T_0 is the reference temperature, and R is the curvature ratio in a beam with length $L>0$; the unknown functions φ, ψ, and w are the vertical, shear angle, and longitudinal displacements; η and ϑ are the temperature deviations from the reference temperature T_0 along the longitudinal and vertical directions.

The main results in [24] go around the stability of system (1.1) in terms of the boundary conditions (1.3) or (1.4), and also taking into account the assumption on equal speeds of wave propagation (EWS for short)

$$k=k_0\quad\Longleftrightarrow\quad E=G.\tag{1.6}$$

More precisely, when (1.6) holds, it is proved in [24, Theorem 3.1] that problem (1.1)–(1.2) with both boundary conditions (1.3)–(1.4) is exponentially stable. Otherwise, if (1.6) does not hold, i.e., in the meaningful physical case $E\neq G$, then problem (1.1)–(1.2) is only polynomial stable with decay rate depending on both the regularity of initial data and boundary conditions (1.3)–(1.4), see [24, Theorem 4.1]. To be even more specific, let us exhibit the latter commented result in case $m=1$ (see on p. 64 therein): *If* $E\neq G$, *then the semigroup solution associated with* (1.1)–(1.2) is *semi-uniformly stable*[1] *with the polynomial-type decay rate*

$$\begin{aligned}&Case1:\ \frac{(\ln t)^{5/4}}{t^{1/4}}\\&\quad\text{for}\quad(1.4)\quad\Big(\approx\frac{1}{t^{1/4-\epsilon}}\ \text{ as }\ t\to+\infty,\ \epsilon\ll 1\Big),\\&Case2:\ \frac{(\ln t)^{9/8}}{t^{1/8}}\\&\quad\text{for}\quad(1.3)\quad\Big(\approx\frac{1}{t^{1/8-\epsilon}}\ \text{ as }\ t\to+\infty,\ \epsilon\ll 1\Big).\end{aligned}\tag{1.7}$$

Since then, several authors dealt with other thermoelastic Bresse–Timoshenko systems by showing similar results on polynomial-like stability, namely, obtaining different polynomial decay rates for different boundary conditions, see e.g., [2,15,16,29,30]. On the other hand, it is common to address Bresse systems with only one boundary condition, especially in cases of mixed Dirichlet–Neumann boundary conditions and avoiding the full Dirichlet one. In this direction, we quote the following references addressing thermoelastic Bresse systems with different couplings and laws for the heat flux of conduction [1,10–12,14,18,21,28].

As far as we have noted, the main technical reason to get distinct polynomial decay rates for each boundary condition falls on the fact of dealing with boundary point-wise terms. Indeed, to handle them, the authors obtain weak estimates. For instance, in the proof by contradictions arguments presented in [24, Sect. 4] the convergence of point-wise boundary terms in case (1.3) drives us to a poorer estimate when compared to (1.4) where such point-wise terms vanish. Conse-

[1] Throughout this paper, the notion of *semi-uniform stability* is always invoked when the stability of the semigroup solution does not occur for all weak initial data (say at the same energy level of solutions), but only for more regular initial data, e.g., data in the domain of the infinitesimal generator of the semigroup.

quently, in the case of different speeds of wave propagation $E \neq G$, a slower decay rate arises for (1.3). The same issue happens in [2,15,16,29,30] where the direct proofs employ a one-dimensional version of the Trace Theorem to handle boundary point-wise terms by reflecting directly in worse estimates than desirable for the case of full Dirichlet boundary condition. In addition, we can ask ourselves whether other boundary conditions could be considered in [1,10–12,14,18,21,28] and still provide the same stability results.

However, there is no consistent motive (mostly physical) to get different decay rates for different conservative boundary conditions as in (1.7). Indeed, going back to the notable work by Liu & Rao [24], we stress the following statement in Remark 4.1 therein:

> "*It is interesting to see that the polynomial decay rate depends on the boundary conditions. Although we can't guarantee that our estimate of the decay rate is optimal since we are only verifying sufficient conditions, the reader will see from the following proof that the best l is chosen in order to get a contradiction. However, we do not have a physical explanation why case one has a faster decay rate than case two.*"

In the above statement, the reader can think of cases one and two as in (1.7). This issue boosted the authors to look for a mathematical method where such a physical dichotomy in terms of boundary conditions can be removed in what concerns the stability results for (thermoelastic) Bresse systems.

Therefore, motivated by [24, Remark 4.1] and also by the fact that none of the aforementioned papers worried about this dichotomy between the two different decay rates for different (but still conservative) boundary conditions, our main goal in this first work of the trilogy is to deal with the thermoelastic Bresse system (1.1)–(1.4) by proving that:

I. In the case where the EWS assumption does not hold ($E \neq G$), problem (1.1)–(1.2) has a unified (semi-uniform) polynomial stability with the same decay rate for both boundary conditions (1.3)–(1.4). Moreover, the decay rate is optimal in the specific case of mixed Dirichlet–Neumann boundary condition (1.4), which prevents any kind of uniform stability on this occasion. These facts are fully detailed in Theorems 3.2 and 3.3, Corollary 3.4, and their subsequent proofs in Sect. 3.

II. In the case of assuming the non-physical EWS assumption (1.6), we still prove the already expected (uniform) exponential stability of problem (1.1)–(1.2) with both boundary conditions (1.3)–(1.4). In particular, it leads to the complete characterization of the stability for (1.1)–(1.2) with boundary condition (1.4). These facts are proved in Theorem 3.5 and Corollary 3.6 (also in Sect. 3).

III. In any case concerning the condition (1.6), we clarify in Sect. 4 that both Theorem 3.2 (polynomial stability) and Theorem 3.5 (exponential stability) can be succeeded to any other tangible boundary condition, which reveals the main advantage of our approach, namely, the polynomial (for $k \neq k_0$) and the exponential (for $k = k_0$) stability results hold true independently of the boundary conditions. Other improvements and novelties are also clarified in the concluding Sect. 4.

IV. Last, but not least, to reach the above three purposes, we develop in Sect. 2 (see Proposition 2.2 and Corollary 2.3) a new *Observability Inequality* and *Extension Result* for linear conservative systems of Bresse type by means of the resolvent equation in a more general framework, namely, for the non-homogeneous Bresse systems where the coefficients are positive functions of the spatial variable. This approach allows us to proceed with a localized resolvent analysis in the subsequent thermoelastic problem and, consequently, to obtain the unified stability results in terms of boundary conditions as stated in Sects. 3 and 4.

In conclusion, we highlight that the whole technique explored in this opening paper brings a new view in the stability of thermoelastic Bresse systems, it is different from [24] (and also from other papers in Bresse beams commented before), and finally it can be extended to several other related Bresse systems (not only thermoelastic ones) as we shall see in forthcoming related works.

2 Observability analysis: conservative problem

We start by considering the following conservative non-homogeneous Bresse system

$$\rho_1\varphi_{tt} - (k(\varphi_x + \psi + lw))_x - k_0 l(w_x - l\varphi) = 0 \quad \text{in} \quad (0, L)\times\mathbb{R}^+, \tag{2.1}$$

$$\rho_2\psi_{tt} - (b\,\psi_x)_x + k(\varphi_x + \psi + lw) = 0 \quad \text{in} \quad (0, L)\times\mathbb{R}^+, \tag{2.2}$$

$$\rho_1 w_{tt} - (k_0(w_x - l\varphi))_x + kl(\varphi_x + \psi + lw) = 0 \quad \text{in} \quad (0, L)\times\mathbb{R}^+, \tag{2.3}$$

with initial conditions

$$\begin{aligned} &\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\ &\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\ &w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \\ &x \in (0, L), \end{aligned} \tag{2.4}$$

and either Dirichlet boundary conditions

$$\varphi(x,t) = \psi(x,t) = w(x,t) = 0, \quad x \in \{0, L\},\ t \geq 0, \tag{2.5}$$

or mixed Dirichlet–Neumann boundary conditions

$$\varphi(x,t) = \psi_x(x,t) = w_x(x,t) = 0, \quad x \in \{0, L\},\ t \geq 0. \tag{2.6}$$

The coefficients $\rho_1, \rho_2, k, k_0, b$ are functions satisfying

$$\begin{aligned} &\rho_1, \rho_2, k, k_0, b \in C^1[0, L], \\ &\rho_1, \rho_2, k, k_0, b > 0 \ \text{in} \ [0, L]. \end{aligned} \tag{2.7}$$

In this case, we can rewrite (2.1)–(2.6) in the following Cauchy problem

$$\begin{cases} \frac{d}{dt}V = \mathscr{A}_j V, & t > 0, \\ V(0) = V_0, \end{cases} \tag{2.8}$$

where

$$V := (\varphi, \Phi, \psi, \Psi, w, W), \quad \Phi := \varphi_t, \quad \Psi := \psi_t, \quad W := w_t,$$

and for $j = 1, 2$,

$$\mathscr{A}_j V = \begin{bmatrix} \Phi \\ \frac{1}{\rho_1}(k(\varphi_x + \psi + lw))_x + \frac{k_0 l}{\rho_1}(w_x - l\varphi) \\ \Psi \\ \frac{1}{\rho_2}(b\,\psi_x)_x - \frac{k}{\rho_2}(\varphi_x + \psi + lw) \\ W \\ \frac{1}{\rho_1}(k_0(w_x - l\varphi))_x - \frac{kl}{\rho_1}(\varphi_x + \psi + lw) \end{bmatrix}^T,$$

$$V_0 = \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \psi_0 \\ \psi_1 \\ w_0 \\ w_1 \end{bmatrix}^T.$$

In order to simplify the notations on functions spaces along the text, we denote

$$L^2 := L^2(0, L), \ H_0^1 := H_0^1(0, L), \quad L_*^2 := L_*^2(0, L), \ H_*^1 := H_*^1(0, L),$$

with standard scalar products and norms, where

$$L_*^2(0, L) = \left\{u \in L^2;\ \frac{1}{L}\int_0^L u(x)\,dx = 0\right\} \ \text{and} \quad H_*^1 = H^1 \cap L_*^2.$$

Under the above notations, we set the Hilbert spaces

$$\mathscr{H}_1 = H_0^1 \times L^2 \times H_0^1 \times L^2 \times H_0^1 \times L^2 \quad \text{for} \quad (2.5),$$

and

$$\mathscr{H}_2 = H_0^1 \times L^2 \times H_*^1 \times L_*^2 \times H_*^1 \times L_*^2 \quad \text{for} \quad (2.6),$$

and then the domain of the operator $\mathscr{A}_j$ is given by

$$D(\mathscr{A}_1) = \left\{U \in \mathscr{H}_1 \ :\ \varphi, \psi, w \in H^2 \cap H_0^1;\ \Phi, \Psi, W \in H_0^1\right\} \ \text{for} \ (2.5),$$

and

$$D(\mathscr{A}_2) = \left\{U \in \mathscr{H}_2 \ :\ \varphi \in H^2;\ \Phi, \psi_x, w_x \in H_0^1;\ \Psi, W \in H_*^1\right\} \ \text{for} \ (2.6).$$

In what follows, we are going to provide the desired observability inequality by means of the resolvent equation corresponding to the conservative problem (2.8). As a consequence, we have an extension result that will be very useful in applications.

To accomplish the above purpose, let us consider the resolvent equation

$$i\beta V - \mathscr{A}_j V = G, \tag{2.9}$$

for $\beta \in \mathbb{R}$ and $G = (g_1, g_2, g_3, g_4, g_5, g_6)$, where we take

$$\begin{cases} (g_1, g_2, g_3, g_4, g_5, g_6) \in \mathscr{H}_1 & \text{for} \quad (2.5), \\ (g_1, g_2, g_3, g_4, g_5, g_6) \in \mathscr{H}_2 & \text{for} \quad (2.6). \end{cases} \tag{2.10}$$

Under the conditions (2.7) and (2.10), it is relatively simple to show that (2.9) has a unique solution $V \in D(\mathscr{A}_j)$, $j = 1, 2$. To this end, the Lax–Milgram theorem and elliptic regularity can be easily employed.

Now, in order to proceed with the necessary computations, let us convert the resolvent Eq. (2.9) in terms of its components as follows

$$i\beta\varphi - \Phi = g_1 \quad \text{in} \quad (0, L), \tag{2.11}$$
$$i\beta\rho_1\Phi - (k(\varphi_x + \psi + lw))_x - k_0 l(w_x - l\varphi) = g_2 \quad \text{in} \quad (0, L), \tag{2.12}$$
$$i\beta\psi - \Psi = g_3 \quad \text{in} \quad (0, L), \tag{2.13}$$
$$i\beta\rho_2\Psi - (b\psi_x)_x + k(\varphi_x + \psi + lw) = g_4 \quad \text{in} \quad (0, L), \tag{2.14}$$
$$i\beta w - W = g_5 \quad \text{in} \quad (0, L), \tag{2.15}$$
$$i\beta\rho_1 W - (k_0(w_x - l\varphi))_x + kl(\varphi_x + \psi + lw) = g_6 \quad \text{in} \quad (0, L), \tag{2.16}$$

Additionally, given any $0 \le a_1 < a_2 \le L$, the notation $\|\cdot\|_{a_1, a_2}$ stands for

$$\|V\|^2_{a_1, a_2} := \int_{a_1}^{a_2} \Big(|\varphi_x + \psi + lw|^2 + |\Phi|^2 + |\psi_x|^2 + |\Psi|^2 + |w_x - l\varphi|^2 + |W|^2 \Big)\, dx,$$

and for $j = 1, 2$, we set

$$I(a_j) := |(\varphi_x + \psi + lw)(a_j)|^2 + |\Phi(a_j)|^2 + |\psi_x(a_j)|^2 + |\Psi(a_j)|^2 + |(w_x - l\varphi)(a_j)|^2 + |W(a_j)|^2.$$

We are finally in position to state and prove the main result of this section. For the sake of didactic reasons, we consider the previous technical lemma which will be used in the proof of the main result.

Lemma 2.1 *Let $q \in C^1[a_1, a_2]$ be a function given by*

$$q(x) = \gamma(x) \int_{a_1}^{x} e^{n\tau}\, d\tau, \quad n \in \mathbb{N},$$

where $\gamma \in C^1[a_1, a_2]$ satisfies $\gamma_0 \le \gamma(x) \le \gamma_1$ for all $x \in [a_1, a_2]$ with $0 < \gamma_0 < \gamma_1$. Then,

$$q'(x) \ge \frac{1}{2}\gamma_0 e^{nx}, \tag{2.17}$$

for all $x \in [a_1, a_2]$ and n large enough.

Proof Trivial. □

Proposition 2.2 *(Observability Inequality) Under the conditions (2.7) and (2.10), let $V = (\varphi, \Phi, \psi, \Psi, w, W)$ be a solution of (2.9). Then, for any numbers $0 \le a_1 < a_2 \le L$, there exist universal constants $C_0, C_1 > 0$ (depending only on $\rho_1, \rho_2, k, k_0, b, l$) such that*

$$I(a_j) \le C_0\|V\|^2_{a_1, a_2} + C_0\|G\|^2_{0, L}, \quad j = 1, 2, \tag{2.18}$$
$$\|V\|^2_{a_1, a_2} \le C_1 I(a_j) + C_1\|G\|^2_{0, L}, \quad j = 1, 2, \tag{2.19}$$

by taking $|\beta| > 1$ large enough.

Proof The proof will be done in three steps as follows. *Step 1. A crucial identity*. Let us start by fixing three functions $q_1, q_2, q_3 \in C^1[a_1, a_2]$.

Initially, taking the multiplier $q_1 k\overline{(\varphi_x + \psi + lw)}$ in (2.12) and integrating on (a_1, a_2), we get

$$\begin{aligned}
\int_{a_1}^{a_2} q_1 k g_2 \overline{(\varphi_x + \psi + lw)}\, dx
&= \underbrace{-\int_{a_1}^{a_2} q_1\rho_1 k\, \Phi\overline{(i\beta(\varphi_x + \psi + lw))}\, dx}_{:=J_1} \\
&\quad \underbrace{-\int_{a_1}^{a_2} q_1(k(\varphi_x + \psi + lw))_x\overline{(k(\varphi_x + \psi + lw))}\, dx}_{:=J_2} \\
&\quad -\int_{a_1}^{a_2} q_1 k_0 l k(w_x - l\varphi)\overline{(\varphi_x + \psi + lw)} dx.
\end{aligned} \tag{2.20}$$

Using Eqs. (2.11), (2.13), and (2.15), integration by parts and taking the real part of J_1 and J_2, it follows that

$$\begin{aligned}
\operatorname{Re} J_1 = &-\frac{1}{2}q_1\rho_1 k\,|\Phi|^2\Big|_{a_1}^{a_2} + \frac{1}{2}\int_{a_1}^{a_2}(q_1\rho_1 k)_x|\Phi|^2\, dx \\
&- \operatorname{Re}\int_{a_1}^{a_2} q_1\rho_1 k\, \Phi\overline{(\Psi + lW)}\, dx \\
&- \operatorname{Re}\int_{a_1}^{a_2} q_1\rho_1 k\, \Phi\overline{(g_{1,x} + g_3 + lg_5)}\, dx,
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Re} J_2 = &-\frac{1}{2}q_1 k^2|\varphi_x + \psi + lw|^2\Big|_{a_1}^{a_2} \\
&+\frac{1}{2}\int_{a_1}^{a_2} q_{1,x}k^2|\varphi_x + \psi + lw|^2\, dx.
\end{aligned}$$

Then, taking the real part of (2.20) we obtain,

$$\begin{aligned}
&-\frac{1}{2}\big(q_1\rho_1 k|\Phi|^2 + q_1 k^2|\varphi_x + \psi + lw|^2\big)\Big|_{a_1}^{a_2} \\
&\quad + \frac{1}{2}\int_{a_1}^{a_2}\big((q_1\rho_1 k)_x|\Phi|^2 + q_{1,x}k^2|\varphi_x + \psi + lw|^2\big)dx \\
&= \operatorname{Re}\int_{a_1}^{a_2} q_1 k g_2\overline{(\varphi_x + \psi + lw)}dx \\
&\quad + \operatorname{Re}\int_{a_1}^{a_2} q_1\rho_1 k\Phi\overline{(g_{1,x} + g_3 + lg_5)}dx \\
&\quad + \operatorname{Re}\int_{a_1}^{a_2} q_1\rho_1 k\Phi\overline{(\Psi + lW)}dx \\
&\quad + \operatorname{Re}\int_{a_1}^{a_2} q_1 k_0 l k(w_x - l\varphi)\overline{(\varphi_x + \psi + lw)}dx.
\end{aligned} \tag{2.21}$$

Secondly, multiplying (2.14) by $q_2 b\, \overline{\psi_x}$ and integrating on (a_1, a_2), we have

$$\begin{aligned}
&\int_{a_1}^{a_2} q_2 b\, g_4 \overline{\psi_x}\, dx\\
&= \underbrace{-\int_{a_1}^{a_2} q_2\rho_2 b\, \Psi \overline{(i\beta\psi_x)}\, dx}_{:=J_3} \underbrace{-\int_{a_1}^{a_2} q_2 (b\,\psi_x)_x \overline{(b\,\psi_x)}\, dx}_{:=J_4}\\
&\underbrace{+\int_{a_1}^{a_2} q_2 bk\,(\varphi_x+\psi+lw)\overline{\psi_x}\, dx}_{:=J_5}. \qquad (2.22)
\end{aligned}$$

Then, using Eq. (2.13) and integrating by parts J_3 and J_4, yields

$$\begin{aligned}
\mathrm{Re}\, J_3 = &-\frac{1}{2} q_2\rho_2 b\, |\Psi|^2\Big|_{a_1}^{a_2}\\
&+\frac{1}{2}\int_{a_1}^{a_2} (q_2\rho_2 b)_x |\Psi|^2\, dx\\
&-\mathrm{Re}\int_{a_1}^{a_2} q_2\rho_2 b\, \Psi \overline{g_{3,x}}\, dx
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Re}\, J_4 = &-\frac{1}{2} q_2 b^2 |\psi_x|^2\Big|_{a_1}^{a_2}\\
&+\frac{1}{2}\int_{a_1}^{a_2} q_{2,x} b^2 |\psi_x|^2\, dx.
\end{aligned}$$

In addition, integration by parts J_5 and using Eqs. (2.12) and (2.13), one has

$$\begin{aligned}
\mathrm{Re}\, J_5 = \mathrm{Re} &\left(q_2 bk\,(\varphi_x+\psi+lw)\overline{\psi}\Big|_{a_1}^{a_2}\right)\\
&+\frac{1}{\beta}\mathrm{Im}\int_{a_1}^{a_2} (q_2 b)_x k(\varphi_x+\psi+lw)\overline{g_3}\, dx\\
&+\frac{1}{\beta}\mathrm{Im}\int_{a_1}^{a_2} (q_2 b)_x k(\varphi_x+\psi+lw)\overline{\Psi}\, dx\\
&+\mathrm{Re}\int_{a_1}^{a_2} q_2 b g_2 \overline{\psi}\, dx\\
&-\frac{1}{\beta}\mathrm{Im}\int_{a_1}^{a_2} q_2 bk_0 l(w_x-l\varphi)\overline{g_3}\, dx\\
&-\frac{1}{\beta}\mathrm{Im}\int_{a_1}^{a_2} q_2 bk_0 l(w_x-l\varphi)\overline{\Psi}\, dx\\
&+\mathrm{Re}\int_{a_1}^{a_2} q_2 b\rho_1 \Phi \overline{g_3}\, dx\\
&+\mathrm{Re}\int_{a_1}^{a_2} q_2 b\rho_1 \Phi \overline{\Psi}\, dx.
\end{aligned}$$

Returning to (2.22), taking its real part and replacing these last three equalities, we deduce

$$\begin{aligned}
&-\frac{1}{2}\Big(q_2\rho_2 b\, |\Psi|^2 + q_2 b^2 |\psi_x|^2\Big)\Big|_{a_1}^{a_2}\\
&\quad+\frac{1}{2}\int_{a_1}^{a_2} \Big((q_2\rho_2 b)_x |\Psi|^2 + q_{2,x} b^2 |\psi_x|^2\Big)\, dx\\
&= \mathrm{Re}\int_{a_1}^{a_2} q_2 b g_4 \overline{\psi_x}\, dx + \mathrm{Re}\int_{a_1}^{a_2} q_2\rho_2 b \Psi \overline{g_{3,x}}\, dx\\
&\quad-\mathrm{Re}\Big(q_2 bk(\varphi_x+\psi+lw)\overline{\psi}\Big)\Big|_{a_1}^{a_2}\\
&\quad-\frac{1}{\beta}\mathrm{Im}\int_{a_1}^{a_2} (q_2 b)_x k(\varphi_x+\psi+lw)\overline{g_3}\, dx\\
&\quad-\frac{1}{\beta}\mathrm{Im}\int_{a_1}^{a_2} (q_2 b)_x k(\varphi_x+\psi+lw)\overline{\Psi}\, dx\\
&-\mathrm{Re}\int_{a_1}^{a_2} q_2 b g_2 \overline{\psi}\, dx + \frac{1}{\beta}\mathrm{Im}\int_{a_1}^{a_2} q_2 bk_0 l(w_x-l\varphi)\overline{g_3}\, dx\\
&\quad-\mathrm{Re}\int_{a_1}^{a_2} q_2\rho_1 b \Phi \overline{g_3}\, dx\\
&\quad+\frac{1}{\beta}\mathrm{Im}\int_{a_1}^{a_2} q_2 bk_0 l(w_x-l\varphi)\overline{\Psi}\, dx\\
&\quad-\mathrm{Re}\int_{a_1}^{a_2} q_2\rho_1 b \Phi \overline{\Psi}\, dx. \qquad (2.23)
\end{aligned}$$

Third, taking the multiplier $q_3 k_0 \overline{(w_x - l\varphi)}$ in (2.16) and integrating on (a_1, a_2), we get

$$\begin{aligned}
&\int_{a_1}^{a_2} q_3 k_0 g_6 \overline{(w_x-l\varphi)}\, dx\\
&= \underbrace{-\int_{a_1}^{a_2} q_3\rho_1 k_0 W \overline{(i\beta(w_x-l\varphi)}\, dx}_{:=J_6}\\
&\quad \underbrace{-\int_{a_1}^{a_2} q_3 (k_0(w_x-l\varphi))_x \overline{(k_0(w_x-l\varphi))}\, dx}_{:=J_7}\\
&\quad+\int_{a_1}^{a_2} q_3 k_0 lk(\varphi_x+\psi+lw)\overline{(w_x-l\varphi)}\, dx.
\end{aligned} \qquad (2.24)$$

Using Eqs. (2.11) and (2.15), integration by parts and taking the real part of J_6 and J_7, it follows that

$$\mathrm{Re}\, J_6 = -\frac{1}{2} q_3\rho_1 k_0\, |W|^2\Big|_{a_1}^{a_2}$$

$$+\frac{1}{2}\int_{a_1}^{a_2}(q_3\rho_1 k_0)_x|W|^2\,dx$$
$$+\operatorname{Re}\int_{a_1}^{a_2} q_3\rho_1 k_0 l\, W\overline{\Phi}\,dx$$
$$-\operatorname{Re}\int_{a_1}^{a_2} q_3\rho_1 k_0\, W\overline{(g_{5,x}-lg_1)}\,dx,$$

and

$$\operatorname{Re} J_7 = -\frac{1}{2}q_3 k_0^2|w_x-l\varphi|^2\Big|_{a_1}^{a_2} + \frac{1}{2}\int_{a_1}^{a_2} q_{3,x}k_0^2|w_x-l\varphi|^2\,dx.$$

Then, taking the real part of (2.24) we obtain,

$$\begin{aligned}
&-\frac{1}{2}\left(q_3\rho_1 k_0|W|^2+q_3k_0^2|w_x-l\varphi|^2\right)\Big|_{a_1}^{a_2}\\
&\quad+\frac{1}{2}\int_{a_1}^{a_2}\left((q_3\rho_1k_0)_x|W|^2+q_{3,x}k_0^2|w_x-l\varphi|^2\right)dx\\
&= \operatorname{Re}\int_{a_1}^{a_2} q_3k_0g_6\overline{(w_x-l\varphi)}\,dx-\operatorname{Re}\int_{a_1}^{a_2}q_3\rho_1k_0lW\overline{\Phi}\,dx\\
&\quad+\operatorname{Re}\int_{a_1}^{a_2}q_3\rho_1k_0lW\overline{(g_{5,x}-lg_1)}\,dx\\
&\quad-\operatorname{Re}\int_{a_1}^{a_2}q_3k_0lk(\varphi_x+\psi+lw)\overline{(w_x-l\varphi)}\,dx.
\end{aligned}\tag{2.25}$$

Finally, combining the identities (2.21), (2.23), and (2.25), we arrive at

$$\begin{aligned}
&\int_{a_1}^{a_2}\Big(q_{1,x}k^2|\varphi_x+\psi+lw|^2+(q_1\rho_1k)_x|\Phi|^2\\
&\quad+q_{2,x}b^2|\psi_x|^2+(q_2\rho_2b)_x|\Psi|^2\Big)\,dx\\
&\quad+\int_{a_1}^{a_2}\left(q_{3,x}k_0^2|w_x-l\varphi|^2+(q_3\rho_1k_0)_x|W|^2\right)dx\\
&=\Big(q_1k^2|\varphi_x+\psi+lw|^2+q_1\rho_1k\,|\Phi|^2+q_2b^2|\psi_x|^2\\
&\quad+q_2\rho_2b\,|\Psi|^2\Big)\Big|_{a_1}^{a_2}\\
&\quad+\left(q_3k_0^2|w_x-l\varphi|^2+q_3\rho_1k_0|W|^2\right)\Big|_{a_1}^{a_2}\\
&\quad+P(a_1,a_2)+J_{10}+J_{11}+J_{12}+J_{13}
\end{aligned}\tag{2.26}$$

for any $q_1, q_2, q_3\in C^1[a_1,a_2]$, which denote

$$P(a_1,a_2)=-2\operatorname{Re}\left(q_2bk\,(\varphi_x+\psi+lw)\overline{\psi}\Big|_{a_1}^{a_2}\right),$$

$$\begin{aligned}
J_{10} &= 2\operatorname{Re}\int_{a_1}^{a_2}(q_1\rho_1k-q_2\rho_1b)\Phi\overline{\Psi}\,dx\\
&\quad+2\operatorname{Re}\int_{a_1}^{a_2}l(q_1\rho_1k-q_3\rho_1k_0)W\overline{\Phi}\,dx,\\
J_{11} &= -\frac{2}{\beta}\operatorname{Im}\int_{a_1}^{a_2}(q_2b)_xk(\varphi_x+\psi+lw)\overline{\Psi}\,dx\\
&\quad+\frac{2}{\beta}\operatorname{Im}\int_{a_1}^{a_2}q_2bk_0l(w_x-l\varphi)\overline{\Psi}\,dx,\\
J_{12} &= -\frac{2}{\beta}\operatorname{Im}\int_{a_1}^{a_2}(q_2b)_xk(\varphi_x+\psi+lw)\overline{g_3}\,dx\\
&\quad+\frac{2}{\beta}\operatorname{Im}\int_{a_1}^{a_2}q_2bk_0l(w_x-l\varphi)\overline{g_3}\,dx\\
&\quad+2\operatorname{Re}\int_{a_1}^{a_2}q_3\rho_1k_0W\overline{(g_{5,x}-lg_1)}\,dx\\
&\quad+2\operatorname{Re}\int_{a_1}^{a_2}q_1\rho_1k\Phi\overline{(g_{1,x}+g_3+lg_5)}\,dx\\
&\quad+2\operatorname{Re}\int_{a_1}^{a_2}q_3k_0g_6\overline{(w_x-l\varphi)}\,dx\\
&\quad+2\operatorname{Re}\int_{a_1}^{a_2}q_1kg_2\overline{(\varphi_x+\psi+lw)}\,dx\\
&\quad+2\operatorname{Re}\int_{a_1}^{a_2}q_2b\left(g_4\overline{\psi_x}+\rho_2\Psi\overline{g_{3,x}}\right.\\
&\quad\left.-\rho_1\Phi\overline{g_3}-g_2\overline{\psi}\right)dx,\\
J_{13} &= 2\operatorname{Re}\int_{a_1}^{a_2}k_0lk(q_1-q_3)\\
&\quad(\varphi_x+\psi+lw)\overline{(w_x-l\varphi)}\,dx.
\end{aligned}$$

Step 2. Conclusion of (2.18)-(2.19) *for* $j=2$. Since (2.26) holds true for any $q_1, q_2, q_3\in C^1[a_1,a_2]$, let us choose them so that

$$(q_1k)(x)=(q_2b)(x)=(q_3k_0)(x)=\int_{a_1}^{x}e^{n\tau}\,d\tau,$$

for $x\in[a_1,a_2]$ and $n\in\mathbb{N}$ to be determined later. Thus, we prompt to have that $J_{10}=0$. Let us estimate the remaining terms in (2.26). Indeed, from (2.7) and Hölder's inequality, there exists a constant $C_n>0$ such that

$$|J_{11}|\le\frac{C_n}{|\beta|}\|V\|^2_{a_1,a_2}\quad\text{and}\quad |J_{12}|\le C_n\|V\|_{a_1,a_2}\|G\|_{0,L}.\tag{2.27}$$

Using (2.13), Hölder and Young inequalities and the embedding $H^1(a_1,a_2)\hookrightarrow L^\infty(a_1,a_2)$, one sees that

$$|P(a_1,a_2)|\le\frac{C_n}{|\beta|}|(\varphi_x+\psi+lw)(a_2)|^2$$

$$+\frac{C_n}{|\beta|}|\Psi(a_2)|^2 + C_n\|G\|^2_{0,\,L}. \tag{2.28}$$

Now, observing that

$$(q_1 - q_3)(x) = \frac{(k_0 - k)(x)}{(k_0k)(x)}\left(\frac{e^{nx} - e^{na_1}}{n}\right),$$

we infer

$$[(lk_0k)(q_1 - q_3)](x) = [l(k_0 - k)](x)\left(\frac{e^{nx} - e^{na_1}}{n}\right).$$

Then, from (2.7) and Young's inequality, there exists a constant $M > 0$ such that

$$|J_{13}| \leq \frac{M}{n}\int_{a_1}^{a_2} e^{nx}\left(|\varphi_x + \psi + lw|^2 + |w_x - l\varphi|^2\right)dx. \tag{2.29}$$

Replacing (2.27)-(2.29) in (2.26), using (2.7) and (2.17), taking n large enough satisfying Lemma 2.1 for the specific functions present in (2.26), there exist constants $C, \alpha_0 > 0$ such that

$$\begin{aligned}\frac{1}{2}\alpha_0 \int_{a_1}^{a_2} e^{nx}&\left(|\varphi_x + \psi + lw|^2 + |\Phi|^2\right.\\ &\left.+|\psi_x|^2 + |\Psi|^2 + |w_x - l\varphi|^2 + |W|^2\right)dx\\ \leq\ & CI(a_2) + \frac{C}{|\beta|}\|V\|^2_{a_1,\,a_2} + C\|G\|^2_{0,\,L}\\ &+ C\|V\|_{a_1,\,a_2}\|G\|_{0,\,L}\\ &+ \frac{M}{n}\int_{a_1}^{a_2} e^{nx}\left(|\varphi_x + \psi + lw|^2 + |w_x - l\varphi|^2\right)dx.\end{aligned}$$

Again, taking $n_0 \in \mathbb{N}$ large enough satisfying Lemma 2.1 and such that

$$\frac{1}{2}\alpha_0 - \frac{M}{n_0} > 0,$$

there exists a constant $C > 0$ such that

$$\begin{aligned}Ce^{n_0a_1}\|V\|^2_{a_1,\,a_2} \leq\ & CI(a_2) + \frac{C}{|\beta|}\|V\|^2_{a_1,\,a_2}\\ &+C\|V\|_{a_1,\,a_2}\|G\|_{0,\,L} + C\|G\|^2_{0,\,L}.\end{aligned}$$

Considering $|\beta| > 1$ large enough and using Young's inequality with $\varepsilon > 0$, there exist a constant $C_1 > 0$ such that

$$\|V\|_{a_1,\,a_2} \leq C_1 I(a_2) + C_1\|G\|^2_{0,\,L},$$

concluding (2.19) for $j = 2$.

To conclude (2.18) for $j = 2$, we recall again the identity (2.26) and, in view of the estimates (2.27)-(2.29) along with the assumption (2.7), there exist a constant $C > 0$ such that

$$\begin{aligned}I(a_2) \leq\ & C\|V\|^2_{a_1,\,a_2} + \frac{C}{|\beta|}|(\varphi_x + \psi + lw)(a_2)|^2\\ &+ \frac{C}{|\beta|}|\Psi(a_2)|^2 + C\|G\|^2_{0,\,L}\\ &+ \frac{C}{|\beta|}\|V\|^2_{a_1,\,a_2} + C\|V\|_{a_1,\,a_2}\|G\|_{0,\,L}\\ &+ Ce^{n_0a_2}\|V\|^2_{a_1,\,a_2},\end{aligned}$$

with $n_0 \in \mathbb{N}$ taken previously. Taking $|\beta| > 1$ large enough and using Young's inequality, there exist a constant $C_0 > 0$ such that

$$I(a_2) \leq C_0\|V\|^2_{a_1,\,a_2} + C_0\|G\|^2_{0,\,L}.$$

This concludes (2.18) $j = 2$.

Step 3. Conclusion of (2.18)-(2.19) *for* $j = 1$. The proof is similar to the case $j = 2$ with minor changes. In fact, in this case, we initially pick up q_1, q_2, and q_3 given by

$$(q_1k)(x) = (q_2b)(x) = (q_3k_0)(x) = -\int_x^{a_2} e^{-n\tau}\,d\tau,$$

for $x \in [a_1, a_2]$ and $n \in \mathbb{N}$. Thus, we still have $J_{10} = 0$ and the estimates (2.27)-(2.29) follow analogously. Therefore, going back to (2.26) and proceeding similarly as above, the estimates (2.18)-(2.19) can be concluded for $j = 1$.

The proof of Proposition 2.2 is complete. □

Remark 2.1 Taking a closer look at the proof of (2.18)-(2.19), one sees that they are proved without any requirement on the boundary conditions for the displacements φ, ψ, and w. Thus, the choices of the boundary conditions (2.5) and (2.6) are simply for reasons of compatibility with the thermoelastic problem in the next section and future works on the subject. In conclusion, Proposition 2.2 can be considered with any other boundary condition concerning the conservative Bresse system provided that (2.9) is satisfied.

The following extension result is a direct consequence of Proposition 2.2. It will be quite useful later in order to recover global estimates in the applications.

Corollary 2.3 *(Extension Result) Under the conditions of Proposition 2.2, let $V = (\varphi, \Phi, \psi, \Psi, w, W)$ be a solution of (2.9). If for some sub-interval $(b_1, b_2) \subset (0, L)$ one has*

$$\|V\|^2_{b_1,b_2} \leq \Lambda, \quad \textit{for some parameter} \quad \Lambda = \Lambda(V,\ G,\ \beta), \tag{2.30}$$

then there exists a (universal) constant $C > 0$ such that

$$\|V\|^2_{0,\,L} \leq C\Lambda + C\|G\|^2_{0,\,L}. \tag{2.31}$$

Proof From (2.18) and (2.30), in particular for (b_1, b_2), we have

$$I(b_j) \leq C_0\Lambda + C_0\|G\|^2_{0,\,L}, \quad j = 1, 2. \tag{2.32}$$

Using (2.19) with $a_1 = 0$, $a_2 = b_2$ and (2.32) with $j = 2$, we obtain

$$\int_0^{b_2} \Big(|\varphi_x + \psi + lw|^2 + |\Phi|^2 + |\psi_x|^2 + |\Psi|^2 + |w_x - l\varphi|^2 + |W|^2\Big)\, dx \leq C_2\,\Lambda + C_2\|G\|^2_{0,\,L}, \tag{2.33}$$

where $C_2 = C_1C_0 + C_1 > 0$. Analogously, using (2.19) with $a_1 = b_2$, $a_2 = L$ and (2.32) with $j = 2$, we also obtain

$$\int_{b_2}^{L} \Big(|\varphi_x + \psi + lw|^2 + |\Phi|^2 + |\psi_x|^2 + |\Psi|^2 + |w_x - l\varphi|^2 + |W|^2\Big)\, dx \leq C_2\,\Lambda + C_2\|G\|^2_{0,\,L}. \tag{2.34}$$

Therefore, adding (2.33) and (2.34), there exists a constant $C > 0$ such that

$$\|V\|^2_{0,\,L} \leq C\Lambda + C\|G\|^2_{0,\,L},$$

which completes the proof of (2.31). □

3 Asymptotic stability: thermoelastic system

In this section, we provide the stability analysis to the partially damped thermoelastic Bresse system (1.1)-(1.4). In order to work in a slightly more general framework, we do not consider necessarily $\rho_3 = \rho_4$, $\gamma_1 = \gamma_2$, and $k_1 = k_2$ as in (1.5). As we shall see in the computations, they can be constants assuming different values. To emphasize this fact, let us rewrite thermoelastic Bresse problem again as follows:

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + k_2 l\eta = 0 \quad \text{in} \quad (0, L) \times \mathbb{R}^+, \tag{3.1}$$

$$\rho_2\psi_{tt} - b\,\psi_{xx} + k(\varphi_x + \psi + lw) + k_1\vartheta_x = 0 \quad \text{in} \quad (0, L) \times \mathbb{R}^+, \tag{3.2}$$

$$\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + k_2\eta_x = 0 \quad \text{in} \quad (0, L) \times \mathbb{R}^+, \tag{3.3}$$

$$\rho_3\vartheta_t - \gamma_1\vartheta_{xx} + k_1\psi_{xt} = 0 \quad \text{in} \quad (0, L) \times \mathbb{R}^+, \tag{3.4}$$

$$\rho_4\eta_t - \gamma_2\eta_{xx} + k_2(w_x - l\varphi)_t = 0 \quad \text{in} \quad (0, L) \times \mathbb{R}^+, \tag{3.5}$$

with initial conditions

$$\begin{aligned} &\varphi(x, 0) = \varphi_0(x),\ \ \varphi_t(x, 0) = \varphi_1(x),\\ &\psi(x, 0) = \psi_0(x),\ \ \psi_t(x, 0) = \psi_1(x),\\ &w(x, 0) = w_0(x),\ \ w_t(x, 0) = w_1(x),\\ &\vartheta(x, 0) = \vartheta_0(x),\\ &\eta(x, 0) = \eta_0(x),\ \ x \in (0, L), \end{aligned} \tag{3.6}$$

and either Dirichlet boundary conditions

$$\varphi(x, t) = \psi(x, t) = w(x, t) = \vartheta(x, t) = \eta(x, t) = 0, \quad x \in \{0, L\},\ t \geq 0, \tag{3.7}$$

or mixed Dirichlet–Neumann boundary conditions

$$\varphi(x, t) = \psi_x(x, t) = w_x(x, t) = \vartheta(x, t) = \eta(x, t) = 0, \quad x \in \{0, L\},\ t \geq 0, \tag{3.8}$$

where physical meaning of the coefficients $\rho_1, \rho_2, \rho_3, \rho_4, k, b, k_0, \gamma_1, \gamma_2, k_1, k_2, l > 0$ are given in the introduction.

3.1 Semigroup setting

Let us initially consider the Hilbert phase spaces

$$\mathcal{H}_1 = H^1_0 \times L^2 \times H^1_0 \times L^2 \times H^1_0 \times L^2 \times L^2 \times L^2 \quad \text{for} \quad (3.7),$$

and

$$\mathcal{H}_2 = H^1_0 \times L^2 \times H^1_* \times L^2_* \times H^1_* \times L^2_* \times L^2 \times L^2 \quad \text{for} \quad (3.8),$$

with inner product

$$(U, U^*)_{\mathcal{H}_j} = \int_0^L \Big[\rho_1\Phi\overline{\Phi^*} + \rho_2\Psi\overline{\Psi^*} + \rho_1 W\overline{W^*} + b\psi_x\overline{\psi^*_x}$$

$$+k(\varphi_x+\psi+lw)\overline{(\varphi_x^*+\psi^*+lw^*)}$$
$$+k_0(w_x-l\varphi)\overline{(w_x^*-l\varphi^*)}$$
$$+\rho_3\vartheta\overline{\vartheta^*}+\rho_4\eta\overline{\eta^*}\Big]\,dx, \tag{3.9}$$

and induced norm

$$\begin{aligned}\|U\|_{\mathcal{H}_j}^2=\int_0^L\Big[&\rho_1|\Phi|^2+\rho_2|\Psi|^2\\&+\rho_1|W|^2+b|\psi_x|^2+k|\varphi_x+\psi+lw|^2\\&+k_0|w_x-l\varphi|^2\\&+\rho_3|\vartheta|^2+\rho_4|\eta|^2\Big]dx,\end{aligned} \tag{3.10}$$

for all $U=(\varphi,\Phi,\psi,\Psi,w,W,\vartheta,\eta)$, $U^*=(\varphi^*,\Phi^*,\psi^*,\Psi^*,w^*,W^*,\vartheta^*,\eta^*)\in\mathcal{H}_j$, $j=1,2$.

Remark 3.1 It is worth mentioning that the bilinear map (3.9) does define an inner product in $\mathcal{H}_1$, whereas in $\mathcal{H}_2$ it is an inner product only if $Ll\neq n\pi$, $n\in\mathbb{Z}$. Therefore, from now on it is implicit that, whenever working with boundary condition (3.8), we are assuming such a condition.

Denoting $\varphi_t=\Phi$, $\psi_t=\Psi$, $w_t=W$, and $U=(\varphi,\Phi,\psi,\Psi,w,W,\vartheta,\eta)$, we can convert the thermoelastic system of second-order (3.1)-(3.8) into the following Cauchy problem

$$\begin{cases}\frac{d}{dt}U=\mathcal{A}_jU, \quad t>0,\\ U(0)=(\varphi_0,\varphi_1,\psi_0,\psi_1,w_0,w_1,\vartheta_0,\eta_0):=U_0,\end{cases} \tag{3.11}$$

where $\mathcal{A}_j:D(\mathcal{A}_j)\subset\mathcal{H}_j\to\mathcal{H}_j$ is defined by

$$\mathcal{A}_jU=\begin{bmatrix}\Phi\\ \frac{k}{\rho_1}(\varphi_x+\psi+lw)_x+\frac{k_0l}{\rho_1}(w_x-l\varphi)-\frac{k_2l}{\rho_1}\eta\\ \Psi\\ \frac{b}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}(\varphi_x+\psi+lw)-\frac{k_1}{\rho_2}\vartheta_x\\ W\\ \frac{k_0}{\rho_1}(w_x-l\varphi)_x-\frac{kl}{\rho_1}(\varphi_x+\psi+lw)-\frac{k_2}{\rho_1}\eta_x\\ \frac{\gamma_1}{\rho_3}\vartheta_{xx}-\frac{k_1}{\rho_3}\Psi_x\\ \frac{\gamma_2}{\rho_4}\eta_{xx}-\frac{k_2}{\rho_4}(W_x-l\Phi)\end{bmatrix},$$
$$U\in D(\mathcal{A}_j),\ j=1,2, \tag{3.12}$$

with domain

$$D(\mathcal{A}_1)=\Big\{U\in\mathcal{H}_1\ :\ \varphi,\psi,w,\vartheta,\eta\in H^2\cap H_0^1;\ \Phi,\Psi,W\in H_0^1\Big\}\ \text{ for } (3.7),$$

and

$$D(\mathcal{A}_2)=\Big\{U\in\mathcal{H}_2\ :\ \varphi,\vartheta,\eta\in H^2;\ \Phi,\ \psi_x,w_x,\vartheta,\eta\in H_0^1;\ \Psi,W\in H_*^1\Big\}\ \text{ for } (3.8).$$

Under the above notations, the existence and uniqueness of a solution to (3.11) and, consequently, to (3.1)-(3.8), reads as follows:

Theorem 3.1 *([24, Theorem 2.1]) Under the above notations, we have:*

(i) *If $U_0\in\mathcal{H}_j$, then problem (3.11) has a unique mild solution*

$$U\in C^0([0,\infty),\mathcal{H}_j),\quad j=1,2.$$

(ii) *If $U_0\in D(\mathcal{A}_j)$, then problem (3.11) has a unique regular solution*

$$U\in C^0([0,\infty),D(\mathcal{A}_j))\cap C^1([0,\infty),\mathcal{H}_j),\quad j=1,2.$$

(iii) *If $U_0\in D(\mathcal{A}_j^n)$, $n\geq 2$ integer, then the solution is more regular*

$$U\in\bigcap_{\nu=0}^{n}C^{n-\nu}([0,\infty),D(\mathcal{A}_j^{\nu})),\quad j=1,2.$$

Proof For the sake of convenience in future computations, we just sketch the proof presented in [24, Theorem 2.1].

It is not difficult to check that $0\in\rho(\mathcal{A}_j)$, where $\rho(\mathcal{A}_j)$ stands for the resolvent set of $\mathcal{A}_j$, $j=1,2$. Also, a straightforward computation shows that $\mathcal{A}_j$ is dissipative with

$$\begin{aligned}\operatorname{Re}(\mathcal{A}_jU,U)_{\mathcal{H}_j}&=-\int_0^L\gamma_1|\vartheta_x|^2dx\\&\quad-\int_0^L\gamma_2|\eta_x|^2\,dx\leq 0,\quad U\in D(\mathcal{A}_j),\ j=1,2.\end{aligned} \tag{3.13}$$

Therefore, employing the Lummer–Philips Theorem (see e.g., [26, Theorem 4.6]) we have that $\mathcal{A}_j$ is the infinitesimal generator of a C_0-semigroup of contractions $S_j(t):=e^{\mathcal{A}_jt}$ on $\mathcal{H}_j$, $j=1,2$. Consequently, the solution of (3.11) satisfying (i)-(iii) is given by

$$U(t)=e^{\mathcal{A}_jt}U_0,\quad t\geq 0,\ j=1,2.$$

□

3.2 Main results

Our first main result asserts that problem (3.1)-(3.8) is, in general, only semi-uniformly stable with the polynomial rate depending on the regularity of initial data. However, it is independent of the boundary conditions. In any case, in accordance with (1.6), the asymptotic stability will depend on the following number

$$\chi_0 := k - k_0. \tag{3.14}$$

Theorem 3.2 *(Semi-uniform Polynomial Decay) Let us assume that $\chi_0 \neq 0$ in (3.14). Then, for every integer $n \geq 1$, there exists a constant $C_n > 0$ independent of $U_0 \in D(\mathcal{A}_j{}^n)$ such that the semigroup solution $U(t) = e^{\mathcal{A}_j t} U_0$ satisfies*

$$\|U(t)\|_{\mathcal{H}_j} \leq \frac{C_n}{t^{n/2}} \|U_0\|_{D(\mathcal{A}_j^n)}, \; j=1,2, \;\; t \to +\infty. \tag{3.15}$$

In other words, the thermoelastic system (3.1)-(3.6) with either boundary conditions (3.7) or (3.8) is (semi-uniformly) polynomially stable with the decay rate depending on the regularity of initial data.

In addition to Theorem 3.2, one can show that the semi-uniform polynomial decay is optimal for the boundary condition (3.8). This is proved for $n = 1$, namely when initial data belong to the domain of the operator. More precisely, we have:

Theorem 3.3 *(Optimality) Let us assume that $\chi_0 \neq 0$ and take $U_0 \in D(\mathcal{A}_2)$. Then, the semi-uniform polynomial rate $1/t^{1/2}$ obtained (3.15) is optimal, that is, there is no constant $\nu_0 > 0$ such that*

$$\|U(t)\|_{\mathcal{H}_2} \leq \frac{C}{t^{\frac{1}{2}+\nu_0}} \|U_0\|_{D(\mathcal{A}_2)}, \quad t \to +\infty. \tag{3.16}$$

In particular, the thermoelastic Bresse system (3.1)-(3.6) with boundary condition (3.8) is not exponentially stable if $\chi_0 \neq 0$.

As an immediate consequence of Theorem 3.3, we deduce the next result.

Corollary 3.4 *(Non-uniform Stability) Under the conditions of Theorem 3.3, then system (3.1)-(3.6) with boundary condition (3.8) is never uniformly stable for initial data $U_0 \in \mathcal{H}_2$. More precisely, there is no positive function $\Upsilon(t)$ vanishing at infinity such that*

$$\begin{aligned} &\|U(t)\|_{\mathcal{H}_2} \\ &\quad \leq C_0 \, \Upsilon(t), \;\; \forall U_0 \in \mathcal{H}_2, \;\; t \to +\infty, \end{aligned} \tag{3.17}$$

where $C_0 = C_0(\|U_0\|_{\mathcal{H}_2}) > 0$ is a constant depending on U_0.

Proof It follows promptly from Theorem 3.3 and [9, Remark 3.1]. □

Our fourth main result in this section deals with the uniform (exponential) stability of system (3.1)-(3.8) when the assumption on equal wave speeds is taken into account.

Theorem 3.5 *(Uniform Exponential Stability) Let us assume that $\chi_0 = 0$ in (3.14). Then, there exist constants $C, \omega > 0$ independent of $U_0 \in \mathcal{H}_j$ such that the semigroup solution $U(t) = e^{\mathcal{A}_j t} U_0, \; j = 1, 2,$ satisfies*

$$\|U(t)\|_{\mathcal{H}_j} \leq Ce^{-\omega t} \|U_0\|_{\mathcal{H}_j}, \quad t > 0. \tag{3.18}$$

In other words, the thermoelastic system (3.1)-(3.6) with either boundary conditions (3.7) or (3.8) is (uniformly) exponentially stable if $\chi_0 = 0$.

Corollary 3.6 *The thermoelastic Bresse system (3.1)-(3.6) with boundary condition (3.8) is exponentially stable if and only if $\chi_0 = 0$.*

Proof Immediately from Theorems 3.3 and 3.5. □

The conclusion of the proofs of Theorems 3.2 to 3.5 will be given at the end of this section. In what follows, we first introduce the needed machinery to this purpose, namely, we provide some technical lemmas with localized estimates employing the resolvent equation and then combine with the observability inequality previously obtained for systems of Bresse type. Hence, the proofs will follow from the general theory in linear semigroup, see e.g., [6,13,17,19,25,27].

3.3 Technical results via resolvent equation

In this case, the resolvent equation associated with problem (3.11) is given by

$$i\beta U - \mathcal{A}_j U = F, \quad j = 1, 2, \tag{3.19}$$

with $U = (\varphi, \Phi, \psi, \Psi, w, W, \vartheta, \eta)$, $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)$ and $\mathcal{A}_j$ defined in (3.12), which in terms of its components takes the form

$$i\beta\varphi - \Phi = f_1, \tag{3.20}$$

$$\begin{aligned} &i\beta\rho_1\Phi - k(\varphi_x + \psi + lw)_x \\ &-k_0 l(w_x - l\varphi) + k_2 l\eta = \rho_1 f_2, \end{aligned} \tag{3.21}$$

$$i\beta\psi - \Psi = f_3, \tag{3.22}$$

$$i\beta\rho_2\Psi - b\psi_{xx} + k(\varphi_x + \psi + lw) + k_1\vartheta_x = \rho_2 f_4, \tag{3.23}$$

$$i\beta w - W = f_5, \tag{3.24}$$

$$i\beta\rho_1 W - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + k_2\eta_x = \rho_1 f_6, \tag{3.25}$$

$$i\beta\rho_3\vartheta - \gamma_1\vartheta_{xx} + k_1\Psi_x = \rho_3 f_7, \tag{3.26}$$

$$i\beta\rho_4\eta - \gamma_2\eta_{xx} + k_2(W_x - l\Phi) = \rho_4 f_8. \tag{3.27}$$

Lemma 3.7 *([24, p. 63]) Under the above notations, we have $i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$, where $\rho(\mathcal{A}_j)$ stands for the resolvent set of $\mathcal{A}_j$, $j = 1, 2$, given in (3.12).*

Proof A proof by contradiction arguments can be found in [24] (see e.g., p. 63 therein). Here, for the sake of completeness, we present an alternative proof involving direct arguments.

From Engel–Nagel [13, Proposition 5.8 and Corollary 1.15], and taking into account that $D(\mathcal{A}_j)$ is compactly embedded in $\mathcal{H}_j$, $j = 1, 2$, then it is enough to show that $i\beta I_d - \mathcal{A}_j$ is injective for every $\beta \in \mathbb{R}$, which in turn is easily obtained by means of the resolvent Eq. (3.19) (with $F = 0$) and the dissipativity (3.13).

Hence, $i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$ as desired. □

Hereafter, to simplify the notations, we will use a parameter $C > 0$ to denote several different positive constants in the computations below. As usual, $\|\cdot\|_2$ stands for the norm in L^2. Hölder and Poincaré's inequalities will be constantly regarded, sometimes implicitly in the estimates without mentioning them to avoid so many repetitions, and also $|\beta| > 1$ large enough can be taken w.l.o.g. in the estimates.

Lemma 3.8 *Under the above notations, there exists a constant $C > 0$ such that*

$$\|\vartheta_x\|_2^2 + \|\eta_x\|_2^2 \leq C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j}, \text{ for } j = 1, 2. \tag{3.28}$$

Proof Estimate (3.28) is a direct consequence of (3.13) and (3.19). □

To the next results, we shall invoke some useful auxiliary cut-off functions in order to get localized estimates. This allows us to work with both boundary conditions at the same time without trouble with possible boundary point-wise terms coming from integration by parts.

Let us consider $l_0 \in (0, L)$ and $\delta > 0$ such that $(l_0 - \delta, l_0 + \delta) \subset (0, L)$. Then, we set $s_1 \in C^2(0, L)$ satisfying

$$\operatorname{supp} s_1 \subset (l_0 - \delta, l_0 + \delta), \quad 0 \leq s_1(x) \leq 1, \quad x \in (0, L), \tag{3.29}$$

and

$$s_1(x) = 1 \quad \text{for} \quad x \in [l_0 - \delta/2, l_0 + \delta/2]. \tag{3.30}$$

Remark 3.2 An explicit example of such a cut-off function is given below.

$$s_1(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq l_0 - \delta, \\ \dfrac{e^{-1}}{e^{(x-(l_0-\delta/2))^2-1}} & \text{if } l_0 - \delta < x \leq l_0 - \frac{\delta}{2}, \\ 1 & \text{if } l_0 - \frac{\delta}{2} \leq x \leq l_0 + \frac{\delta}{2}, \\ \dfrac{e^{-1}}{e^{(x-(l_0+\delta/2))^2-1}} & \text{if } l_0 + \frac{\delta}{2}, \leq x < l_0 + \delta, \\ 0 & \text{if } l_0 + \delta \leq x \leq L. \end{cases}$$

The geometric idea of s_1 can be seen, e.g., in [8] (see Figure 1 therein). We mention that cut-off functions have shown very effective for local estimates in Timoshenko systems, cf. [3–5,8,20]. However, as far as we know, this is the first time they are employed in computations for the Bresse system via components of the resolvent Eqs. (3.20)–(3.27).

Lemma 3.9 *Under the above notations, there exists a constant $C > 0$ such that*

$$\int_{l_0-\delta}^{l_0+\delta} s_1|w_x - l\varphi|^2\,dx \leq \frac{C}{|\beta|}\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + \frac{C}{|\beta|}\|\eta_x\|_2\|U\|_{\mathcal{H}_j} + \frac{C}{|\beta|}\|F\|_{\mathcal{H}_j}^2 + C\|\eta_x\|_2\left(\int_{l_0-\delta}^{l_0+\delta} s_1|W|^2\,dx\right)^{1/2}, \quad j = 1, 2. \tag{3.31}$$

Proof From (3.20), (3.24), and (3.27), we have

$$i\beta\rho_4\eta - \gamma_2\eta_{xx} + i\beta k_2(w_x - l\varphi) = \rho_4 f_8 + k_2(f_{5,x} - lf_1). \tag{3.32}$$

Taking the multiplier $k_0 s_1 \overline{(w_x - l\varphi)}$ in (3.32) and performing integration by parts, we get

$$
\begin{aligned}
& i\beta k_0 k_2 \int_0^L s_1 |w_x - l\varphi|^2 \, dx \\
& = \underbrace{-\gamma_2 \int_0^L s_1 \eta_x \overline{(k_0(w_x - l\varphi))} \, dx}_{:=I_1} \\
& \quad - \gamma_2 k_0 \int_0^L s_1' \eta_x \overline{(w_x - l\varphi)} \, dx \\
& \quad \underbrace{+ k_0 \rho_4 \int_0^L s_1 \eta \overline{(i\beta(w_x - l\varphi))} \, dx}_{:=I_2} \\
& \quad + k_0 \rho_4 \int_0^L s_1 f_8 \overline{(w_x - l\varphi)} \, dx \\
& \quad + k_2 k_0 \int_0^L s_1 (f_{5,x} - l f_1) \overline{(w_x - l\varphi)} \, dx. \qquad (3.33)
\end{aligned}
$$

Using (3.25),

$$
\begin{aligned}
I_1 = & \, i\beta\gamma_2\rho_1 \int_0^L s_1 \eta_x \overline{W} \, dx \\
& - \gamma_2 k l \int_0^L s_1 \eta_x \overline{(\varphi_x + \psi + l w)} \, dx \\
& - \gamma_2 k_2 \int_0^L s_1 |\eta_x|^2 \, dx \\
& + \gamma_2 \rho_1 \int_0^L s_1 \eta_x \overline{f_6} \, dx.
\end{aligned}
$$

In addition, applying (3.20) and (3.24),

$$
\begin{aligned}
I_2 = & -k_0 \rho_4 \int_0^L [s_1 \eta]_x \overline{W} \, dx \\
& - k_0 l \rho_4 \int_0^L s_1 \eta \overline{\Phi} \, dx \\
& + k_0 \rho_4 \int_0^L s_1 \eta \overline{(f_{5,x} - l f_1)} \, dx.
\end{aligned}
$$

Replacing these two last identities in (3.33), we obtain

$$
\begin{aligned}
& i\beta k_0 k_2 \int_0^L s_1 |w_x - l\varphi|^2 \, dx \\
& = i\beta\gamma_2\rho_1 \int_0^L s_1 \eta_x \overline{W} \, dx + I_3, \qquad (3.34)
\end{aligned}
$$

where

$$
\begin{aligned}
I_3 = & - \gamma_2 k_2 \int_0^L s_1 |\eta_x|^2 \, dx \\
& - \gamma_2 k_0 \int_0^L s_1' \eta_x \overline{(w_x - l\varphi)} \, dx \\
& - k_0 \rho_4 \int_0^L [s_1 \eta]_x \overline{W} \, dx \\
& + k_0 \rho_4 \int_0^L s_1 \eta \overline{(f_{5,x} - l f_1)} \, dx \\
& + \gamma_2 \rho_1 \int_0^L s_1 \eta_x \overline{f_6} \, dx \\
& - \gamma_2 k l \int_0^L s_1 \eta_x \overline{(\varphi_x + \psi + l w)} \, dx \\
& + k_0 \rho_4 \int_0^L s_1 f_8 \overline{(w_x - l\varphi)} \, dx \\
& + k_2 k_0 \int_0^L s_1 (f_{5,x} - l f_1) \overline{(w_x - l\varphi)} \, dx \\
& - k_0 l \rho_4 \int_0^L s_1 \eta \overline{\Phi} \, dx.
\end{aligned}
$$

Using (3.28), the condition (3.29) about the function s_1 and the Hölder and Young inequalities, there exists a constant $C > 0$ such that

$$
\begin{aligned}
|I_3| \leq & \, C\|\eta_x\|_2 \|U\|_{\mathcal{H}_j} + C\|\eta_x\|_2 \|F\|_{\mathcal{H}_j} \\
& + C\|U\|_{\mathcal{H}_j} \|F\|_{\mathcal{H}_j}.
\end{aligned}
$$

Going back to (3.34) and using condition (3.29) on s_1, we conclude

$$
\begin{aligned}
& |\beta| \int_{l_0-\delta}^{l_0+\delta} s_1 |w_x - l\varphi|^2 \, dx \\
& \leq C\|U\|_{\mathcal{H}_j} \|F\|_{\mathcal{H}_j} + C\|\eta_x\|_2 \|U\|_{\mathcal{H}_j} \\
& \quad + C\|\eta_x\|_2 \|F\|_{\mathcal{H}_j} \\
& \quad + C|\beta| \int_{l_0-\delta}^{l_0+\delta} s_1 |\theta_x||W| \, dx.
\end{aligned}
$$

Moreover, applying Hölder and Young inequalities and estimate (3.28), we obtain

$$
\begin{aligned}
& \int_{l_0-\delta}^{l_0+\delta} s_1 |w_x - l\varphi|^2 \, dx \\
& \leq \frac{C}{|\beta|} \|U\|_{\mathcal{H}_j} \|F\|_{\mathcal{H}_j} + \frac{C}{|\beta|} \|\eta_x\|_2 \|U\|_{\mathcal{H}_j} \\
& \quad + \frac{C}{|\beta|} \|F\|_{\mathcal{H}_j}^2
\end{aligned}
$$

$$+ C\|\eta_x\|_2 \left(\int_{l_0-\delta}^{l_0+\delta} s_1 |W|^2 \, dx \right)^{1/2}. \tag{3.35}$$

□

Lemma 3.10 *Under the above notations, there exists a constant $C > 0$ such that*

$$\int_{l_0-\delta}^{l_0+\delta} |W|^2 \, dx \leq \frac{C}{|\beta|} \|\eta_x\|_2 \|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j} \|F\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2} \|U\|^2_{\mathcal{H}_j}, \tag{3.36}$$

for $j = 1, 2$.

Proof Taking the multiplier $-s_1\overline{w}$ in (3.25), performing integration by parts, and using (3.24), we get

$$\begin{aligned} \rho_1 \int_0^L s_1 |W|^2 \, dx &= k_0 \int_0^L s_1 |w_x - l\varphi|^2 \, dx + k_0 l \int_0^L s_1 (w_x - l\varphi) \overline{\varphi} \, dx \\ &\quad \underbrace{-\rho_1 \int_0^L s_2 [f_6 \overline{w} + W \overline{f_5}] \, dx + \frac{i}{\beta} k_2 \int_0^L s_1 \eta_x \overline{(W + f_5)} \, dx}_{:=I_4} \\ &\quad \underbrace{-kl^2 \int_0^L s_1 |\varphi|^2 - kl \int_0^L s_1' \varphi \overline{w} \, dx + kl \int_0^L s_1 (\psi + lw) \overline{w} \, dx}_{:=I_5} \\ &\quad - kl \int_0^L s_1 \varphi \overline{(w_x - l\varphi)} \, dx \underbrace{+ k_0 \int_0^L s_1' (w_x - l\varphi) \overline{w} \, dx}_{:=I_6}. \end{aligned} \tag{3.37}$$

Using Hölder's inequality, there exists a constant $C > 0$ such that

$$|I_4| \leq \frac{C}{|\beta|} \|\eta_x\|_2 \|U\|_{\mathcal{H}_j} + C\|\eta_x\|_2 \|F\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j} \|F\|_{\mathcal{H}_j}. \tag{3.38}$$

On the other hand, from Eqs. (3.20), (3.22), (3.24), it follows that

$$|I_5| \leq \frac{C}{|\beta|^2} \|U\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2} \|F\|^2_{\mathcal{H}_j}, \tag{3.39}$$

for some constant $C > 0$.

Besides, applying (3.20), (3.24), and integration by parts,

$$|\mathrm{Re} I_6| \leq \frac{C}{|\beta|^2} \|U\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2} \|F\|^2_{\mathcal{H}_j}, \tag{3.40}$$

for some constant $C > 0$. Thus, taking the real part of (3.37), using (3.38), (3.39), (3.40) and the condition (3.29) on s_1, we obtain

$$\begin{aligned} \int_{l_0-\delta}^{l_0+\delta} s_1 |W|^2 \, dx &\leq C \int_{l_0-\delta}^{l_0+\delta} s_1 |w_x - l\varphi|^2 \, dx + C \int_{l_0-\delta}^{l_0+\delta} s_1 |w_x - l\varphi| |\varphi| \, dx \\ &\quad + \frac{C}{|\beta|} \|\eta_x\|_2 \|U\|_{\mathcal{H}_j} + \frac{C}{|\beta|} \|\eta_x\|_2 \|F\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j} \|F\|_{\mathcal{H}_j} \\ &\quad + \frac{C}{|\beta|^2} \|U\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2} \|F\|^2_{\mathcal{H}_j}, \end{aligned}$$

for some constant $C > 0$. Using Hölder's inequality and (3.20),

$$\begin{aligned} \int_{l_0-\delta}^{l_0+\delta} s_1 |W|^2 \, dx &\leq C \int_{l_0-\delta}^{l_0+\delta} s_1 |w_x - l\varphi|^2 \, dx + \frac{C}{|\beta|} \left(\int_{l_0-\delta}^{l_0+\delta} s_1 |w_x - l\varphi|^2 \, dx \right)^{1/2} \|U\|_{\mathcal{H}_j} \\ &\quad + \frac{C}{|\beta|} \|\eta_x\|_2 \|U\|_{\mathcal{H}_j} + \frac{C}{|\beta|} \|\eta_x\|_2 \|F\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j} \|F\|_{\mathcal{H}_j} \\ &\quad + \frac{C}{|\beta|^2} \|U\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2} \|F\|^2_{\mathcal{H}_j}, \end{aligned}$$

for some constant $C > 0$. Therefore, from Lemmas 3.8 and 3.9, using Young's inequality, we conclude

$$\int_{l_0-\delta}^{l_0+\delta} |W|^2 \, dx \leq \frac{C}{|\beta|} \|\eta_x\|_2 \|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j} \|F\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2} \|U\|^2_{\mathcal{H}_j}.$$

□

Corollary 3.11 *Under the above notations, there exists a constant $C > 0$ such that*

$$\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |w_x - l\varphi|^2\, dx \leq \frac{C}{|\beta|}\|\eta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j}, \tag{3.41}$$

for $j = 1, 2$.

Proof Combining (3.31) and (3.36), using the condition (3.30) on s_1, Young's inequality, and Lemma 3.8, we obtain (3.11). □

Lemma 3.12 *Under the above notations, there exists a constant $C > 0$ such that*

$$\int_{l_0-\delta}^{l_0+\delta} s_1|\psi_x|^2\, dx \leq \frac{C}{|\beta|}\|\vartheta_x\|_2\|U\|_{\mathcal{H}_j} + \frac{C}{|\beta|}\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + \frac{C}{|\beta|}\|F\|^2_{\mathcal{H}_j} + C\|\vartheta_x\|_2\left(\int_{l_0-\delta}^{l_0+\delta} s_1|\Psi|^2\, dx\right)^{1/2}, \text{ for } j = 1, 2. \tag{3.42}$$

Proof Deriving (3.22) and replacing in (3.26), we have

$$i\beta k_1\psi_x = \gamma_1\vartheta_{xx} - i\beta\rho_3\vartheta + k_1 f_{3,x} + \rho_3 f_7. \tag{3.43}$$

Multiplying (3.43) by $bs_1\overline{\psi_x}$ and integrating in $(0, L)$, we get

$$i\beta k_1 b\int_0^L s_1|\psi_x|^2\, dx = \gamma_1 b\int_0^L s_1\vartheta_{xx}\overline{\psi_x}\, dx - i\beta\rho_3 b\int_0^L s_1\vartheta\overline{\psi_x}\, dx + b\int_0^L s_1[k_1 f_{3,x} + \rho_3 f_7]\overline{\psi_x}\, dx.$$

Integrating by parts, using Eqs. (3.22) and (3.23), it follows that

$$i\beta k_1 b\int_0^L s_1|\psi_x|^2\, dx = i\beta\gamma_1\rho_2\int_0^L s_1\vartheta_x\overline{\Psi}\, dx - \rho_3 b\int_0^L s_1\vartheta_x\overline{\Psi}\, dx + I_7, \tag{3.44}$$

where

$$I_7 = -\gamma_1\int_0^L s_1\vartheta_x\overline{[k(\varphi_x+\psi+lw) + k_1\vartheta_x - \rho_2 f_4]}\, dx - \rho_3 b\int_0^L s_1\vartheta_x\overline{f_3}\, dx - \rho_3 b\int_0^L s_1'\vartheta\overline{(\Psi + f_3)}\, dx + b\int_0^L s_1[k_1 f_{3,x} + \rho_3 f_7]\overline{\psi_x}\, dx - \gamma_1 b\int_0^L s_1'\vartheta_x\overline{\psi_x}\, dx.$$

From condition (3.29) on s_1, Hölder and Young inequalities, and Lemma 3.8, we have that

$$|I_7| \leq C\|\vartheta_x\|_2\|U\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j},$$

for some constant $C > 0$ and for $j = 1, 2$. Therefore, taking the module in (3.44), using Hölder's inequality along with the condition (3.29) on s_1, we conclude

$$\int_{l_0-\delta}^{l_0+\delta} s_1|\psi_x|^2 dx \leq \frac{C}{|\beta|}\|\vartheta_x\|_2\|U\|_{\mathcal{H}_j} + \frac{C}{|\beta|}\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|}\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + C\|\vartheta_x\|_2\left(\int_{l_0-\delta}^{l_0+\delta} s_1|\Psi|^2\, dx\right)^{1/2},$$

for some constant $C > 0$ and $j = 1, 2$, concluding 3.42. □

Lemma 3.13 *Under the above notations, there exists a constant $C > 0$ such that*

$$\int_{l_0-\delta}^{l_0+\delta} s_1|\Psi|^2\, dx \leq \frac{C}{|\beta|}\|\vartheta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j}, \tag{3.45}$$

for $j = 1, 2$.

Proof Multiplying (3.23) by $-s_1\overline{\psi}$ and integrating on $(0, L)$, we have

$$-i\beta\rho_2\int_0^L s_1\Psi\overline{\psi}\, dx = -b\int_0^L s_1\psi_{xx}\overline{\psi}\, dx + k\int_0^L s_1(\varphi_x + \psi + lw)\overline{\psi}\, dx + \int_0^L s_1[k_1\vartheta_x + \rho_2 f_4]\overline{\psi}\, dx.$$

Integrating by parts, from (3.22) and (3.23), we have that

$$\rho_2 \int_0^L s_1|\Psi|^2\,dx = b\int_0^L s_1|\psi_x|^2\,dx + \underbrace{b\int_0^L s_1'\psi_x\overline{\psi}\,dx}_{:=I_8} + I_9, \tag{3.46}$$

where

$$I_9 = k\int_0^L s_1(\varphi_x+\psi+lw)\overline{\psi}\,dx + \int_0^L s_1 k_1\vartheta_x + \rho_2 f_4]\overline{\psi}\,dx - \rho_1\int_0^L s_1\Psi\overline{f_3}\,dx.$$

Performing some calculations, it is easy to check

$$|I_9| \le C\int_{l_0-\delta}^{l_0+\delta} s_1|\psi_x|^2\,dx + \frac{C}{|\beta|}\|\vartheta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j}, \tag{3.47}$$

and

$$|\mathrm{Re} I_8| \le \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2}\|F\|^2_{\mathcal{H}_j}, \tag{3.48}$$

for some constant $C>0$ and $j=1,2$. Therefore, taking the real part of (3.46) and using (3.47), (3.48), it follows that

$$\int_{l_0-\delta}^{l_0+\delta} s_1|\Psi|^2\,dx \le C\int_{l_0-\delta}^{l_0+\delta} s_1|\psi_x|^2\,dx + \frac{C}{|\beta|}\|\vartheta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j}, \tag{3.49}$$

for some constant $C>0$ and $j=1,2$. Using Lemma 3.12, Young's inequality, and Lemma 3.8, then

$$\int_{l_0-\delta}^{l_0+\delta} s_1|\Psi|^2\,dx \le \frac{C}{|\beta|}\|\vartheta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j},$$

for some constant $C>0$ and $j=1.2$, concluding 3.45. ◻

Corollary 3.14 *Under the above notations, there exists a constant $C>0$ such that*

$$\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} \left(|\psi_x|^2+|\Psi|^2\right)dx \le \frac{C}{|\beta|}\|\vartheta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j}, \tag{3.50}$$

for $j=1,2$.

Proof Combining (3.42) and (3.45), using the condition (3.30) on s_1, Young's inequality, and Lemma 3.8, we arrive at (3.50). ◻

Now, we consider another auxiliary cut-off function $s_2\in C^2(0,L)$ satisfying

$$\operatorname{supp} s_2 \subset (l_0-\delta/2,\, l_0+\delta/2), \quad 0\le s_2(x)\le 1, \quad x\in(0,L), \tag{3.51}$$

and

$$s_2(x)=1 \quad \text{for} \quad x\in[l_0-\delta/3,\, l_0+\delta/3]. \tag{3.52}$$

A prototype of such a function can be considered in a similar way as done for s_1 in Remark 3.2. See also [8, Figure 1].

Lemma 3.15 *Under the above notations, there exists a constant $C>0$ such that*

$$\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|\varphi_x+\psi+lw|^2\,dx \le C|\beta||k_0-k|\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|w_x-l\varphi||\Phi|\,dx + C\|\eta_x\|_2\|U\|_{\mathcal{H}_j} + C\|U\|_{\mathcal{H}_j}\left(\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}}|w_x-l\varphi|^2\,dx\right)^{1/2} + C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j}, \tag{3.53}$$

for $j=1,2$.

Proof Multiplying (3.21) by $s_2\frac{kl}{\rho_1}\overline{\varphi}$ and integrating on $(0,L)$, we get

$$kl\int_0^L s_2 f_2\overline{\varphi}\,dx$$

$$
\begin{aligned}
&= i\beta kl \int_0^L s_2 \Phi \overline{\varphi}\, dx \\
&\quad - \frac{k^2 l}{\rho_1} \int_0^L s_2 (\varphi_x + \psi + lw)_x \overline{\varphi}\, dx \\
&\quad - \frac{k_0 k l^2}{\rho_1} \int_0^L s_2 (w_x - l\varphi) \overline{\varphi}\, dx \\
&\quad + \frac{k_2 k l^2}{\rho_1} \int_0^L s_2 \eta \overline{\varphi}\, dx.
\end{aligned}
$$

Performing integration by parts and adding appropriate terms, we have

$$
\begin{aligned}
&\frac{k^2 l}{\rho_1} \int_0^L s_2 |\varphi_x + \psi + lw|^2\, dx \\
&= i\beta k \int_0^L s_2 (-l\Phi) \overline{\varphi}\, dx \\
&\quad - \frac{k^2 l}{\rho_1} \int_0^L s_2' (\varphi_x + \psi + lw) \overline{\varphi}\, dx \\
&\quad + \frac{k_0 k l^2}{\rho_1} \int_0^L s_2 (w_x - l\varphi) \overline{\varphi}\, dx \\
&\quad - \frac{k_2 k l^2}{\rho_1} \int_0^L s_2 \eta \overline{\varphi}\, dx \\
&\quad + kl \int_0^L s_2 f_2 \overline{\varphi}\, dx \\
&\quad + \frac{k^2 l}{\rho_1} \int_0^L s_2 (\varphi_x + \psi + lw) \overline{(\psi + lw)}\, dx. \quad (3.54)
\end{aligned}
$$

On the other hand, deriving (3.24), multiplying by $s_2 \frac{k}{\rho_1} \overline{\varphi}$ and integrating on $(0, L)$, we get

$$
\begin{aligned}
&k \int_0^L s_2 f_{6,x} \overline{\varphi}\, dx \\
&= i\beta k \int_0^L s_2 W_x \overline{\varphi}\, dx - \frac{k}{\rho_1} \int_0^L s_2 (k_0 (w_x - l\varphi)_{xx}) \overline{\varphi}\, dx \\
&\quad + \frac{k^2 l}{\rho_1} \int_0^L s_2 (\varphi_x + \psi + lw)_x \overline{\varphi}\, dx \\
&\quad + \frac{k_2 k}{\rho_1} \int_0^L s_2 \eta_{xx} \overline{\varphi}\, dx.
\end{aligned}
$$

Integrating by parts and adding appropriate terms,

$$
\begin{aligned}
&\frac{k^2 l}{\rho_1} \int_0^L s_2 |\varphi_x + \psi + lw|^2\, dx \\
&= i\beta k \int_0^L s_2 W_x \overline{\varphi}\, dx + \frac{k}{\rho_1} \int_0^L s_2' (k_0 (w_x - l\varphi)_x) \overline{\varphi}\, dx \\
&\quad + \frac{k}{\rho_1} \int_0^L s_2 (k_0 (w_x - l\varphi)_x) \overline{\varphi_x}\, dx \\
&\quad - \frac{k^2 l}{\rho_1} \int_0^L s_2' (\varphi_x + \psi + lw) \overline{\varphi}\, dx \\
&\quad + \frac{k^2 l}{\rho_1} \int_0^L s_2 (\varphi_x + \psi + lw) \overline{(\psi + lw)}\, dx \\
&\quad - \frac{k_2 k}{\rho_1} \int_0^L \eta_x [s_2 \overline{\varphi}]_x\, dx \\
&\quad + k \int_0^L f_6 [s_2 \overline{\varphi}]_x\, dx. \qquad (3.55)
\end{aligned}
$$

Now, adding (3.54) and (3.55), we obtain

$$
\begin{aligned}
&\frac{2k^2 l}{\rho_1} \int_0^L s_2 |\varphi_x + \psi + lw|^2\, dx \\
&= \underbrace{i\beta k \int_0^L s_2 (W_x - l\Phi) \overline{\varphi}\, dx}_{:= I_{10}} \\
&\quad - \frac{2k^2 l}{\rho_1} \int_0^L s_2' (\varphi_x + \psi + lw) \overline{\varphi}\, dx \\
&\quad + \frac{k_0 k l^2}{\rho_1} \int_0^L s_2 (w_x - l\varphi) \overline{\varphi}\, dx - \frac{k_2 k l^2}{\rho_1} \int_0^L s_2 \eta \overline{\varphi}\, dx \\
&+ kl \int_0^L s_2 f_2 \overline{\varphi}\, dx - \frac{k_2 k}{\rho_1} \int_0^L \eta_x [s_2 \overline{\varphi}]_x\, dx \\
&\quad + k \int_0^L f_6 [s_2 \overline{\varphi}]_x\, dx \\
&\quad \underbrace{+ \frac{k}{\rho_1} \int_0^L s_2 (k_0 (w_x - l\varphi)_x) \overline{\varphi_x}\, dx}_{:= I_{11}} \\
&\quad \underbrace{+ \frac{k}{\rho_1} \int_0^L s_2' (k_0 (w_x - l\varphi)_x) \overline{\varphi}\, dx}_{:= I_{12}} \\
&+ \frac{2k^2 l}{\rho_1} \int_0^L (\varphi_x + \psi + lw) \overline{(\psi + lw)}\, dx \qquad (3.56)
\end{aligned}
$$

Note that, from (3.20), (3.24), and integrating by parts,

$$
\begin{aligned}
I_{10} &= -i\beta k \int_0^L s_2 (w_x - l\varphi) \overline{\Phi}\, dx \\
&\quad + k \int_0^L s_2 (f_{5,x} - l f_1) \overline{\Phi}\, dx + k \int_0^L W [s_2 \overline{f_1}]_x\, dx \\
&\quad + kl \int_0^L s_2 \Phi \overline{f_1}\, dx.
\end{aligned}
$$

In addition, again integrating by parts and using (3.21),

$$
\begin{aligned}
I_{11} &= -\frac{k_0 k}{\rho_1} \int_0^L s_2' (w_x - l\varphi) \overline{\varphi_x}\, dx \\
&\quad + \frac{k_0 k}{\rho_1} \int_0^L s_2 (w_x - l\varphi) \overline{\psi_x}\, dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{k_0kl}{\rho_1}\int_0^L s_2(w_x-l\varphi)\overline{w_x}\,dx\\
&+i\beta k_0\int_0^L s_2(w_x-l\varphi)\overline{\Phi}\,dx\\
&+\frac{k_0l^2}{\rho_1}\int_0^L s_2(w_x-l\varphi)\overline{(w_x-l\varphi)}\,dx\\
&-\frac{k_0k_2l}{\rho_1}\int_0^L s_2(w_x-l\varphi)\overline{\eta}\,dx\\
&+k_0\int_0^L s_2(w_x-l\varphi)\overline{f_2}\,dx,
\end{aligned}
$$

and

$$
\begin{aligned}
I_{12}=&-\frac{k_0k}{\rho_1}\int_0^L s_2''(w_x-l\varphi)\overline{\varphi}\,dx\\
&-\frac{k_0k}{\rho_1}\int_0^L s_2'(w_x-l\varphi)\overline{\varphi_x}\,dx.
\end{aligned}
\tag{3.57}
$$

Replacing the three last identities in (3.56), we arrive at

$$
\begin{aligned}
&\frac{2k^2l}{\rho_1}\int_0^L s_2|\varphi_x+\psi+lw|^2\,dx\\
&=i\beta(k_0-k)\int_0^L s_2(w_x-l\varphi)\overline{\Phi}\,dx+I_{13}+I_{14}+I_{15}\\
&+\frac{2k^2l}{\rho_1}\int_0^L s_2(\varphi_x+\psi+lw)\overline{(\psi+lw)}\,dx,
\end{aligned}
\tag{3.58}
$$

where

$$
\begin{aligned}
I_{13}=&\,k\int_0^L W[s_2\overline{f_1}]_x\,dx+kl\int_0^L s_2\Phi\overline{f_1}\,dx\\
&+k\int_0^L s_2(f_{5,x}-lf_1)\overline{\Phi}\,dx\\
&-\frac{k_2kl^2}{\rho_1}\int_0^L s_2\eta\overline{\varphi}\,dx+kl\int_0^L s_2f_2\overline{\varphi}\,dx\\
&+\frac{k_2k}{\rho_1}\int_0^L \eta_x[s_2\overline{\varphi}]_x\,dx\\
&+k\int_0^L f_6[s_2\overline{\varphi}]_x\,dx-\frac{k_0k_2l}{\rho_1}\int_0^L s_2(w_x-l\varphi)\overline{\eta}\,dx\\
&+k_0\int_0^L s_2(w_x-l\varphi)\overline{f_6}\,dx,\\
I_{14}=&\,\frac{k_0kl^2}{\rho_1}\int_0^L s_2(w_x-l\varphi)\overline{\varphi}\,dx\\
&-\frac{k_0k}{\rho_1}\int_0^L s_2(w_x-l\varphi)\overline{\varphi}\,dx\\
&+\frac{k_0k}{\rho_1}\int_0^L s_2(w_x-l\varphi)\overline{\psi_x}\,dx\\
&+\frac{k_0kl}{\rho_1}\int_0^L s_2(w_x-l\varphi)\overline{w_x}\,dx\\
&-\frac{k_0k}{\rho_1}\int_0^L s_2''(w_x-l\varphi)\overline{\varphi}\,dx\\
&-\frac{k_0k}{\rho_1}\int_0^L s_2'(w_x-l\varphi)\overline{\varphi_x}\,dx\\
&+\frac{k_0^2l}{\rho_1}\int_0^L s_2(w_x-l\varphi)\overline{(w_x-l\varphi)}\,dx,
\end{aligned}
$$

and

$$
I_{15}=-\frac{2k^2l}{\rho_1}\int_0^L s_2'(\varphi_x+\psi+lw)\overline{\varphi}\,dx.
$$

Now, using the Hölder, Poincaré and Young inequalities, along with Lemma 3.8,

$$
\begin{aligned}
|I_{13}|\le&\, C\|\eta_x\|_2\|U\|_{\mathcal{H}_j}\\
&+C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j}+C\|F\|^2_{\mathcal{H}_j},
\end{aligned}
\tag{3.59}
$$

for some constant $C>0$ and $j=1,2$. Additionally, from condition (3.51) on s_2 and invoking again Hölder and Poincaré's inequalities,

$$
|I_{14}|\le C\|U\|_{\mathcal{H}_j}\left(\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}}|w_x-l\varphi|^2\,dx\right)^{1/2},
\tag{3.60}
$$

for some constant $C>0$ and $j=1,2$. Moreover, from (3.20), (3.22), (3.24), and applying Hölder and Young's inequalities,

$$
\begin{aligned}
|\mathrm{Re} I_{15}|\le&\,\frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j}\\
&+\frac{C}{|\beta|^2}\|F\|_{\mathcal{H}_j},
\end{aligned}
\tag{3.61}
$$

for some constant $C>0$ and $j=1,2$. Going back to (3.58), taking the real part, using (3.59), (3.60), (3.61), and condition (3.51) on s_2,

$$
\begin{aligned}
&\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|\varphi_x+\psi+lw|^2\,dx\\
&\le C|\beta||k_0-k|\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|w_x-l\varphi||\Phi|\,dx\\
&\quad+C\|\eta_x\|_2\|U\|_{\mathcal{H}_j}+C\|U\|_{\mathcal{H}_j}\\
&\quad\left(\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}}|w_x-l\varphi|^2\,dx\right)^{1/2}\\
&\quad+C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j}+C\|F\|^2_{\mathcal{H}_j}\\
&\quad+\frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j}
\end{aligned}
$$

$$+ C\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|\varphi_x+\psi+lw||\psi+lw|\,dx,$$

for some constant $C>0$ and $j=1,2$. Finally, using (3.22), (3.24), Hölder and Young's inequalities, we conclude that

$$\begin{aligned}
&\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|\varphi_x+\psi+lw|^2\,dx\\
&\quad\le C|\beta||k_0-k|\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|w_x-l\varphi||\Phi|\,dx\\
&\qquad+C\|\eta_x\|_2\|U\|_{\mathcal{H}_j}+C\|U\|_{\mathcal{H}_j}\\
&\qquad\left(\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}}|w_x-l\varphi|^2\,dx\right)^{1/2}\\
&\qquad+C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j}+C\|F\|^2_{\mathcal{H}_j}\\
&\qquad+\frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j},
\end{aligned}$$

for some constant $C>0$ and $j=1,2$. This concludes the proof of (3.53). □

Lemma 3.16 *Under the above notations, there exists a constant $C>0$ such that*

$$\begin{aligned}
&\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|\Phi|^2\,dx\\
&\quad\le C|\beta||k_0-k|\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|w_x-l\varphi||\Phi|\,dx+C\|U\|_{\mathcal{H}_j}\\
&\qquad\left(\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}}|w_x-l\varphi|^2\,dx\right)^{1/2}\\
&\qquad+C\|\eta_x\|_2\|U\|_{\mathcal{H}_j}+C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j}+C\|F\|^2_{\mathcal{H}_j}\\
&\qquad+\frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j},
\end{aligned}\tag{3.62}$$

for $j=1,2$.

Proof Taking the multiplier $s_2\overline{\varphi}$ in (3.21), performing integration by parts, using (3.20), and adding appropriate terms, we get

$$\begin{aligned}
&\int_0^L s_2|\Phi|^2\,dx\\
&\quad=-\int_0^L s_2\Phi\overline{f_1}\,dx+\underbrace{\frac{k}{\rho_1}\int_0^L s_2'(\varphi_x+\psi+lw)\overline{\varphi}\,dx}_{:=I_{16}}\\
&\qquad\underbrace{+\frac{k}{\rho_1}\int_0^L s_2|\varphi_x+\psi+lw|^2\,dx-\frac{k}{\rho_1}\int_0^L s_2(\varphi_x+\psi+lw)\overline{(\psi+lw)}\,dx}_{:=I_{17}}\\
&\qquad-\frac{k_0 l}{\rho_1}\int_0^L s_2(w_x-l\varphi)\overline{\varphi}\,dx\\
&\qquad+\frac{k_2 l}{\rho_1}\int_0^L s_2\eta\overline{\varphi}\,dx-\int_0^L s_2 f_2\overline{\varphi}\,dx.
\end{aligned}\tag{3.63}$$

From (3.20), (3.22), (3.24), performing integration by parts, and applying the Hölder and Young inequalities,

$$|\mathrm{Re}I_{16}|\le\frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j}+\frac{C}{|\beta|^2}\|F\|^2_{\mathcal{H}_j},\tag{3.64}$$

for some constant $C>0$ and $j=1,2$. In addition, using (3.22), (3.24), the Hölder and Young inequalities, and the condition (3.51) on s_2, we infer

$$\begin{aligned}
|I_{17}|\le{}& C\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|\varphi_x\\
&+\psi+lw|^2\,dx+\frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j}\\
&+\frac{C}{|\beta|^2}\|F\|^2_{\mathcal{H}_j},
\end{aligned}\tag{3.65}$$

for some constant $C>0$ and $j=1,2$.

Therefore, taking the real part in (3.63), using Hölder's inequality, and employing (3.64)-(3.65), Lemma 3.15, and condition (3.51) on s_2, we conclude

$$\begin{aligned}
&\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|\Phi|^2\,dx\\
&\quad\le C|\beta||k_0-k|\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} s_2|w_x-l\varphi||\Phi|\,dx\\
&\qquad+C\|U\|_{\mathcal{H}_j}\left(\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}}|w_x-l\varphi|^2\,dx\right)^{1/2}\\
&\qquad+C\|\eta_x\|_2\|U\|_{\mathcal{H}_j}+C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j}\\
&\qquad+C\|F\|^2_{\mathcal{H}_j}+\frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}_j},
\end{aligned}$$

for some constant $C>0$ and $j=1,2$, which proves (3.62). □

Corollary 3.17 *Let $\varepsilon>0$ be given. Under the above results, we have:*

(*i*) *If* $\chi_0 \neq 0$ *in* (3.14)*, then there exists a constant* $C_\varepsilon > 0$ *such that*

$$\int_{l_0-\frac{\delta}{3}}^{l_0+\frac{\delta}{3}} \left(|\varphi_x + \psi + lw|^2 + |\Phi|^2\right) dx \leq \varepsilon \|U\|^2_{\mathcal{H}_j} + C_\varepsilon |\beta|^4 \|F\|^2_{\mathcal{H}_j}. \tag{3.66}$$

(*ii*) *If* $\chi_0 = 0$ *in* (3.14)*, then there exists a constant* $C_\varepsilon > 0$ *such that*

$$\int_{l_0-\frac{\delta}{3}}^{l_0+\frac{\delta}{3}} \left(|\varphi_x + \psi + lw|^2 + |\Phi|^2\right) dx \leq \varepsilon \|U\|^2_{\mathcal{H}_j} + C_\varepsilon \|F\|^2_{\mathcal{H}_j}. \tag{3.67}$$

Proof Let $\varepsilon > 0$. Adding (3.53) and (3.62), using Young's inequality with $\varepsilon > 0$ and conditions (3.51) and (3.52) on s_2, we obtain

$$\begin{aligned}
&\int_{l_0-\frac{\delta}{3}}^{l_0+\frac{\delta}{3}} \left(|\varphi_x + \psi + lw|^2 + |\Psi|^2\right) dx \\
&\leq C_\varepsilon |\beta|^2 |k_0 - k|^2 \int_{l_0-\frac{\delta}{3}}^{l_0+\frac{\delta}{3}} |w_x - l\varphi|^2\, dx + \varepsilon \|U\|^2_{\mathcal{H}_j} \\
&\quad + C_\varepsilon \int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |w_x - l\varphi|^2\, dx + C\|\eta_x\|_2 \|U\|_{\mathcal{H}_j} \\
&\quad + C\|U\|_{\mathcal{H}_j} \|F\|_{\mathcal{H}_j} + C\|F\|^2_{\mathcal{H}_j} + \frac{C}{|\beta|^2} \|U\|^2_{\mathcal{H}_j},
\end{aligned}$$

for some constants $C, C_\varepsilon > 0$ and $j = 1, 2$. Again, using Young's inequality, from Lemma 3.8 and $|\beta| > 1$ large enough, we have

$$\begin{aligned}
&\int_{l_0-\frac{\delta}{3}}^{l_0+\frac{\delta}{3}} \left(|\varphi_x + \psi + lw|^2 + |\Psi|^2\right) dx \\
&\leq C_\varepsilon |\beta|^2 |k_0 - k|^2 \int_{l_0-\frac{\delta}{3}}^{l_0+\frac{\delta}{3}} |w_x - l\varphi|^2\, dx + \varepsilon \|U\|^2_{\mathcal{H}_j} \\
&\quad C_\varepsilon \int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |w_x - l\varphi|^2\, dx + C_\varepsilon \|F\|^2_{\mathcal{H}_j},
\end{aligned} \tag{3.68}$$

for some constant $C_\varepsilon > 0$ and $j = 1, 2$.

Therefore, if $\chi_0 \neq 0$, that is, $k_0 - k \neq 0$, we conclude from (3.41) and (3.68), using Young's inequality with $\varepsilon > 0$, and Lemma 3.8, that

$$\int_{l_0-\frac{\delta}{3}}^{l_0+\frac{\delta}{3}} \left(|\varphi_x + \psi + lw|^2 + |\Psi|^2\right) dx \leq \varepsilon \|U\|^2_{\mathcal{H}_j} + C_\varepsilon |\beta|^4 \|F\|^2_{\mathcal{H}_j},$$

for some constant $C_\varepsilon > 0$ and $j = 1, 2$, which proves the desired estimate (3.66). Now, if $\chi_0 = 0$, using (3.41) and (3.68), Young's inequality with $\varepsilon > 0$, and Lemma 3.8, we obtain more easily

$$\int_{l_0-\frac{\delta}{3}}^{l_0+\frac{\delta}{3}} \left(|\varphi_x + \psi + lw|^2 + |\Psi|^2\right) dx \leq \varepsilon \|U\|^2_{\mathcal{H}_j} + C_\varepsilon \|F\|^2_{\mathcal{H}_j},$$

for some constant $C_\varepsilon > 0$ and $j = 1, 2$, concluding the estimate (3.67). □

3.4 Conclusion of the proofs (completion)

From the previous sections, we have finally gathered all ingredients to conclude the proofs of Theorems 3.2, 3.3, and 3.5. For the sake of logistic, we are going to conclude initially Theorems 3.2 and 3.5, and then Theorem 3.3.

3.4.1 Proof of Theorem 3.2

In this case, we have assumed $\chi_0 \neq 0$. Thus, given $\varepsilon > 0$, employing (3.66) together with estimates (3.36), (3.41), (3.50), and using Young's inequality, we get

$$\mathcal{I}_{\frac{\delta}{3}} \leq \varepsilon \|U\|^2_{\mathcal{H}_j} + C_\varepsilon |\beta|^4 \|F\|^2_{\mathcal{H}_j} := \Lambda, \tag{3.69}$$

for some constant $C_\varepsilon > 0$ and $j = 1, 2$, where we set the notation

$$\begin{aligned}
\mathcal{I}_{\frac{\delta}{3}} := \int_{l_0-\frac{\delta}{3}}^{l_0+\frac{\delta}{3}} &\left(|\varphi_x + \psi + lw|^2\right. \\
&\left. + |\Phi|^2 + |\psi_x|^2 + |\Psi|^2 + |w_x - l\varphi|^2 + |W|^2\right) dx.
\end{aligned} \tag{3.70}$$

This is the exact moment where we employ the observability inequality provided in Sect. 2 through its consequence given by Corollary 2.3.

Indeed, from the resolvent Eqs. (3.20)–(3.25) we can see that $V := (\varphi, \Phi, \psi, \Psi, w, W)$ is a solution of (2.11)-(2.16) with $G := (g_1, g_2, g_3, g_4, g_5, g_6)$ given by

$$g_1 := f_1, \quad g_2 := \rho_1 f_2 - k_2 l\eta, \quad g_3 := f_3,$$
$$g_4 := \rho_2 f_4 - k_1 \vartheta_x, \quad g_5 := f_5, \quad g_6 := \rho_1 f_6 - k_2 \eta_x.$$

In addition, considering

$$b_1 := l_0 - \delta/3 \quad \text{and} \quad b_2 := l_0 + \delta/3,$$

and taking into account the estimate (3.69), then we are in conditions to apply Corollary 2.3, Lemma 3.8, and

Young's inequality, to arrive at

$$\int_0^L \left(|\varphi_x+\psi+lw|^2+|\Phi|^2+|\psi_x|^2+|\Psi|^2 + |w_x-l\varphi|^2+|W|^2\right)dx \le \varepsilon C\|U\|^2_{\mathcal{H}_j}+C_\varepsilon|\beta|^4\|F\|^2_{\mathcal{H}_j},$$

for some constants C, $C_\varepsilon>0$ and $j=1,2$. Again from Lemma 3.8, we have

$$\|U\|^2_{\mathcal{H}_j}\le \varepsilon C\|U\|^2_{\mathcal{H}_j}+C_\varepsilon|\beta|^4\|F\|^2_{\mathcal{H}_j},$$

and choosing $\varepsilon>0$ small enough we finally obtain

$$\|(i\beta I_d-\mathcal{A}_j)^{-1}F\|_{\mathcal{H}_j}\le C\,|\beta|^2\|F\|_{\mathcal{H}_j},\quad |\beta|\to+\infty, \tag{3.71}$$

for some constant $C>0$. From Lemma 3.7 and (3.71), we conclude by virtue of Borichev–Tomilov's Theorem (cf. [6, Theorem 2.4]) that

$$\|U(t)\|_{\mathcal{H}_j}\le\frac{C}{t^{1/2}}\|U_0\|_{D(\mathcal{A}_j)},\quad t\to+\infty,$$

for $U_0\in D(\mathcal{A}_j)$, $j=1,2$, which proves (3.15) for $n=1$. The remaining decay rates in (3.15) follow by using induction over $n\ge 2$.

This completes the proof of Theorem 3.2. □

3.4.2 *Proof of Theorem 3.5*

In this case, we have assumed $\chi_0=0$. Thus, given $\varepsilon>0$, invoking now the estimate (3.67) along with (3.36), (3.41), and (3.50), we have

$$\mathcal{I}_{\frac{\delta}{3}}\le\varepsilon\|U\|^2_{\mathcal{H}_j}+C_\varepsilon\|F\|^2_{\mathcal{H}_j}:=\Lambda, \tag{3.72}$$

for some constant $C_\varepsilon>0$ and $j=1,2$, where $\mathcal{I}_{\frac{\delta}{3}}$ is given in (3.70). Similarly as done before, applying the Corollary 2.3, Lemma 3.8, and Young's inequality, we deduce

$$\int_0^L \left(|\varphi_x+\psi+lw|^2+|\Phi|^2+|\psi_x|^2+|\Psi|^2 + |w_x-l\varphi|^2+|W|^2\right)dx\le\varepsilon C\|U\|^2_{\mathcal{H}_j}+C_\varepsilon\|F\|^2_{\mathcal{H}_j}, \tag{3.73}$$

for some constants C, $C_\varepsilon>0$ and $j=1,2$. Combining Lemma 3.8 and (3.73), we obtain

$$\|U\|^2_{\mathcal{H}_j}\le\varepsilon C\|U\|^2_{\mathcal{H}_j}+C_\varepsilon\|F\|^2_{\mathcal{H}_j},$$

and taking $\varepsilon>0$ small enough, we conclude

$$\|(i\beta I_d-\mathcal{A}_j)^{-1}F\|_{\mathcal{H}_j}\le C\|F\|_{\mathcal{H}_j},\quad |\beta|\to+\infty,\ j=1,2. \tag{3.74}$$

Therefore, using once again Lemma 3.7 and (3.74), we conclude the exponential decay (3.18) through the classical Gearhart–Huang-Prüss characterization of exponential stability of C_0-semigroups on Hilbert spaces (see, for instance, [25, Theorem 1.3.2]).

The proof of Theorem 3.5 is ended. □

3.4.3 *Proof of Theorem 3.3*

Let us consider $\chi_0\neq 0$ and fix $U_0\in D(\mathcal{A}_2)$. In order to prove the desired optimality, we shall argue by contraction.

Indeed, let us suppose that there exists a constant $\nu_0>0$ such that (3.16) holds true. Therefore, by taking $\nu=2-\frac{2}{1+2\nu_0}\in(0,2)$, we get the following equivalent (to (3.16)) estimate

$$\|U(t)\|_{\mathcal{H}_2}\le\frac{C}{t^{\frac{1}{2-\nu}}}\|U_0\|_{D(\mathcal{A}_2)},\quad t\to+\infty.$$

From this and equivalence coming from the Borichev–Tomilov Theorem, cf. [6, Theorem 2.4], there exists a constant $C>0$ such that

$$\frac{1}{|\beta|^{2-\nu}}\|(i\beta I_d-\mathcal{A}_2)^{-1}\|_{\mathcal{L}(\mathcal{H}_2)}\le C,\quad |\beta|\to+\infty. \tag{3.75}$$

On the other hand, if given a bounded sequence $(F_\mu)_{\mu\in\mathbb{N}}\subset\mathcal{H}_2$, we can find a real sequence $(\beta_\mu)_{\mu\in\mathbb{N}}\subset\mathbb{R}^+$, satisfying $\lim\limits_{\mu\to\infty}\beta_\mu=+\infty$, such that

$$\lim_{\mu\to+\infty}\frac{1}{|\beta_\mu|^{2-\nu}}\|(i\beta_\mu I_d-\mathcal{A}_2)^{-1}F_\mu\|_{\mathcal{H}_2}=+\infty, \tag{3.76}$$

we conclude the desired contradiction with (3.75).

In what follows, we are going to proceed with the proof of (3.76). W.l.o.g. let us take $L=\pi$ and consider $F_\mu\in\mathcal{H}_2$ as

$$F_\mu(x)=\left(0,\ \frac{1}{\rho_1}\sin(\mu x),\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right),\quad \mu\in\mathbb{N}.$$

Then, $F_\mu\in\mathcal{H}_2$ and since $i\mathbb{R}\subseteq\rho(A_2)$, let $U_\mu\in D(\mathcal{A}_2)$ be the solution of the resolvent equation

$$(i\beta_\mu I_d-\mathcal{A}_2)U_\mu=F_\mu \Leftrightarrow\quad U_\mu=(i\beta_\mu I_d-\mathcal{A}_2)^{-1}F_\mu. \tag{3.77}$$

Rewriting (3.77) in terms of its components, where denoting $U_\mu := (\varphi, \Phi, \psi, \Psi, w, W, \vartheta, \eta)$ and $\beta_\mu := \beta$ to simplify the notations, we get

$$
\begin{aligned}
&i\beta\varphi - \Phi = 0,\\
&i\beta\rho_1\Phi - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + k_2 l\eta = \sin(\mu x),\\
&i\beta\psi - \Psi = 0,\\
&i\beta\rho_2\Psi - b\psi_{xx} + k(\varphi_x + \psi + lw) + k_1\vartheta_x = 0,\\
&i\beta w - W = 0,\\
&i\beta\rho_1 W - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + k_2\eta_x = 0,\\
&i\beta\rho_3\vartheta - \gamma_1\vartheta_{xx} + k_1\Psi_x = 0,\\
&i\beta\rho_4\eta - \gamma_2\eta_{xx} + k_2(W_x - l\Phi) = 0.
\end{aligned}
$$

From the first, third and fifth equations of the above system, we can consider the following reduced system in terms of $\varphi, \psi, w, \vartheta, \eta$

$$
\begin{aligned}
&-\beta^2\rho_1\varphi - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + k_2 l\eta = \sin(\mu x),\\
&-\beta^2\rho_2\psi - b\psi_{xx} + k(\varphi_x + \psi + lw) + k_1\vartheta_x = 0,\\
&-\beta^2\rho_1 w - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + k_2\eta_x = 0,\\
&i\beta\rho_3\vartheta - \gamma_1\vartheta_{xx} + i\beta k_1\psi_x = 0,\\
&i\beta\rho_4\eta - \gamma_2\eta_{xx} + i\beta k_2(w_x - l\varphi) = 0.
\end{aligned}
\tag{3.78}
$$

This is the precise moment where we need to work with boundary conditions (3.8). In fact, due to the symmetry of the above system in compatibility with the boundary conditions, we can look for solutions to (3.78) of the form

$$
\begin{aligned}
&\varphi = A\sin(\mu x),\ \psi = B\cos(\mu x),\ w = C\cos(\mu x),\\
&\vartheta = D\sin(\mu x),\ \eta = E\sin(\mu x),\ x \in [0, \pi],
\end{aligned}
$$

where $A = A_\mu$, $B = B_\mu$, $C = C_\mu$, $D = D_\mu$ and $E = E_\mu$ will be determined later.

In this way, to solve (3.78) is equivalent to find a solution (A, B, C, D, E) for the algebraic system

$$
\begin{aligned}
(-\beta^2\rho_1 + k\mu^2 + k_0 l^2)A + k\mu B + (k + k_0)l\mu C + k_2 l E &= 1,\\
k\mu A + (-\beta^2\rho_2 + b\mu^2 + k)B + klC + k_1\mu D &= 0,\\
(k + k_0)l\mu A + klB + (-\beta^2\rho_1 + k_0\mu^2 + kl^2)C + k_2\mu E &= 0,\\
-i\beta k_1\mu B + (i\beta\rho_3 + \gamma_1\mu^2)D &= 0,\\
-i\beta k_2 l A - i\beta k_2\mu C + (i\beta\rho_4 + \gamma_2\mu^2)E &= 0.
\end{aligned}
\tag{3.79}
$$

We denote the matrix of coefficients in (3.79) by

$$
M = \begin{pmatrix}
P_1 & k\mu & (k + k_0)l\mu & 0 & k_2 l\\
k\mu & P_2 & kl & k_1\mu & 0\\
(k + k_0)l\mu & kl & P_3 & 0 & k_2\mu\\
0 & -i\beta k_1\mu & 0 & P_4 & 0\\
-i\beta k_2 l & 0 & -i\beta k_2\mu & 0 & P_5
\end{pmatrix}
\tag{3.80}
$$

where

$$
\begin{cases}
P_1 = -\beta^2\rho_1 + k\mu^2 + k_0 l^2,\\
P_2 = -\beta^2\rho_2 + b\mu^2 + k,\\
P_3 = -\beta^2\rho_1 + k_0\mu^2 + kl^2,\\
P_4 = i\beta\rho_3 + \gamma_1\mu^2,\\
P_5 = i\beta\rho_4 + \gamma_2\mu^2,
\end{cases}
\tag{3.81}
$$

are functions of β. Recalling that our goal is to evaluate the behavior of

$$\|(i\beta I_d - \mathcal{A}_2)^{-1}F_\mu\|_{\mathcal{H}_2}, \quad \mu \to +\infty,$$

let us note that

$$
\begin{aligned}
\|(i\beta I_d - \mathcal{A}_2)^{-1}F_\mu\|^2_{\mathcal{H}_2} &= \|U_\mu\|^2_{\mathcal{H}_2} \ge \rho_1\|\Phi\|_2^2\\
&= \rho_1|\beta|^2\|\varphi\|_2^2 = \frac{\pi}{2}\rho_1|\beta|^2|A|^2.
\end{aligned}
$$

Thus, we should evaluate the behavior of A when $\mu \to +\infty$. For this, using Cramer's Rule, we have

$$
A = \frac{\det M_1}{\det M},
\tag{3.82}
$$

where

$$
M_1 = \begin{pmatrix}
1 & k\mu & (k + k_0)l\mu & 0 & k_2 l\\
0 & P_2 & kl & k_1\mu & 0\\
0 & kl & P_3 & 0 & k_2\mu\\
0 & -i\beta k_1\mu & 0 & P_4 & 0\\
0 & 0 & -i\beta k_2\mu & 0 & P_5
\end{pmatrix}.
$$

Then, a simple calculation shows that

$$
\begin{aligned}
\det M = {} & P_1P_2P_3P_4P_5 - k^2\mu^2P_3P_4P_5\\
& - (k + k_0)^2l^2\mu^2P_2P_4P_5 + i\beta k_2^2\mu^2P_1P_2P_4\\
& + i\beta k_1^2\mu^2P_1P_3P_5 - \beta^2k_1^2k_2^2\mu^4P_1\\
& - i\beta(k + k_0)^2k_1^2l^2\mu^4P_5 + i\beta k_2^2l^2P_2P_3P_4\\
& - 2i\beta(k + k_0)k_2^2l^2\mu^2P_2P_4\\
& - i\beta k^2k_2^2\mu^4P_4 + 2\beta^2(k + k_0)k_1^2k_2^2l^2\mu^4\\
& - \beta^2k_1^2k_2^2l^2\mu^2P_3\\
& + 2(k + k_0)k^2l^2\mu^2P_4P_5\\
& - k^2l^2P_1P_4P_5 + 2i\beta k^2k_2^2l^2\mu^2P_4 - i\beta k^2k_2^2l^4P_4.
\end{aligned}
\tag{3.83}
$$

and

$$\det M_1 = P_2P_3P_4P_5 + i\beta k_2^2 P_2P_4 + i\beta k_1^2\mu^2 P_3P_5 \\ -\beta^2 k_1^2 k_2^2 \mu^4 - k^2 l^2 P_4 P_5. \quad (3.84)$$

Now, choosing the sequence $(\beta_\mu)_{\mu\in\mathbb{N}} \subset \mathbb{R}^+$ such that

$$\beta_\mu = \sqrt{\frac{1}{\rho_1}(k\mu^2 + k_0 l^2 - \Xi)}, \quad (3.85)$$

where Ξ is a constant to be determined later, we get

$\det M \neq 0$ and $\det M_1 \neq 0$,

for $\mu > 0$ large enough. In addition, note that $\det M$ and $\det M_1$ are polynomials in the variable μ of degree ≤ 10 and degree ≤ 8, respectively. Our intention is to reduce the degree of $\det M$ without interfering in the degree of $\det M_1$. Indeed, from (3.83) and (3.85), we can analyze the first three terms of $\det M$, which have a higher degree,

$$\begin{aligned} &P_1P_2P_3P_4P_5 - k^2\mu^2 P_3P_4P_5 \\ &\quad - (k+k_0)^2 l^2\mu^2 P_2P_4P_5 \\ &= \Big(P_1P_2P_3 - k^2\mu^2 P_3 \\ &\quad -(k+k_0)^2 l^2 \mu^2 P_2\Big) P_4P_5. \end{aligned}$$

From (3.81) and (3.85), we have

$$\begin{aligned} P_1 &= \Xi, \\ P_2 &= \left(b - \frac{\rho_2}{\rho_1}k\right)\mu^2 + k - \frac{\rho_2}{\rho_1}k_0 l^2 + \frac{\rho_2}{\rho_1}\Xi, \\ P_3 &= (k_0 - k)\mu^2 + (k - k_0)l^2 - \Xi. \end{aligned}$$

Now, in view of $\chi_0 \neq 0$ in (3.14) and keeping in mind (1.5)-(1.6), we get

$$k \neq k_0 \iff IE \neq IG \iff b \neq \frac{\rho_2}{\rho_1}k.$$

Then, by setting $\chi := b - \frac{\rho_2}{\rho_1}k$, we also deduce that $\chi \neq 0$, and thus

$$\begin{aligned} &P_1P_2P_3 - k^2\mu^2 P_3 - (k+k_0)^2 l^2 \mu^2 P_2 \\ &\quad \approx \sigma_1\mu^4 + \sigma_2\mu^3, \ \ \sigma_1, \sigma_2 > 0, \ \ \mu \to +\infty, \end{aligned}$$

where

$$\sigma_1 = \chi\chi_0\Xi - k^2\chi_0 - l^2(k+k_0)^2\chi.$$

Therefore, we pick up

$$\Xi = \frac{k^2\chi_0 + l^2(k+k_0)^2\chi}{\chi\chi_0},$$

in the sequence $(\beta_\mu)_{\mu\in\mathbb{N}} \subset \mathbb{R}^+$ given in (3.85), yields

$$\begin{aligned} &\det M \approx \sigma_3\mu^7, \ \ \sigma_3 > 0, \ \ \mu \to +\infty, \\ &\det M_1 \approx \sigma_4\mu^8, \ \ \sigma_4 > 0, \ \ \mu \to +\infty. \end{aligned}$$

From (3.82), we get

$$|A_\mu| \approx \sigma_5\mu, \ \ \sigma_5 > 0, \ \ \mu \to +\infty.$$

Therefore,

$$\begin{aligned} &\|(i\beta_\mu I_d - \mathcal{A}_2)^{-1}F_\mu\|_{\mathcal{H}_2}^2 = \|U_\mu\|_{\mathcal{H}_2}^2 \\ &\geq \frac{\pi}{2}\rho_1|\beta_\mu|^2|A_\mu|^2 \to +\infty, \ \ \mu \to +\infty, \end{aligned} \quad (3.86)$$

which implies

$$\begin{aligned} &|\beta_\mu|^{\nu-2}\|(i\beta_\mu I_d - \mathcal{A}_2)^{-1}F_\mu\|_{\mathcal{H}_2} \\ &\geq \sqrt{\frac{\pi}{2}}\rho_1|\beta_\mu|^{\nu-1}|A_\mu| \approx \sigma_6\mu^\nu, \ \ \sigma_6 > 0, \ \ \mu \to +\infty, \end{aligned}$$

that is,

$$\lim_{\mu\to+\infty} \frac{1}{|\beta|^{2-\nu}}\|(i\beta_\mu I_d - \mathcal{A}_2)^{-1}F_\mu\|_{\mathcal{H}_2} = +\infty.$$

This completes the proof of (3.76), which provides the desired conclusion on optimality.

In particular, going back to (3.86) one sees that

$$\begin{aligned} &\lim_{\mu\to+\infty} \|(i\beta_\mu I_d - \mathcal{A}_2)^{-1}F_\mu\|_{\mathcal{H}_2} \\ &= \lim_{\mu\to+\infty} \|U_\mu\|_{\mathcal{H}_2} = +\infty, \end{aligned}$$

which ensures the lack of exponential stability of the C_0-semigroup $\{e^{\mathcal{A}_2 t}\}$ on $\mathcal{H}_2$ if $\chi_0 \neq 0$, cf. [25, Theorem 1.3.2]. Hence, the semigroup solution $U(t) = e^{\mathcal{A}_2 t}U_0$ is not exponentially stable as well.

This finishes the proof of Theorem 3.3. □

4 Concluding remarks

Let us consider some final remarks on Theorems 3.2, 3.3, and 3.5 as follows, by clarifying the novelties of this work. We also express some brief comments on the linear physical modeling as well as on the connection with nonlinear-related systems.

I. Polynomial stability. The semi-uniform polynomial decay rate

$$\left(\frac{1}{t}\right)^{n/2}, \ \ n \in \mathbb{N}, \quad (4.1)$$

achieved in (3.15) for $U_0 \in D(\mathcal{A}_j{}^n)$ and $\chi_0 \neq 0$ is the same independently of the boundary conditions (3.7) ($j = 1$) or (3.8) ($j = 2$). Therefore, in both cases w.r.t. boundary conditions, Theorem 3.2 constitutes an improvement in the stability result [24, Theorem 4.1] where the slower decay is obtained for $n \in \mathbb{N}$:

$$\begin{cases} ln\, t \left(\dfrac{\ln t}{t}\right)^{n/8} & \text{for (3.7)}, \\ ln\, t \left(\dfrac{\ln t}{t}\right)^{n/4} & \text{for (3.8)}. \end{cases}$$

Moreover, since the semi-uniform stability (4.1) is the same whatever boundary condition is taken into account, then Theorem 3.2 gives the precise answer to the issue raised in [24, Remark 4.1] on the faster decay rate for (3.8) than that one for case (3.7).

II. Optimality. Besides improving and clarifying the statements in [24, Sect. 4.1], Theorem 3.3 states what is the optimal polynomial decay rate in (3.15) for $U_0 \in D(\mathcal{A}_2)$, namely, (4.1) for $n = 1$. In particular, the proof reveals us the lack of exponential stability as well as it prevents any other uniform decay patterns (Corollary 3.4) when $\chi_0 \neq 0$. Unfortunately, since the technique employed in the proof of Theorem 3.3 requires compatibility between the symmetry of the system and boundary conditions, the optimality only works for the mixed boundary condition (3.8). An analogous approach does not work well for (3.7). However, due to the conservative nature of both boundary conditions, one might expect the optimality in case (3.7). This fact is still open.

III. Exponential stability. Although Theorem 3.5 is already proved in [24, Theorem 3.1] when $\chi_0 = 0$, here we consider it as a complementary result by the simple fact that it can be extended to any other boundary conditions, some exemplified below. Indeed, the main advantage of computations provided in Sect. 3.3 relies on the fact that all localized estimates are given by means of auxiliary cut-off multipliers, and no further information on boundary point-wise terms is required. This is quite different from the proofs by contradictions arguments presented in [24, Sect. 3], see for instance on pages 63-64, wherein the case of the boundary condition (3.7), the convergence of boundary terms must be analyzed in (3.26) therein.

IV. Boundary conditions. It is worth mentioning again that all local estimates in Sect. 3.3 are independent of any boundary conditions. As a consequence, no boundary point-wise terms must be handled and the proofs as in Sect. 3.4 will follow in the same way. Hence, the polynomial and exponential stability, namely, Theorems 3.2 and 3.5, respectively, can be similarly reproduced for other boundary conditions. Among them, we highlight the following where the existence result holds true.

$$\begin{aligned}
&\varphi(x,t) = \psi_x(x,t) = w(x,t) = \vartheta(x,t) \\
&\quad = \eta_x(x,t) = 0, \qquad x \in \{0, L\},\ t \geq 0, \\
&\varphi(x,t) = \psi(x,t) = w_x(x,t) = \vartheta_x(x,t) \\
&\quad = \eta(x,t) = 0, \qquad x \in \{0, L\},\ t \geq 0, \\
&\varphi(x,t) = \psi(x,t) = w(x,t) = \vartheta_x(x,t) \\
&\quad = \eta_x(x,t) = 0, \qquad x \in \{0, L\},\ t \geq 0, \\
&\varphi_x(x,t) = \psi(x,t) = w(x,t) = \vartheta(x,t) \\
&\quad = \eta(x,t) = 0, \qquad x \in \{0, L\},\ t \geq 0, \\
&\varphi_x(x,t) = \psi(x,t) = w(x,t) = \vartheta_x(x,t) \\
&\quad = \eta_x(x,t) = 0, \qquad x \in \{0, L\},\ t \geq 0, \\
&\varphi_x(x,t) = \psi(x,t) = w(x,t) = \vartheta(x,t) \\
&\quad = \eta_x(x,t) = 0, \qquad x \in \{0, L\},\ t \geq 0, \\
&\varphi_x(x,t) = \psi(x,t) = w(x,t) = \vartheta_x(x,t) \\
&\quad = \eta(x,t) = 0, \qquad x \in \{0, L\},\ t \geq 0,
\end{aligned}$$

and variations so on. Moreover, any other mixed boundary conditions such as Dirichlet (or Neumann) at $x = 0$ and Neumann (or Dirichlet) on $x = L$ can be also considered provided that the existence (in particular the dissipative condition (3.13)) is satisfied.

V. Invariance. Under the above statements, one sees that Theorem 3.2 (for $\chi_0 \neq 0$) and Theorem 3.5 (for $\chi_0 = 0$) are invariant with respect to boundary conditions linked to the thermoelastic Bresse system (3.1)-(3.5). We recall that (3.1)-(3.5) represents the Bresse system with temperature deviations along the longitudinal and vertical directions, here with couplings on the axial force and bending moment as a first case. In addition, we advance that in the next two forthcoming works of the trilogy the analogous results (to Theorems 3.2 and 3.5) will be proved for Bresse systems with thermal couplings on the shear force and bending moment (second case) and also on the shear and axial forces (third case). Therefore, we will be able to conclude that the results on polynomial and exponential stability of the whole three cases are invariant not only under the boundary conditions but also under the thermal couplings on two displacements of the system.

VI. Physical setup. We also stress that a deep physical setting (including visual interpretations) of the three aforementioned thermoelastic couplings, namely,

- axial force and bending moment (related to problem (1.1));
- shear force and bending moment (coming case);
- shear and axial forces (final case);

will be provided in the last related paper of the trilogy, where a precise justification of the linear thermoelastic models will be made by means of constitutive laws in mathematical-physics. To this purpose, we shall use the theory developed by [22,23] in combination with the classical Bresse model, see [7], whose governing equations are very well-known, see for instance [28,29] (Eqs. (1.1)–(1.3) therein), among several others. For a nice picture describing the variables of the Bresse system, we refer to [10,14] (Fig. 1 therein).

VII. Nonlinear generalizations. As we know, the linear thermoelastic Bresse systems can be derived by themselves with no mention to a more general nonlinear configuration. However, as mentioned in the introduction the thermoelastic problem (1.1) (among others) can also be seen as a linear system coming from more general nonlinear settings. Indeed, relevant nonlinearities could appear in the modeling of thin thermoelastic curved beams when forces like twist and warping are taken into account. For example, in [22, Sects. 3–5] a list of nonlinear thermoelastic Bresse beams is presented along the sections under several situations. See also [23, Chapt. III]. Therefore, potential generalizations of (1.1) include nonlinear problems that could be studied by regarding a more refined nonlinear analysis on the dynamics of solutions. Other ways to connect problem (1.1) with nonlinear thermoelastic Bresse systems is to consider a similar approach as in [14].

Compliance with ethical standards

Conflicts of Interests. The authors have no conflicts of interest to declare that are relevant to the content of this article.

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