Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 290 (2021) 197–222

Journal of Differential Equations

www.elsevier.com/locate/jde

Stability for extensible beams with a single degenerate nonlocal damping of Balakrishnan-Taylor type

M.M. Cavalcanti [a,1], V.N. Domingos Cavalcanti [a,2], M.A. Jorge Silva [b,*,3], V. Narciso [c,4]

[a] *Department of Mathematics, State University of Maringá, 87020-900, Maringá, PR, Brazil*
[b] *Department of Mathematics, State University of Londrina, 86057-970, Londrina, PR, Brazil*
[c] *Center of Exact Sciences, State University of Mato Grosso do Sul, 79804-970, Dourados, MS, Brazil*

Received 12 December 2020; accepted 26 April 2021

Abstract

In this paper, motivated by recent papers on the stabilization of evolution problems with nonlocal degenerate damping terms, we address an extensible beam model with degenerate nonlocal damping of Balakrishnan-Taylor type. We discuss initially on the well-posedness with respect to weak and regular solutions. Then we show for the first time how hard is to guarantee the stability of the energy solution (related to regular solutions) in the scenarios of constant and non-constant coefficient of extensibility. The degeneracy (in time) of the single nonlocal damping coefficient and the methodology employed in the stability approach are the main novelty for this kind of beam models with degenerate damping.

MSC: 35B35; 35L35; 35L76; 35Q74; 74K20

Keywords: Beam equation; Stability; Degenerate dissipation; Balakrishnan-Taylor damping

* Corresponding author.
E-mail addresses: mmcavalcanti@uem.br (M.M. Cavalcanti), vndcavalcanti@uem.br (V.N. Domingos Cavalcanti), marcioajs@uel.br (M.A. Jorge Silva), vnarciso@uems.br (V. Narciso).
[1] Research of Marcelo M. Cavalcanti partially supported by the CNPq Grant 300631/2003-0.
[2] Research of Valéria N. Domingos Cavalcanti partially supported by the CNPq Grant 304895/2003-2.
[3] Research of Marcio A. Jorge Silva partially supported by the CNPq Grant 301116/2019-9.
[4] Research of Vando Narciso partially supported by the FUNDECT Grant 219/2016.

https://doi.org/10.1016/j.jde.2021.04.028

1. Introduction

In the recent paper by Pucci and Saldi [41], the following Kirchhoff system governed by the fractional p-Laplacian operator and nonlocal damping is considered

$$\begin{cases} u_{tt} + (-\Delta)^s_p u + \mu |u|^{p-2}u + \varrho(t) M\left([u]_s^p\right)|u_t|^{p-2}u_t \\ \qquad\qquad\qquad + Q(t,x,u,u_t) + f(t,x,u) = 0 & \text{in} \quad \mathbb{R}_0^+ \times \Omega, \\ u = 0 & \text{on} \quad \mathbb{R}_0^+ \times (\mathbb{R}^n \backslash \Omega), \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n > ps)$, $u = (u_1, \dots, u_N)$ represents a vectorial displacement $(N \geq 1)$, $\mu \geq 0$, the p-Laplacian is defined by

$$(-\Delta)^s_p \varphi(x) = \int_{\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|^{p-2}[\varphi(x) - \varphi(y)]}{|x-y|^{n+ps}} dy$$

for any function $\varphi \in C_0^\infty(\Omega)$, Q represents a nonlinear damping and f is an external force both given under suitable assumptions, ϱ is a non-negative scalar function lying in $L^1_{\text{loc}}\left(\mathbb{R}_0^+\right)$, M is a dissipative (possibly degenerate) function under proper conditions, and the bracket $[u]_s$ is a nonlocal term given by

$$[u]_s = \left(\|u\|_\Omega^p + 2 \int_\Omega |u(x)|^p dx \int_{\mathbb{R}^n \backslash \Omega} |x-y|^{-(n+ps)} dy \right)^{1/p}, \tag{1.2}$$

where $\| \cdot \|_\Omega$ represents the fractional Gagliardo norm given by

$$\|u\|_\Omega = \left(\iint_{\Omega \times \Omega} |u(x) - u(y)|^p |x-y|^{-(n+ps)} dx dy \right)^{1/p}$$

for all $u \in W_0^{s,p}\left(\Omega; \mathbb{R}^N\right)$. For precise details on the functional spaces, we refer to [41, Section 2]. We still refer to Pan et al. [39] for other types of degenerate Kirchhoff waves involving the fractional Laplacian with nonlinear damping and source terms.

Under the above statements, one sees that the nonlocal nonlinear damping term expressed by $\varrho(t) M\left([u]_s^p\right)|u_t|^{p-2}u_t$ is a prototype of possibly degenerate dissipation to the system, where the nonlocal term is driven by (1.2). Consequently, the stability results have revealed to be very intricate concerning problem (1.1). For global and local asymptotic stability results with respect to (1.1), we refer [41, Sections 3 and 4].

Motivated by the inspiring work [41], and since the stability of evolution models by means of nonlocal damping terms (which could be physically well established) has been attracted the attention recently of several researchers, we consider here a different evolution problem whose nonlocal degenerate damping term comes from a different prototype of dissipative systems. Indeed, as designed below, we extract a nonlocal extensible beam equation from the Balakrishnan-Taylor theory whose model is governed by the bi-harmonic operator. Therefore, when compared to [41], we provide a different technique in order to deal with the corresponding stability result, which is the main novelty for this kind of beam systems with degenerate damping.

1.1. The nonlocal model

In 1989 Balakrishnan and Taylor [5] proposed the following new model for flight structures with nonlinear nonlocal damping in the one dimensional case

$$u_{tt} - 2\zeta\sqrt{\lambda}u_{xx} + \lambda u_{xxxx} - \gamma \left[\int_{-L}^{L} \left(\lambda|u_{xx}|^2 + |u_t|^2\right) dx \right] u_{xxt} = 0, \tag{1.3}$$

where $u = u(x,t)$ represents the transversal deflection of a beam with length $2L > 0$ in the rest position, $\gamma > 0$ is a (small) damping coefficient, ζ is a constant appearing in the approximation of Krylov-Bogoliubov and $\lambda = \frac{2\zeta w}{\sigma^2}$ with w being the mode frequency and σ^2 the spectral density of a Gaussian external force. We refer to [5, Sect. 4] for the modeling of (1.3) (see eq. (4.2) with $q = 1$ therein). The stability of (1.3) is very little known in the literature once we must deeply analyze the character of the degenerate nonlocal damping $-\gamma \left[\int_{-L}^{L} \left(\lambda|u_{xx}|^2 + |u_t|^2\right) dx\right] u_{xxt}$. We observe that the latter is a damping with two nonlocal components. Since γ is small, then neglecting the nonlocal term corresponding to the velocity, problem (1.3) turns into the following equation

$$u_{tt} - 2\zeta\sqrt{\lambda}u_{xx} + \lambda u_{xxxx} - \gamma\lambda \left[\int_{-L}^{L} |u_{xx}|^2 \, dx \right] u_{xxt} = 0. \tag{1.4}$$

Moreover, in order to consider a more challenging work (at least from stability point of view), it would be relevant to take a nonlocal frictional damping instead of the viscous one in (1.4). In this direction, we may also consider the following beam model with nonlocal frictional damping

$$u_{tt} - 2\zeta\sqrt{\lambda}u_{xx} + \lambda u_{xxxx} + \gamma\lambda \left[\int_{-L}^{L} |u_{xx}|^2 \, dx \right] u_t = 0. \tag{1.5}$$

Our main goal in this paper is to analyze the stability of both degenerate problems (1.4) and (1.5), paying more attention to the case where nonlocal frictional damping is taken into account. More precisely, we shall consider the above models (1.4) and (1.5) in the N-dimensional case with extensibility coefficient $2\zeta\sqrt{\lambda}$ replaced by a nonlocal function (encompassing the constant case) and normalizing the remaining constants because they do not change the core of the computations, namely, we study the model

$$u_{tt} + \Delta^2 u - M\left(||\nabla u||_2^2\right)\Delta u + ||\Delta u||_2^2 \, A u_t = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.6}$$

with $A = -\Delta$ or $A = I$ (identity), where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $||\cdot||_2$ stands for the norm in $L^2(\Omega)$ and M corresponds to a non-constant function of extensibility which appears in models of extensible beams, see e.g. Woinowsky-Krieger [52] and Berger [14].

To our best knowledge, there is no result on stability for the degenerate beam model (1.6). Below we consider the literature on related problems.

1.2. A related literature overview

The stabilization of systems like (1.6) is considered by some authors just in the case of non-degenerate damping. In fact, in Lange and Perla Menzala [35], Cavalcanti et al. [18], Jorge Silva and Narciso [30,31], are considered some beam/plate models with the following nonlocal damping

$$N(\|\nabla u\|_2^2)u_t, \tag{1.7}$$

where $N > 0$ is assumed to be a C^1-function on $[0, \infty)$. In all these works the damping coefficient N is bounded from below, that is, there exists a constant $\lambda_0 > 0$ such that $N(s) \geq \lambda_0,\ s \geq 0$, which implies that a full damping is taken into account. In such cases, as provided in [31, Remark 7], we can easily conclude the uniform stabilization of the energy by using multipliers. For papers dealing with asymptotic behavior of beam/plate models by using standard linear or nonlinear full damping we also refer to [6–8,18,19,23–25,34,38,50,51,54] and references therein. But this is not our situation once we are considering a beam equation with degenerate damping (in time) and the multiplier technique is not enough to achieve the stability of the energy solution of (1.6). Indeed, trying direct computations in the present case we are not able to estimate "bad" terms in terms of the damping $\int_0^T ||\Delta u(t)||_2^2||\nabla u_t(t)||_2^2\,dt$ in the case $A = -\Delta$ or $\int_0^T ||\Delta u(t)||_2^2||u_t(t)||_2^2\,dt$ in the case $A = I$ when dealing with (1.6). This statement is clarified in Appendix A at the end of this article. There, we also note that the same difficulty also appears in the case of wave models with degenerate nonlocal damping.

On the other hand, for papers concerning beam models with locally distributed damping in the spatial variable, a pioneer work we found is due to Tucksnak [49] where it is considered the following problem

$$u_{tt} + \Delta^2 u - b\,||\nabla u||_2^2\,\Delta u + a(x)u_t = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.8}$$

where $b > 0$ is a constant and $a \in L^\infty(\Omega),\ a(x) \geq 0$ a.e. in Ω. Assuming that there exists an open set $\omega \subset \Omega$ and $a_0 > 0$ such that

$$\omega \subset \{x \in \Omega;\ a(x) \geq a_0 > 0\}$$

satisfies proper geometrical hypotheses (see assumptions (A1)-(A2) on p. 899 in [49]), the author shows the exponential stabilization of (1.8) when initial data are uniformly bounded in the weak phase space, see for instance [49, Theorem 3.1]. The proof relies on a (direct) multiplier technique that also generates "bad" terms unquantified by a damping of the form $\int_0^T ||\Delta u(t)||_2^2||u_t(t)||_2^2\,dt$, which is our case. We refer again Appendix A for the reader's convenience. This comes from the fact that our damping is not locally distributed in the spatial variable or even degenerates in some points of the domain Ω as considered in the papers by Tucsnak et al. [1,2,9,49], Pazoto et al. [46,20], Tebou [48], Bucci and Toundykov [16], Han and Wang [29], Bortot et al. [15], Geredeli and Webster [28], Wu [53] and Khanmamedov et al. [47,3], among others. Therefore, the uniform stability of beam models with degenerate nonlocal damping $||\Delta u(t)||_2^2 u_t$ seems to be a hard task.

1.3. The main stability result

In order to overcome the above negative attempts in finding multipliers to show the stabilization of the energy $E_u(t)$ associated with (1.6) we have used similar ideas as given in Dehman et al. [26,27] and inspired in the recent work by Cavalcanti et al. [17] for wave models. Our main result is given in Theorem 3.1 that states: In the case of constant coefficient of extensibility (say $M \equiv c > 0$), then for every $R > 0$, there exist constants $K = K(R) > 0$ and $\gamma = \gamma(R) > 0$ such that inequality

$$E_u(t) \le K\ E_u(0)\, e^{-\gamma t}, \quad t > 0, \tag{1.9}$$

holds for every *regular solution* u of problem (1.6) with initial data (u_0, u_1) satisfying

$$\|(u_0, u_1)\|_{(H^4(\Omega)\cap H_0^2(\Omega))\times H_0^2(\Omega)} \le R.$$

In addition, in the case of non-constant extensibility coefficient, we still prove that the energy $E_u(t)$ is stable when t goes to infinity, that is,

$$E_u(t) \longrightarrow 0 \quad \text{as} \quad t \to \infty. \tag{1.10}$$

Actually, this is a *local stability result* since (1.9) and (1.10) are only uniform on every ball with radius $R > 0$ in the strong topology $(H^4(\Omega) \cap H_0^2(\Omega)) \times H_0^2(\Omega)$, but they are not independent of the initial data. The only drawback we have is that we are not able to prove this *local stability result* in the weak topology $H_0^2(\Omega) \times L^2(\Omega)$ by taking initial data uniformly bounded in $H_0^2(\Omega) \times L^2(\Omega)$, as considered e.g. in [26,27]. However, it seems to be the pioneer result for such kinds of nonlocal damped beam models whose degeneracy of the damping coefficient depends upon the solution in time. The methodology used in its proof was inspired by [17] where an observability inequality is proved relying on contradiction arguments. In Section 3 we give a detailed proof of these statements.

1.4. An additional literature overview

It is also worth mentioning that there are some papers addressing plate models with nonlinear displacement-dependent damping $\sigma(u)u_t$, where $\sigma \in C^1(\mathbb{R})$ is a positive function. See, for instance, Kolbasin [33], Chueshov and Kolbasin [21,22] and Khanmamedov [32] where the authors study the long-time dynamics of a class of plate models with (strictly) positive damping coefficient, which is not our case. On the other hand, in what concerns nonlinear wave models with (possibly) degenerate damping we refer to articles by Barbu et al. [10–13] where their results are mainly given with respect to existence, uniqueness and blow up of solutions. Indeed, in [13] it is considered the model

$$u_{tt} - \Delta u + |u|^k \partial j(u_t) = |u|^{p-1}u \quad \text{in} \quad \Omega \times (0, T),\ T > 0,$$

where j is a continuous convex function defined on $\mathbb{R}$ and ∂j is its sub-differential operator. According to the authors the damping term degenerates which contrasts to the existing literature. Under suitable conditions on function j and parameters k and p, they prove existence and uniqueness of weak and regular solutions. In addition, the last issue approached in [13] deals with blow up of the weak solution in a finite time, see e.g. [13, Theorem 1.13]. Therefore, we can not compare our stability result to their nonexistence of global solution result.

We finish this introduction with remarkable earlier problems we found in the literature concerning dissipations degenerating explicitly in time. In fact, to this matter we refer the papers by Autuori and Pucci [4], Martinez [37], Pucci and Serrin [42–45]. For instance, in [37] the author studies the following semilinear wave equation with a time-dependent damping

$$u_{tt} - \Delta u + \rho(t, u_t) = 0 \text{ in } \Omega \times \mathbb{R}_+ \quad \text{and} \quad u = 0 \text{ in } \partial\Omega \times \mathbb{R}_+. \tag{1.11}$$

It is assumed the following condition on the damping term ρ: there exist a non-increasing function $\sigma(t)$ and a strictly increasing odd function $g(v)$ such that $\rho(t, v)$ satisfies

$$\sigma(t) g(|v|) \leq |\rho(t, v)| \leq g^{-1}\left(\frac{|v|}{\sigma(t)}\right), \quad \forall\, t \geq 0,\ \forall\, v \in \mathbb{R}. \tag{1.12}$$

We note that assumption (1.12) has its degeneracy in time through function $\sigma(t)$ but not on $\sigma(u(t))$ (where $u(x,t)$ is the solution of (1.11)) as in the present work. Explicit decay rates estimates can be seen in [37, Theorem 1]. A similar result on stability can be seen in [42, Theorem 3.1], but without an explicit decay rate as in (1.10). Analogous results are proved by Pucci et al. [4,43–45] in the same direction. However, their arguments are not useful in the present problem since degeneracy upon the solution in time concerns another kind of dissipation. Summarizing, it seems that stability results for beam models with Balakrishnan-Taylor damping like in (1.6) are not explored in the literature so far, unless the wave equation recently addressed by Cavalcanti et al. [17] where the degenerate damping coefficient depends upon the average of the gradient of u like (1.7).

The remaining paper is organized as follows. In section 2 we prove the well-posedness to problem (1.6) in the case $A = I$. In Section 3 we prove the stability of the corresponding energy $E_u(t)$ for regular solutions of (1.6). The case $A = -\Delta$ is analyzed at the end of this paper (see Remark 2) as a simpler case than the first one. We end this paper with Appendix A to clarify our difficulty in finding multipliers for degenerate nonlocal wave and beam models.

2. Well-Posedness

In this section we address the well-posedness to the following extensible beam model with degenerate nonlocal weak damping

$$\begin{cases} u_{tt} + \Delta^2 u - M\left(||\nabla u||_2^2\right) \Delta u + ||\Delta u||_2^2 u_t = 0 \quad \text{in} \quad \Omega \times (0, \infty), \\ u = \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, ∂_ν represents the normal derivative and $M(\cdot)$ corresponds to the extensibility coefficient. Throughout this paper we use the following notations on the function spaces

$$L^p(\Omega), \quad ||u||_p^p = \int_\Omega |u(x)|^p \, dx, \quad p \geq 1;$$

$$
\begin{aligned}
&L^2(\Omega), \quad (u,v)=\int_\Omega u(x)v(x)\,dx, \quad ||u||_2^2=\int_\Omega |u(x)|^2\,dx;\\
&H_0^1(\Omega), \quad (u,v)_{H_0^1(\Omega)}=(\nabla u,\nabla v), \quad ||u||_{H_0^1(\Omega)}=||\nabla u||_2,\\
&H_0^2(\Omega), \quad (u,v)_{H_0^2(\Omega)}=(\Delta u,\Delta v), \quad ||u||_{H_0^2(\Omega)}=||\Delta u||_2.
\end{aligned}
$$

The parameter $\lambda_1>0$ corresponds to the following embedding inequalities:

$$
||u||_2^2\le \lambda_1^{-1}||\Delta u||_2^2, \;\; ||\nabla u||_2^2\le \lambda_1^{-1/2}||\Delta u||_2^2, \quad \forall\, u\in H_0^2(\Omega). \tag{2.2}
$$

We also set the following phase spaces

$$
\begin{aligned}
\mathcal{H}&=H_0^2(\Omega)\times L^2(\Omega), & \|(u,v)\|_{\mathcal{H}}^2&=\|\Delta u\|_2^2+\|v\|_2^2,\\
\mathcal{H}_1&=(H^4(\Omega)\cap H_0^2(\Omega))\times H_0^2(\Omega), & \|(u,v)\|_{\mathcal{H}_1}^2&=\|\Delta^2 u\|_2^2+\|\Delta v\|_2^2.
\end{aligned}
$$

The following assumption on M is considered:

Assumption 2.1. $M\in C^1([0,\infty))$ with $M(\tau)\ge-\beta_1$ for all $\tau\ge0$, where $0\le\beta_1<\lambda_1^{1/2}$.

The well-posedness of problem (2.1) can be stated analogously to the papers [18,30,31]. More precisely, we have:

Theorem 2.1 *(Well-Posedness - Part I). Under* **Assumption 2.1** *we have:*

(i) *If $(u_0,u_1)\in\mathcal{H}$, then problem* (2.1) *has a unique weak solution in the class*

$$
(u,u_t)\in C([0,T],\mathcal{H}) \;\; \textit{and} \;\; u_{tt}\in L^\infty(0,T;H^{-2}(\Omega)), \quad \forall\, T>0.
$$

(ii) *If $(u_0,u_1)\in\mathcal{H}_1$, then problem* (2.1) *has a unique regular solution in the class*

$$
(u,u_t)\in L^\infty(0,T;\mathcal{H}_1)\cap C([0,T],\mathcal{H}) \;\; \textit{and} \;\; u_{tt}\in L^\infty(0,T;L^2(\Omega)), \quad \forall\, T>0.
$$

Remark 1. The proof of Theorem 2.1 can be done through Faedo-Galerkin method by following similar arguments as in Cavalcant et al. [18,19] or Jorge Silva and Narciso [30,31]. However, is this work we give an alternative (and more dynamic) proof of the Hadamard well-posedness of (2.1) by using semigroup approach as follows.

2.1. Semigroup solution

We first consider the abstract Cauchy problem related to (2.1). Indeed, denoting by U the vector-valued function

$$
U=(u,v) \quad \text{with} \quad v=u_t,
$$

we can rewrite system (2.1) in the following first order problem

$$\begin{cases} U_t = AU + B(U), \quad t > 0, \\ U(0) = (u_0, u_1) := U_0, \end{cases} \tag{2.3}$$

where $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is a linear operator defined by

$$AU = \begin{pmatrix} v \\ -\Delta^2 u \end{pmatrix}^{\perp}, \quad U = (u, v) \in D(A) = \mathcal{H}_1, \tag{2.4}$$

and $B : \mathcal{H} \to \mathcal{H}$ is the nonlinear operator

$$B(U) = \begin{pmatrix} 0 \\ M\left(||\nabla u||_2^2\right) \Delta u - ||\Delta u||_2^2\, v \end{pmatrix}^{\perp}, \quad U = (u, v) \in \mathcal{H}. \tag{2.5}$$

Thus, the well-posedness result for (2.3), and consequently for the system (2.1), reads as follows:

Theorem 2.2 *(Well-Posedness - Part II). Under* **Assumption 2.1** *we have:*

(i) *If $U_0 \in \mathcal{H}$, then there exists $T_{\max} > 0$ such that problem (2.3) has a unique mild solution $U \in C([0, T_{\max}), \mathcal{H})$, which is given by*

$$U(t) = e^{At} U_0 + \int_0^t e^{A(t-s)} B(U(s))\, ds, \quad t \in [0, T_{\max}).$$

(ii) *If $U_0 \in D(A)$, then the above mild solution U is regular one.*

In both cases, we have that $T_{\max} = +\infty$.

Proof. It is simple to prove that $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ given in (2.4) is the infinitesimal generator of a C_0-semigroup of contractions e^{At} and, in view of Assumption 2.1, $B : \mathcal{H} \to \mathcal{H}$ defined in (2.5) is a locally Lipschitz continuous operator on $\mathcal{H}$. Hence, the proof of (i)-(ii) follows as an immediate consequence of Theorems 1.4 and 1.6 in Pazy's book [40, Chapter 6].

Remains to check that both mild and regular solutions are globally defined, that is, $T_{\max} = +\infty$. Indeed, the energy functional $E_u(t) = E(u(t), u_t(t))$ associated with problem (2.1) is given by

$$E_u(t) = \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\widehat{M}\left(||\nabla u(t)||_2^2\right), \quad t \geq 0, \tag{2.6}$$

where $\widehat{M}(\tau) := \int_0^\tau M(s)\, ds$ is the primitive of M. Then, a straightforward computation gives

$$\frac{d}{dt} E_u(t) = -||\Delta u(t)||_2^2 ||u_t(t)||_2^2 \leq 0, \tag{2.7}$$

from where it follows that $E_u(t)$ is non-increasing with $E_u(t) \leq E_u(0)$ for all $t \in [0, T_{\max})$. Moreover, from Assumption 2.1, the initial energy $E_u(0)$ is a constant depending only on initial data in $\mathcal{H}$ and also, from (2.2), we obtain

$$E_u(t) \geq \frac{\beta}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_2^2 \geq \frac{\beta}{2}\|U(t)\|_{\mathcal{H}}^2,$$

where $\beta = 1 - \beta_1\lambda_1^{-1/2} > 0$, that is,

$$\frac{\beta}{2}\|U(t)\|_{\mathcal{H}}^2 \leq E_u(t) \leq E_u(0), \quad \forall\, t \in [0, T_{\max}). \tag{2.8}$$

Estimate (2.8) implies that any (regular or mild) solution is globally bounded in time. Therefore, from Pazy [40, Theorem 1.4] we conclude that $T_{\max} = +\infty$.

The proof of Theorem 2.2 is now complete. □

3. Stability

In this section we prove the stability and exponential stability (depending on the extensibility coefficient M) of the energy $E_u(t)$ set in (2.6) corresponding to regular solutions of (2.1). We also consider the following "linear" energy $E_u^l(t)$ coming from (2.1) when $M \equiv 0$, namely,

$$E_u^l(t) := \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_2^2 = \frac{1}{2}\|U(t)\|_{\mathcal{H}}^2, \quad t \geq 0. \tag{3.1}$$

Before stating our main result on exponential stability we establish a Lemma that provides a relationship between the energy $E_u(t)$ and linear energy $E_u^l(t)$. To do so, we consider the following additional assumption on M and its primitive $\widehat{M}$.

Assumption 3.1. M and $\widehat{M}$ satisfy

$$\widehat{M}(\tau) \leq M(\tau)\tau + \beta_1\tau \quad \text{for all} \quad \tau \geq 0,$$

where $0 \leq \beta_1 < \lambda_1^{1/2}$.

Lemma 3.1. *Under* **Assumptions** 2.1 *and* 3.1, *the following inequalities hold:*

$$\beta E_u^l(t) \leq E_u(t) \leq \beta_0 E_u^l(t), \quad t \geq 0, \tag{3.2}$$

for any (mild or regular) solution u of (2.1), *where* $\beta = 1 - \beta_1\lambda_1^{-1/2} > 0$ *and*

$$\beta_0 = 1 + \beta_1\lambda_1^{-1/2} + 2\lambda_1^{-1/2}\max_{\tau\in I_0}|M(\tau)| \quad \textit{with} \quad I_0 := \left[0, 2\beta^{-1}\lambda_1^{-1/2}E_u(0)\right]. \tag{3.3}$$

Proof. The first inequality in (3.2) follows promptly from (2.8) and (3.1). The second one is a consequence of (2.2), (2.8) and Assumption 3.1. □

Our main result ensures that the energy $E_u(t)$ is (exponential) stable for every regular solution whose initial data is taken uniformly bounded in the regular phase space $\mathcal{H}_1$.

Theorem 3.1. *Let* **Assumptions** 2.1 *and* 3.1 *be in force. If $M(s) = c$, where c is a positive constant, then for every $R > 0$, there exist constants $K = K(R) > 0$ and $\gamma = \gamma(R) > 0$ such that*

$$E_u(t) \le K \, E_u(0) \, e^{-\gamma t}, \quad \forall \, t > 0, \tag{3.4}$$

for every regular solution u with initial data satisfying $\|(u_0, u_1)\|_{\mathcal{H}_1} \le R$. In addition, in case where $M(s)$ is more general, the energy $E_u(t)$ goes to zero as t goes to infinity, that is,

$$E_u(t) \longrightarrow 0 \quad as \quad t \to \infty. \tag{3.5}$$

Proof. Identity (2.7) implies that $E_u(t)$ is non-increasing and verifies

$$E_u(T) - E_u(0) = -\int_0^T \|\Delta u(t)\|_2^2 \|u_t(t)\|_2^2 \, dt, \quad T > 0. \tag{3.6}$$

Therefore, we claim that it is sufficient to prove the estimate

$$E_u(0) \le C \int_0^T \|\Delta u(t)\|_2^2 \|u_t(t)\|_2^2 \, dt, \tag{3.7}$$

for some $C > 0$, for all $T > 0$, and every initial data verifying $\|(u_0, u_1)\|_{\mathcal{H}_1} \le R$. Indeed, combining (3.6) and (3.7), since $E_u(T) \le E_u(0)$, and recalling that the solution of (2.1) satisfies the semigroup property, then the exponential decay (3.4) as well as the decay (3.5) will follow in a standard way. For the sake of the readers, these facts will be explained at the end of the proof. In what follows our aim is to prove (3.7).

We argue by contradiction. Let us suppose that (3.7) does not hold. Then, there exist a time $T > 0$ and a sequence of regular solutions u^n of (2.1) such that

$$\frac{E_{u^n}(0)}{\int_0^T \|\Delta u^n(t)\|_2^2 \|u_t^n(t)\|_2^2 \, dt} \longrightarrow \infty \quad \text{when} \quad n \to \infty, \tag{3.8}$$

with

$$\|(u_0^n, u_1^n)\|_{\mathcal{H}_1} \le R, \quad n \in \mathbb{N}. \tag{3.9}$$

Using Assumption 3.1 and (3.9) there exists a constant $C_R > 0$ such that

$$E_{u^n}(0) \le C_R, \quad n \in \mathbb{N}. \tag{3.10}$$

From (3.8) and (3.10) it follows that

$$\int_0^T \|\Delta u^n(t)\|_2^2 \|u_t^n(t)\|_2^2 \, dt \longrightarrow 0 \quad \text{when} \quad n \to \infty. \tag{3.11}$$

On the other hand, since we are dealing with regular solutions whose initial data are taken into uniformly bounded sets, then using standard Galerkin's estimates (see e.g. [18,19,31]) we obtain

$$u^n \rightharpoonup u \quad \text{weakly star in} \quad L^\infty(0,T;H^4(\Omega)\cap H_0^2(\Omega)), \tag{3.12}$$

$$u_t^n \rightharpoonup u_t \quad \text{weakly star in} \quad L^\infty(0,T;H_0^2(\Omega)), \tag{3.13}$$

$$u_{tt}^n \rightharpoonup u_{tt} \quad \text{weakly star in} \quad L^\infty(0,T;L^2(\Omega)). \tag{3.14}$$

Moreover, in the particular case $M(s)=c>0$, higher Galerkin estimates can be done globally in time, that is, one can replace T by ∞ in the limits (3.12)-(3.14). Indeed, in this situation we note that for each n the solution u^n satisfies

$$\begin{cases} u_{tt}^n + \Delta^2 u^n - c\Delta u^n + ||\Delta u^n||_2^2 u_t^n = 0 \quad \text{in} \quad \Omega\times(0,\infty), \\ u^n = \partial_\nu u^n = 0 \quad \text{on} \quad \partial\Omega\times(0,\infty), \\ u^n(x,0)=u_0^n(x),\ u_t^n(x,0)=u_1^n(x), \quad x\in\Omega. \end{cases} \tag{3.15}$$

Taking the multiplies u_t^n, $-\Delta u_t^n$ and $\Delta^2 u_t^n$ with (3.15) in a Faedo-Galerkin scheme, and following similar arguments as in [30,31], then standard computations yield

$$\begin{aligned} &\|u_t^n(t)\|_2^2 + \|\Delta u^n(t)\|_2^2 + c\|\nabla u^n(t)\|_2^2 + 2\int_0^t ||\Delta u^n(s)||_2^2\|u_t^n(s)\|_2^2 ds \\ &\qquad \le \|u_1^n\|_2^2 + \|\Delta u_0^n\|_2^2 + c\|\nabla u_0^n\|_2^2, \\ &\|\nabla u_t^n(t)\|_2^2 + \|\nabla\Delta u^n(t)\|_2^2 + c\|\Delta u^n(t)\|_2^2 + 2\int_0^t ||\Delta u^n(s)||_2^2\|\nabla u_t^n(s)\|_2^2 ds \\ &\qquad \le \|\nabla u_1^n\|_2^2 + \|\nabla\Delta u_0^n\|_2^2 + c\|\Delta u_0^n\|_2^2, \\ &\|\Delta u_t^n(t)\|_2^2 + \|\Delta^2 u^n(t)\|_2^2 + c\|\nabla\Delta u^n(t)\|_2^2 + 2\int_0^t ||\Delta u^n(s)||_2^2\|\Delta u_t^n(s)\|_2^2 ds \\ &\qquad \le \|\Delta u_1^n\|_2^2 + \|\Delta^2 u_0^n\|_2^2 + c\|\nabla\Delta u_0^n\|_2^2, \end{aligned} \tag{3.16}$$

and from (3.9) we obtain the uniform boundedness of higher order for solutions independent of time, which allows us to achieve the limits (3.12)-(3.14) on $[0,\infty)$ with respect to time.

In addition, since embeddings $H^4(\Omega)\cap H_0^2(\Omega)\hookrightarrow H_0^2(\Omega)\hookrightarrow L^2(\Omega)$ are compact, then applying Aubin-Lions Theorem and passing a subsequence if necessary, we also have

$$u^n \to u \quad \text{strongly in} \quad L^2(0,T;H_0^2(\Omega)), \tag{3.17}$$

$$u_t^n \to u_t \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{3.18}$$

Further, due to continuity of solutions and Arzelá-Ascoli Theorem, we infer

$$||\Delta u^n(\cdot)||_2^2 \;\to\; ||\Delta u(\cdot)||_2^2 \;\text{ uniformly in }\; [0,T], \tag{3.19}$$

$$||u_t^n(\cdot)||_2^2 \;\to\; ||u_t(\cdot)||_2^2 \;\text{ uniformly in }\; [0,T]. \tag{3.20}$$

Indeed, regarding convergences (3.12)-(3.14), we get

$$\{||\Delta u^n(\cdot)||_2^2\}, \{||u_t^n(\cdot)||_2^2\} \;\text{ are bounded in }\; H^2([0,T]) \hookrightarrow C^1([0,T]). \tag{3.21}$$

Moreover, convergences (3.17)-(3.18) and the regularity of the functions u^n imply

$$||\Delta u^n(t)||_2^2 \;\to\; ||\Delta u(t)||_2^2 \;\text{ for every }\; t \in [0,T], \tag{3.22}$$

$$||u_t^n(t)||_2^2 \;\to\; ||u_t(t)||_2^2 \;\text{ for every }\; t \in [0,T]. \tag{3.23}$$

Noting that $\frac{d}{dt}||\Delta u^n(t)||_2^2 = 2(\Delta u^n(t), \Delta u_t^n(t))| \le 2||\Delta u^n(t)||_2||\Delta u_t^n(t)||_2$ and

$$||\Delta u^n(t_1)||_2^2 - ||\Delta u^n(t_2)||_2^2 = \int_{t_2}^{t_1} \frac{d}{ds}||\Delta u^n(s)||_2^2\,ds, \quad t_1, t_2 \in [0,T],$$

then Galerkin's estimates yield, for some positive constant L_1,

$$\Big| ||\Delta u^n(t_1)||_2^2 - ||\Delta u^n(t_2)||_2^2 \Big| \le L_1|t_1 - t_2|, \quad t_1, t_2 \in [0,T]. \tag{3.24}$$

Analogously, one has

$$\Big| ||u_t^n(t_1)||_2^2 - ||u_t^n(t_2)||_2^2 \Big| \le L_2|t_1 - t_2|, \quad t_1, t_2 \in [0,T]. \tag{3.25}$$

The above estimates (3.21) and (3.24)-(3.25) show that $\{||\Delta u^n(\cdot)||_2^2\}, \{||u_t^n(\cdot)||_2^2\}$ are uniformly bounded and equicontinuous on $[0,T]$ and, therefore, in view of Arzelá-Ascoli Theorem together with convergences (3.22)-(3.23) establish the desired convergences (3.19)-(3.20). Therefore, in light of these uniformly limits (3.19)-(3.20) and the assumptions on M (see **Assumptions 2.1** and **3.1**) we have, in particular, the following convergence for $t = 0$:

$$\lim_{n\to+\infty} E_{u^n}(0) = E_u(0) \ge 0, \tag{3.26}$$

where $E_u(t)$, $t \ge 0$, is the energy defined in (2.6), which is associated with the original problem (2.1). We note that we have two possibilities, namely, $E_u(0) = 0$ (where the limit solution has the energy equal identically zero) or else $E_u(0) > 0$ for non-trivial limit solution. But in both cases we are going to see that the desired contradiction happens. The first one ($E_u(0) = 0$) is more delicate and a normalization of the problem is necessary whereas the second one ($E_u(0) > 0$) it is easier to examine. In what follows, we will analyze these two cases separately.

Case 1. *Limit solution with energy identically zero.* In this case, let us first define the following sequences:

$$\alpha_n = [E_{u^n}(0)]^{\frac{1}{2}} \quad \text{ and } \quad v_n = \frac{u^n}{\alpha_n}. \tag{3.27}$$

Then, from (3.26) one has $\alpha_n \to 0$. We also observe that for null initial data, the only solution is the trivial one and nothing needs to be done. Thus, to the next considerations we consider initial data $(u_0, u_1) \neq (0, 0)$. This implies that $\|\Delta u_0\|_2^2 > 0$ or else $\|u_1\|_2^2 > 0$.

If we have $\|\Delta u_0\|_2^2 > 0$, then we choose $\epsilon_0 > 0$ small enough verifying $\|\Delta u_0\|_2^2 > \epsilon_0 > 0$. The uniform convergence (3.19) and the continuity of the map $t \mapsto \|\Delta u(t)\|_2^2$ guarantee that for all $\eta > 0$ there exists $T_1 > 0$ such that

$$\left| \|\Delta u^n(t)\|_2^2 - \|\Delta u_0\|_2^2 \right| < \eta, \quad \forall\, t \in [0, T_1),\ \forall\, n \text{ large enough.}$$

In particular, considering $\eta = \frac{\epsilon_0}{2} > 0$, we obtain

$$\|\Delta u^n(t)\|_2^2 > \frac{\epsilon_0}{2} > 0, \quad \forall\, t \in [0, T_1),\ \forall\, n \text{ large enough.} \tag{3.28}$$

Now, we observe that since u^n satisfies problem

$$u_{tt}^n + \Delta^2 u^n - M\left(||\nabla u^n(t)||_2^2\right) \Delta u^n + \|\Delta u^n(t)\|_2^2\, u_t^n = 0, \quad n \in \mathbb{N}, \tag{3.29}$$

then, in view of (3.27), v^n satisfies the following sequence of problems

$$v_{tt}^n + \Delta^2 v^n - \frac{1}{\alpha_n} M\left(||\nabla u^n(t)||_2^2\right) \Delta u^n + \|\Delta u^n(t)\|_2^2\, v_t^n = 0, \quad n \in \mathbb{N}, \tag{3.30}$$

with corresponding energies

$$E_{v^n}(t) = \frac{1}{2}\|\Delta v^n(t)\|_2^2 + \frac{1}{2}\|v_t^n(t)\|_2^2 + \frac{1}{2{\alpha_n}^2}\widehat{M}\left(||\nabla u^n(t)||_2^2\right)$$

and linear one

$$E_{v^n}^l(t) = \frac{1}{2}\|\Delta v^n(t)\|_2^2 + \frac{1}{2}\|v_t^n(t)\|_2^2 = \frac{E_{u^n}^l(t)}{{\alpha_n}^2}.$$

Similarly to (2.7) one can also check

$$\frac{d}{dt}E_{v^n}(t) = -||\Delta u^n(t)||_2^2||v_t^n(t)||_2^2, \quad t > 0. \tag{3.31}$$

Firstly, we claim that there exists a constant $0 < \widetilde{C}_R < 1$ such that

$$E_{v^n}(0) \geq \beta\widetilde{C}_R > 0, \quad \forall\, n \in \mathbb{N}. \tag{3.32}$$

Indeed, using Assumption 2.1, (2.2) and (3.2), it follows that

$$E_{v^n}(t) \geq \beta\, E_{v^n}^l(t) = \beta\, \frac{E_{u^n}^l(t)}{\alpha_n^2} \geq \beta\left(\frac{1}{\beta_0}\right)\frac{E_{u^n}^l(t)}{\alpha_n^2}, \quad \forall\, t \geq 0,\ n \in \mathbb{N}.$$

Thus, taking $t=0$ in the above inequality and $\widetilde{C}_R = \frac{1}{\beta_0} > 0$, the desired inequality (3.32) follows by noting that from (3.10) the constant $\beta_0 > 1$ defined in (3.3) is finite.

On the other hand, combining (3.8) with (3.28) and definition of α_n in (3.27), we infer

$$\frac{\epsilon_0}{2}\int_0^{T_1} \|v_t^n(t)\|_2^2\, dt \longrightarrow 0 \quad \text{as} \quad n\to\infty,$$

that is,

$$v_t^n \to 0 \quad \text{strongly in} \quad L^2(0,T_1;L^2(\Omega)). \tag{3.33}$$

In addition, from inequality (3.2) and since $E_{u^n}(t) \le E_{u^n}(0)$ we observe that

$$\|\Delta v^n(t)\|_2^2 + \|v_t^n(t)\|_2^2 = 2E_{v^n}^l(t) = 2\frac{E_{u^n}^l(t)}{{\alpha_n}^2} \le \frac{2}{\beta}\frac{E_{u^n}(t)}{{\alpha_n}^2} \le \frac{2}{\beta}, \quad \forall\, t\ge 0,\ n\in\mathbb{N}. \tag{3.34}$$

Thus, passing to a subsequence if necessary, we have

$$v^n \rightharpoonup v \quad \text{weakly star in} \quad L^\infty(0,T_1;H_0^2(\Omega)), \tag{3.35}$$

$$v_t^n \rightharpoonup v_t = 0 \quad \text{weakly star in} \quad L^\infty(0,T_1;L^2(\Omega)), \tag{3.36}$$

$$v^n \to v \quad \text{strongly in} \quad L^2(0,T_1;L^2(\Omega)). \tag{3.37}$$

Now, multiplying (3.30) by $\varphi(t)v^n$, where φ is defined in (3.44), and integrating over $\Omega\times(0,T_1)$, we get the identity

$$\begin{aligned}
&-\int_0^{T_1}\int_\Omega \varphi'(t)v^n v_t^n\,dxdt - \int_0^{T_1}\int_\Omega \varphi(t)|v_t^n|^2\,dxdt + \int_0^{T_1}\int_\Omega \varphi(t)|\Delta v^n|^2\,dxdt \\
&\quad + \int_0^{T_1}\int_\Omega \varphi(t)M(||\nabla u^n(t)||_2^2)|\nabla v^n|^2\,dxdt + \int_0^{T_1}\int_\Omega \varphi(t)\|\Delta u^n(t)\|_2^2\, v^n v_t^n\,dxdt = 0,
\end{aligned}$$

where we omit the parameter (x,t) for convenience. Since $\{v^n\}$ is bounded in $L^2(0,T_1;L^2(\Omega))$ and $\{u^n\}$ is bounded in $L^\infty(0,T;H_0^2(\Omega))$, then passing the above expression to the limit and taking into account (3.33) we obtain

$$\int_0^{T_1} \varphi(t)\big[||\Delta v^n(t)||_2^2 + M(||\nabla u^n(t)||_2^2)||\nabla v^n(t)||_2^2\big]\,dt \longrightarrow 0.$$

Applying again Assumption 2.1 and (2.2) one has

$$\beta \int_{\varepsilon}^{T_1-\varepsilon} ||\Delta v^n(t)||_2^2\, dt \le \int_0^{T_1} \varphi(t)\big[||\Delta v^n(t)||_2^2 - \beta_1 \lambda_1^{-1/2} ||\Delta v^n(t)||_2^2\big]\, dt$$

$$\le \int_0^{T_1} \varphi(t)\big[||\Delta v^n(t)||_2^2 + M(||\nabla u^n(t)||_2^2)||\nabla v^n(t)||_2^2\big]\, dt \longrightarrow 0,$$

from where it follows

$$v^n \to 0 \quad \text{strongly in } L^2(0, T_1; H_0^2(\Omega)). \tag{3.38}$$

The strong limits (3.33) and (3.38) along with Assumption 3.1 are enough to obtain

$$\int_0^{T_1} E_{v^n}(t)\, dt \to 0 \quad \text{when} \quad n \to \infty.$$

Going back to (3.31) and integrating it on $[0, T_1]$ it follows that

$$T_1 E_{v^n}(0) = \int_0^{T_1} E_{v^n}(t)\, dt + \int_0^{T_1}\int_0^{t} ||\Delta u^n(s)||_2^2 ||v_t^n(s)||_2^2\, ds\, dt,$$

and from (3.8) it is easy to see that the last term of the above equality goes to zero when $n \to \infty$. Therefore, $E_{v^n}(0) \to 0$ which is a contradiction with (3.32).

It remains to analyze the situation $\|u_1\|_2^2 > 0$. We will proceed similarly as above.

If $\|u_1\|_2^2 > 0$, we choose $\epsilon_1 > 0$ such that $\|u_1\|_2^2 > \epsilon_1$. From (3.20) and since the map $t \mapsto \|u_t(t)\|_2^2$ is also continuous, for all $\eta > 0$ there exists $T_1 > 0$ such that

$$\big|\|u_t^n(t)\|_2^2 - \|u_1\|_2^2\big| < \eta, \quad \forall\, t \in [0, T_1),\ \forall\, n \text{ large enough.}$$

In particular, choosing $\eta = \frac{\epsilon_1}{2} > 0$, we obtain

$$\|u_t^n(t)\|_2^2 > \frac{\epsilon_1}{2} > 0, \quad \forall\, t \in [0, T_1),\ \forall\, n \text{ large enough.} \tag{3.39}$$

Let us consider the same sequences (α_n) and (v_n) as defined in (3.27). Thus, we remember that (3.32) still holds once we are considering the same sequence v^n. On the other hand, combining (3.8) and (3.39), and observing definitions of α_n, v_n in (3.27), it results

$$\frac{\epsilon_1}{2}\int_0^{T_1} \|\Delta v^n(t)\|_2^2\, dt \longrightarrow 0 \quad \text{when} \quad n \to \infty,$$

that is,

$$v^n \to 0 \quad \text{strongly} \quad \text{in} \quad L^2(0,T_1;H_0^2(\Omega)). \tag{3.40}$$

In addition, $E_{v^n}(t)$ is bounded for all $t \geq 0$ (see (3.34)) and then

$$\begin{aligned} v^n &\rightharpoonup v=0 \quad \text{weakly* in} \quad L^\infty(0,T_1;H_0^2(\Omega)),\\ v_t^n &\rightharpoonup v_t \quad \text{weakly* in} \quad L^\infty(0,T_1;L^2(\Omega)),\\ v^n &\to 0 \quad \text{strongly in} \quad L^2(0,T_1;L^2(\Omega)). \end{aligned}$$

Multiplying the following sequence of problems (namely, problem (3.30))

$$v_{tt}^n + \Delta^2 v^n - \frac{1}{\alpha_n} M\left(||\nabla u^n(t)||_2^2\right)\Delta u^n + \|\Delta u^n(t)\|_2^2\, v_t^n = 0$$

by $\varphi(t)v^n$, where φ is defined in (3.44), and integrating over $\Omega \times (0,T_1)$, we have

$$\begin{aligned} &-\int_0^{T_1}\int_\Omega \varphi'(t)v^n v_t^n\,dxdt - \int_0^{T_1}\int_\Omega \varphi(t)|v_t^n|^2\,dxdt + \int_0^{T_1}\int_\Omega \varphi(t)|\Delta v^n|^2\,dxdt\\ &\quad + \int_0^{T_1}\int_\Omega \varphi(t)M(||\nabla u^n(t)||_2^2)|\nabla v^n|^2\,dxdt + \int_0^{T_1}\int_\Omega \varphi(t)\|\Delta u^n(t)\|_2^2\, v^n v_t^n\,dxdt = 0. \end{aligned}$$

Similar to (3.48) and from (3.40) we infer

$$\int_0^{T_1}\int_\Omega \varphi(t)|\Delta v^n|^2\,dxdt + \int_0^{T_1}\int_\Omega \varphi(t)M(||\nabla u^n(t)||_2^2)|\nabla v^n|^2\,dxdt \longrightarrow 0,$$

and

$$\int_0^{T_1}\int_\Omega \varphi'(t)v^n v_t^n\,dx,\ \int_0^{T_1}\int_\Omega \varphi(t)\|\Delta u^n(t)\|_2^2\, v^n v_t^n\,dxdt \longrightarrow 0,$$

from where we conclude

$$v_t^n \to 0 \quad \text{strongly} \quad \text{in} \quad L^2(0,T_1;L^2(\Omega)). \tag{3.41}$$

The convergences (3.40)-(3.41) ensure that $E_{v^n}(0) \to 0$, which is a contradiction with (3.32). Hence, in both situations within the Case 1, we conclude that estimate (3.7) holds true.

Case 2. *Limit solution with energy non identically zero.* This is the easier case to be analyzed and from (3.26)-(3.27) one gets $\alpha_n \to \alpha > 0$ with $\alpha := [E_u(0)]^{1/2}$. We still note that for null initial data, there is nothing to do. Thus, let us consider non-null initial data so that $\|\Delta u_0\|_2^2 > 0$ or $\|u_1\|_2^2 > 0$.

If $\|\Delta u_0\|_2^2 > 0$, we pick up $\epsilon_0 > 0$ small enough so that $\|\Delta u_0\|_2^2 > \epsilon_0 > 0$. Then we remember that (3.28) can be taken into account and, therefore, going back to (3.11) and choosing T_1 sufficiently small such that $0 < T_1 \le T$, we infer from (3.28) that

$$\frac{\epsilon_0}{2}\int_0^{T_1} \|u_t^n(t)\|_2^2\, dt \longrightarrow 0 \quad \text{when} \quad n \to \infty,$$

that is,

$$u_t^n \to 0 \quad \text{strongly} \quad \text{in} \quad L^2(0, T_1; L^2(\Omega)). \tag{3.42}$$

In this case, we just take the sequence of problems in u^n (no normalization is necessary)

$$u_{tt}^n + \Delta^2 u^n - M\left(||\nabla u^n(t)||_2^2\right)\Delta u^n + \|\Delta u^n(t)\|_2^2\, u_t^n = 0, \quad n \in \mathbb{N}, \tag{3.43}$$

and the cut-off function $\varphi \in C_0^\infty(0, T_1)$ satisfying

$$\varphi(t) \ge 0, \quad \varphi(t) = 1, \quad t \in (\varepsilon, T_1 - \varepsilon), \quad 0 < \varepsilon < T_1. \tag{3.44}$$

Multiplying (3.43) by $\varphi(t)u^n$ and integrating on $\Omega \times (0, T_1)$, we get

$$\begin{aligned}
&-\int_0^{T_1}\int_\Omega \varphi'(t)u^n u_t^n\, dxdt - \int_0^{T_1}\int_\Omega \varphi(t)|u_t^n|^2\, dxdt + \int_0^{T_1}\int_\Omega \varphi(t)|\Delta u^n|^2\, dxdt \\
&\quad + \int_0^{T_1}\int_\Omega \varphi(t)M(||\nabla u^n(t)||_2^2)|\nabla u^n|^2\, dxdt + \int_0^{T_1}\int_\Omega \varphi(t)\|\Delta u^n(t)\|_2^2\, u^n u_t^n\, dxdt = 0,
\end{aligned}$$

where we omit the parameter (x, t) for convenience. Passing the above expression to the limit when $n \to \infty$, noting that $\{u^n\}$ is bounded in $L^2(0, T_1; L^2(\Omega))$ and in $L^\infty(0, T_1; H_0^2(\Omega))$ and applying convergence (3.42), then it holds for every $\varepsilon > 0$ that

$$\int_0^{T_1} \varphi(t)\big[||\Delta u^n(t)||_2^2 + M(||\nabla u^n(t)||_2^2)||\nabla u^n(t)||_2^2\big]\, dt \longrightarrow 0.$$

From Assumption 2.1 we have $M(||\nabla u^n(t)||_2^2) \ge -\beta_1$ on $[0, \infty)$. Thus, using (2.2) and regarding that $\beta = 1 - \beta_1\lambda_1^{-1/2} > 0$, we obtain

$$\beta \int_{\varepsilon}^{T_1-\varepsilon} ||\Delta u^n(t)||_2^2 \, dt \leq \int_0^{T_1} \varphi(t)\big[||\Delta u^n(t)||_2^2 - \beta_1 \lambda_1^{-1/2} ||\Delta u^n(t)||_2^2\big] \, dt$$

$$\leq \int_0^{T_1} \varphi(t)\big[||\Delta u^n(t)||_2^2 + M(||\nabla u^n(t)||_2^2)||\nabla u^n(t)||_2^2\big] \, dt \longrightarrow 0,$$

as n goes to infinity. Therefore, due to arbitrariness of ε we conclude

$$u^n \to 0 \quad \text{strongly in} \quad L^2(0, T_1; H_0^2(\Omega)). \tag{3.45}$$

The limits (3.42) and (3.45), along with Assumption 3.1 and embedding $H_0^2(\Omega) \hookrightarrow H_0^1(\Omega)$, are sufficient to conclude that

$$\int_0^{T_1} E_{u^n}(t) \, dt \to 0 \quad \text{when} \quad n \to \infty.$$

On the other hand, since (3.6) holds true for every solution u^n and time $t > 0$, then integrating it over $[0, T_1]$ we have

$$T_1 E_{u^n}(0) = \int_0^{T_1} E_{u^n}(t) \, dt + \int_0^{T_1} \int_0^t \|\Delta u^n(s)\|_2^2 \|u_t^n(s)\|_2^2 \, ds \, dt. \tag{3.46}$$

Taking into account (3.11) it is possible to check that the last term of (3.46) also goes to zero when $n \to \infty$. This implies that $E_{u^n}(0) \to 0$, which contradicts the condition $E_{u^n}(0) = \alpha_n^2 \to \alpha^2 > 0$.

If $\|u_1\|_2^2 > 0$, we also choose $\epsilon_1 > 0$ such that $\|u_1\|_2^2 > \epsilon_1$. In such case, we can also regard (3.39) to the next estimates. Indeed, from (3.39), returning to (3.11) and choosing (without loss of generality) $0 < T_1 \leq T$, then

$$\frac{\epsilon_1}{2} \int_0^{T_1} \|\Delta u^n(t)\|_2^2 \, dt \longrightarrow 0 \quad \text{when} \quad n \to \infty,$$

that is to say

$$u^n \to 0 \quad \text{strongly in} \quad L^2(0, T_1; H_0^2(\Omega)). \tag{3.47}$$

In addition, multiplying the sequence of problems

$$u_{tt}^n + \Delta^2 u^n - M\left(||\nabla u^n(t)||_2^2\right) \Delta u^n + \|\Delta u^n(t)\|_2^2 u_t^n = 0, \quad n \in \mathbb{N},$$

by $\varphi(t) u^n$, where the cut-off function φ is defined in (3.44) and integrating over $\Omega \times (0, T_1)$, yields

$$
-\int_0^{T_1}\int_\Omega \varphi'(t)u^n u_t^n\,dxdt - \int_0^{T_1}\int_\Omega \varphi(t)|u_t^n|^2\,dxdt + \int_0^{T_1}\int_\Omega \varphi(t)|\Delta u^n|^2\,dxdt
$$
$$
+\int_0^{T_1}\int_\Omega \varphi(t)M(||\nabla u^n(t)||_2^2)|\nabla u^n|^2\,dxdt + \int_0^{T_1}\int_\Omega \varphi(t)\|\Delta u^n(t)\|_2^2\,u^n u_t^n\,dxdt = 0.
$$

Since $\{u_t^n\}$ is bounded in $L^2(0,T_1;L^2(\Omega))$ and $\{u^n\}$ is bounded in $L^\infty(0,T_1;H_0^2(\Omega))$, and taking the convergence (3.47) into account, then the first and last terms in the above identity go to zero when $n\to\infty$. Moreover, from (2.2), Assumption 2.1 and (3.10), we obtain

$$
\begin{aligned}
0 &\le \int_0^{T_1}\int_\Omega \varphi(t)|\Delta u^n|^2\,dxdt + \int_0^{T_1}\int_\Omega \varphi(t)M(||\nabla u^n(t)||_2^2)|\nabla u^n|^2\,dxdt \\
&= \int_0^{T_1} \varphi(t)\big[||\Delta u^n(t)||_2^2 + M(||\nabla u^n(t)||_2^2)||\nabla u^n(t)||_2^2\big]\,dt \qquad (3.48)\\
&\le \overline{C}_R\int_0^{T_1} \|\Delta u^n(t)\|_2^2\,dt \;\longrightarrow\; 0,
\end{aligned}
$$

for some constant $\overline{C}_R>0$. Then, it holds that

$$
\int_0^{T_1}\int_\Omega \varphi(t)|\Delta u^n|^2\,dxdt + \int_0^{T_1}\int_\Omega \varphi(t)M(||\nabla u^n(t)||_2^2)|\nabla u^n|^2\,dxdt \;\longrightarrow\; 0.
$$

From the above convergences and due to the arbitrariness of ε it follows that

$$
u_t^n \;\to\; 0 \quad \text{strongly in} \quad L^2(0,T_1;L^2(\Omega)). \qquad (3.49)
$$

Therefore, from (3.47)-(3.49) we obtain $E_{u^n}(0)\to 0$ when $n\to\infty$, which also contradicts the fact $E_{u^n}(0)=\alpha_n^2\to\alpha^2>0$.

Hence, in both situations within the Case 2, we can also conclude that (3.7) holds true.

Completion of the proof. Fix $T_0>0$. Then, from inequality (3.7) there exists a constant $C=C(R,T_0)$ such that

$$
E_u(0)\le C(R,T_0)\int_0^{T_0} ||\Delta u(t)||_2^2||u_t(t)||_2^2\,dt. \qquad (3.50)
$$

From the energy identity (2.7) yields

$$\int_0^{T_0} ||\Delta u(t)||_2^2 ||u_t(t)||_2^2 \, dt = -E_u(T_0) + E_u(0). \tag{3.51}$$

Combining (3.50) and (3.51) and since $E_u(T_0) \le E_u(0)$ we infer

$$E_u(T_0)\,(1 + C(R, T_0)) \le C(R, T_0) E_u(0),$$

from which we conclude that

$$E_u(T_0) \le \left(\frac{C(R, T_0)}{1 + C(R, T_0)} \right) E_u(0),$$

and, consequently, since the map $t \mapsto E_u(t)$ is non-increasing we deduce

$$E_u(T) \le \gamma_1 E_u(0), \quad \forall\, T > T_0, \ \text{ where } \gamma_1 := \left(\frac{1}{\tilde{C}_0 + 1} \right), \tag{3.52}$$

and $\tilde{C}_0 = \tilde{C}_0(R, T_0)$. From the boundedness $||(u(T), u_t(T))||_{\mathcal{H}_1} \le C_1(T)$ and proceeding as above we can conclude that

$$E_u(2T) \le \gamma_2 E_u(T), \quad \forall\, T > T_0, \ \text{ where } \gamma_2 := \left(\frac{1}{\tilde{C}_1 + 1} \right) \tag{3.53}$$

and $\tilde{C}_1 = \tilde{C}_1(C_1(T), T_0)$. Thus, from (3.52) and (3.53) we arrive at

$$E_u(2T) \le (\gamma_1 \gamma_2)\, E_u(0), \quad \forall\, T > T_0, \ \text{with } \gamma_1, \gamma_2 < 1,$$

and, recursively, we obtain the following estimate for all $n \in \mathbb{N}$:

$$E_u(nT) \le (\gamma_1 \gamma_2 \cdots \gamma_n)\, E_u(0), \quad \forall\, T > T_0, \ \text{ with } \gamma_1, \gamma_2, \cdots, \gamma_n < 1. \tag{3.54}$$

Therefore, we claim that (3.5) holds true. Indeed, if we assume by contradiction that it does not happen, then $E_u(t)$ is bounded from below by a positive constant $\gamma_0 > 0$, that is, $E_u(t) \ge \gamma_0$ for all $t > 0$. But from (3.54) it follows that $E_u(nT) \le \xi^n\, E_u(0)$ for some $\xi < 1$, from where we obtain a contradiction for n large enough. Consequently, $E_u(t)$ goes to zero when t goes to infinity.

In the particular case when one has $M(s) = c > 0$, we note that similar to (3.16) we have the uniform boundedness

$$||(u(T), u_t(T)||_{\mathcal{H}_1} \le C(||(u_0, u_1)||_{\mathcal{H}_1}) \le C_R < \infty, \quad \forall\, T > 0,$$

where C_R is a positive constant depending only on the size of initial data in the strong topology of $\mathcal{H}_1$, but it does not depend on T. In this case, the exponential decay (3.4) follows easily. Indeed, from the above estimate (3.54) we deduce

$$E_u(nT) \le \xi^n E_u(0), \quad \forall\, T > T_0,$$

for some $0<\xi<1$ depending only on R, T_0. Thus, for any $t>T_0$, we can write $t=nT_0+r$ for $0\le r<T_0$ and, therefore,

$$E_u(t)\le E_u(t-r)=E_u(nT_0)\le \xi^n E_u(0)=\xi^{\frac{t-r}{T_0}}E_u(0)=e^{\frac{t-r}{T_0}\ln\xi}E_u(0),$$

which implies the desired exponential stability. $\square$

Remark 2. The same results (Theorems 2.1 and 3.1) can be easily extended to the original problem with Balakrishnan-Taylor (strong) damping

$$\begin{cases} u_{tt}+\Delta^2 u-M\left(||\nabla u||_2^2\right)\Delta u-||\Delta u||_2^2\,\Delta u_t=0 \quad \text{in} \quad \Omega\times(0,\infty),\\ u=\partial_\nu u=0 \quad \text{or} \quad u=\Delta u=0 \quad \text{on} \quad \partial\Omega\times(0,\infty),\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\quad x\in\Omega. \end{cases} \tag{3.55}$$

Indeed, to the existence part we can use Faedo-Galerkin method whereas in the stability part we have essentially the same estimates by achieving the convergence of the velocity u_t^n (and v_t^n) in $L^2(0,T;H_0^1(\Omega))$ which implies, in particular, the desired limit for such sequences in the energy space for the velocity, namely, in $L^2(0,T;L^2(\Omega)),\ T>0$.

Remark 3. We can also conclude our main results at an abstract level encompassing problems (2.1) and (3.55) in the case of hinged boundary condition. Indeed, let V and H be Hilbert spaces such that V is dense in H and $V\hookrightarrow H$ is compactly embedding. Consider the linear operator A defined by the triple $\{V,H,a(u,v)\}$, where $a(u,v)$ is the inner product in V, and the fractional powers operators A^θ, $\theta\in\mathbb{R}$, defined through spectral theory in functional analysis. Thus, the same methodology used in this paper can be easily extended to the following abstract problem

$$\begin{cases} u_{tt}(t)+Au(t)+M\left(||A^{1/4}u||_2^2\right)A^{1/2}u+||A^{1/2}u(t)||_H^2A^\theta u_t(t)=0, \quad t>0,\\ u(0)=u_0,\quad u_t(0)=u_1, \end{cases} \tag{3.56}$$

for $0\le\theta\le\frac{1}{2}$. A more challenging problem related to (3.56) is to consider the same results by taking a lower power in the damping coefficient such as $||A^\kappa u||_H^2A^\theta u_t$ for $0\le\kappa<1/2$.

Appendix A. Looking for multipliers

Motivated by the statements on multipliers given in Section 1 we shall conclude below that the usual multipliers technique are not enough to deal with degenerate (in time) nonlocal problems.

A.1. The wave model

We start with the simple case of degenerate wave models. Let us consider the following damped problem

$$\begin{cases} u_{tt}-\Delta u+||\nabla u||_2^2\,u_t=0 \quad \text{in} \quad \Omega\times(0,\infty),\\ u=0 \quad \text{on} \quad \partial\Omega\times(0,\infty),\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\quad x\in\Omega. \end{cases} \tag{A.1}$$

The total energy is defined by

$$E(t) = \frac{1}{2}\int_{\Omega} \left[|u_t(x,t)|^2 + |\nabla u(x,t)|^2\right] dx, \quad t \geq 0.$$

Similarly to the case of localized (on space) problems, we are going to try the combination of the multipliers u and $m \cdot \nabla u$, where $m(x) := x - x^0$ for a fixed x^0, but we will see that the resulting multiplier does solve our case. Indeed, let us take the usual multiplier $Nu := 2(m \cdot \nabla u) + (N-1)u$ in (A.1) and integrating over $\Omega \times (0,T)$, $T > 0$, we infer

$$\begin{aligned} \int_0^T E_u(t)\,dt \leq |\chi(t)|_0^T &+ \int_0^T ||\nabla u(t)||_2^2 \int_{\Omega} \partial_t u (m \cdot \nabla u)\,dxdt \\ &+ \frac{N-1}{2}\int_0^T ||\nabla u(t)||_2^2 \int_{\Omega} \partial_t u\,dxdt + \frac{1}{2}\int_0^T \int_{\Gamma(x^0)} (m(x)\cdot \nu(x))\,(\partial_\nu u)^2\,d\gamma dt, \end{aligned} \tag{A.2}$$

where $\nu(x)$ is the unit outward normal vector field,

$$\chi(t) := \int_{\Omega} \partial_t u (m \cdot \nabla u)\,dx + \frac{N-1}{2}\int_{\Omega} \partial_t u u\,dx,$$

and

$$\Gamma(x^0) := \{x \in \partial\Omega;\ m(x)\cdot \nu(x) > 0\}.$$

In Fig. 1 we give an example of a domain Ω satisfying the above geometric set, where ω is a neighborhood of the boundary containing the closure of $\Gamma(x^0)$.

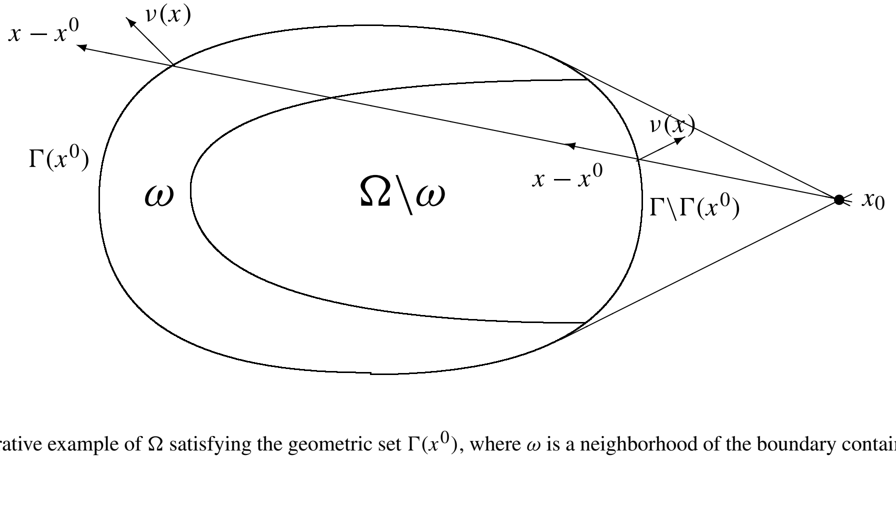

Fig. 1. Illustrative example of Ω satisfying the geometric set $\Gamma(x^0)$, where ω is a neighborhood of the boundary containing $\overline{\Gamma(x^0)}$.

The next step is to estimate the term $\int_0^T \int_{\Gamma(x^0)} (\partial_\nu u)^2 d\gamma\, dt$ in terms of the damping term $\int_0^T ||\nabla u(t)||_2^2 ||u_t(t)||_2^2\, dt$. As stated in Lions [36, Lemma 2.3] we construct a neighborhood ω_ε of $\overline{\Gamma(x^0)}$ such that $(\overline{\omega_\varepsilon} \cap \Omega) \subset \omega$, and a vector field $h \in (C^1(\overline{\Omega}))^N$ satisfying

$$h = \nu \ \text{ on } \ \Gamma(x^0), \quad h \cdot \nu \geq 0 \ \text{ a.e. in } \ \Gamma := \partial\Omega, \quad h = 0 \ \text{ on } \ \Omega \backslash \omega_\varepsilon. \tag{A.3}$$

See Fig. 2 below.

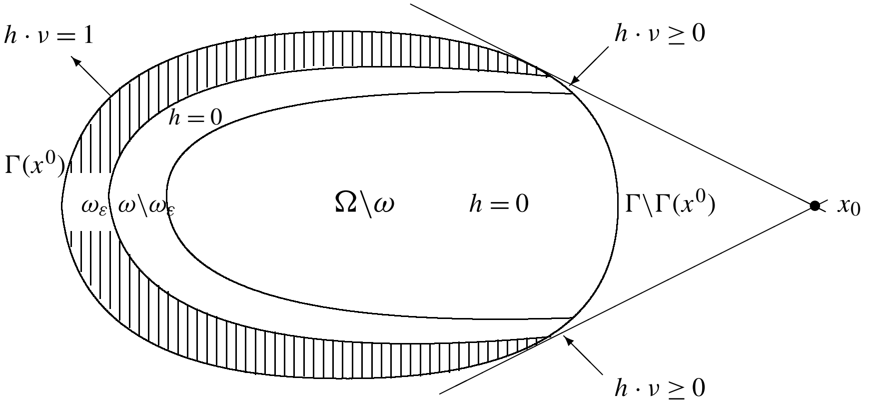

Fig. 2. An illustrative neighborhood ω_ε of $\overline{\Gamma(x^0)}$ such that $(\overline{\omega_\varepsilon} \cap \Omega) \subset \omega$, and a vector field $h \in (C^1(\overline{\Omega}))^N$ satisfying (A.3).

A straightforward computation shows that

$$\begin{aligned} &\frac{1}{2} \int_0^T \int_{\Gamma(x^0)} (\partial_\nu u)^2 d\gamma\, dt \\ &\qquad \leq \frac{1}{2} \int_0^T \int_{\Gamma(x^0)} \underbrace{(h \cdot \nu)}_{=1} (\partial_\nu u)^2 d\gamma\, dt + \frac{1}{2} \int_0^T \int_{\Gamma \backslash \Gamma(x^0)} \underbrace{(h \cdot \nu)}_{\geq 0} (\partial_\nu u)^2 d\gamma\, dt \\ &\qquad + \left[\int_{\omega_\varepsilon} u_t (h \cdot \nabla u)\, dx \right]_0^T + \int_0^T \int_{\omega_\varepsilon} \nabla u \cdot \nabla h \cdot \nabla u\, dx\, dt \\ &\qquad + \frac{1}{2} \int_0^T \int_{\omega_\varepsilon} \operatorname{div}(h) [u_t^2 - |\nabla u|^2]\, dx\, dt \\ &\qquad + \int_0^T \int_{\omega_\varepsilon} ||\nabla u(t)||_{L^2(\Omega)} u_t (h \cdot \nabla u)\, dx\, dt. \end{aligned} \tag{A.4}$$

The multipliers technique will work if we are able to quantify the term $\frac{1}{2}\int_0^T\int_{\omega_\varepsilon} \operatorname{div}(h)\, u_t^2\, dx\, dt$ in terms of the damping $\int_0^T ||\nabla u(t)||_2^2||u_t(t)||_2^2\, dt$ in the identity (A.4), which is not the case in our problem. The same occurs if we try to use the unusual multiplier $||\nabla u(t)||_2^2 u$. Therefore, in any case, we are not able to recover the energy in the right hand side of the equation (A.2).

A.2. The beam model

The situation is much more delicate for the beam model (2.1). In fact, in order to consider a suitable multiplier similar to Nu defined above we still need to estimate the problematic term $\frac{1}{2}\int_0^T\int_\Omega \operatorname{div}(h)\, u_t^2\, dx\, dt$ as in (A.4). To illustrate this case we tried the same multiplier as in Tucsnak [49]. See also Lions [36].

Let us consider a vector field $h = (h_1, \ldots, h_N) \in (W^{2,\infty}(\Omega))^N$. Multiplying (2.1) by $h(x)\cdot\nabla u(x,t)$, integrating by parts over $\Omega\times(0,T)$, $T>0$, and arguing similar to [49, Lemma 3.1] or [36, p. 244], we arrive at

$$
\begin{aligned}
&(u_t(t), h\cdot\nabla u(t))\Big|_0^T + \frac{1}{2}\int_0^T\int_\Omega \operatorname{div}(h)\left[|u_t(t)|^2 - |\Delta u(t)|^2\right] dx\, dt \\
&\qquad + 2\sum_{j,k}\int_0^T\int_\Omega \frac{\partial h_k}{\partial x_j}\Delta u(t)\frac{\partial^2 u(t)}{\partial x_k \partial x_j}\, dx\, dt + \int_0^T\int_\Omega (\Delta h\cdot\nabla u(t))\,\Delta u(t)\, dx\, dt \\
&\qquad - \int_0^T M\left(||\nabla u(t)||_2^2\right)\int_\Omega (h\cdot\nabla u(t))\,\Delta u(t)\, dx\, dt \\
&\qquad + \int_0^T ||\Delta u(t)||_2^2 \int_\Omega (h\cdot\nabla u(t))\, u_t(t)\, dx\, dt \\
&= \frac{1}{2}\int_0^T\int_{\partial\Omega} (h\cdot\nu)(\Delta u(t))^2\, dS\, dt.
\end{aligned}
$$

Therefore, even for particular choices of the vector field h in the above identity, we need to quantify the term $\frac{1}{2}\int_0^T\int_\Omega \operatorname{div}(h) u_t^2\, dxdt$ in terms of the damping term $\int_0^T ||\Delta u(t)||_2^2||u_t(t)||_2^2\, dt$, which is not the case again. The same problem also appears when we try to apply the (spatial) multipliers considered by Pazoto et al. [46] (see Lemmas 4.2 and 4.3 therein) or Bortot et al. [15, Section 5]. This means that we have the same difficulty to recover the energy by using multipliers in this case.

References

[1] K. Ammari, M. Tucsnak, Stabilization of Bernoull-Euler beams by means of a pointwise feedback force, SIAM J. Control Optim. 39 (4) (2000) 1160–1181.

[2] K. Ammari, Z. Liu, M. Tucsnak, Decay rates for a beam with pointwise force and moment feedback, Math. Control Signals Syst. 15 (3) (2002) 229–255.

[3] Z. Arat, A. Khanmamedov, S. Simsek, A unique continuation result for the plate equations and an application, Math. Methods Appl. Sci. 39 (4) (2016) 744–761.

[4] G. Autuori, P. Pucci, Asymptotic stability for Kirchhoff systems in variable exponent Sobolev spaces, Complex Var. Elliptic Equ. 56 (7–9) (2011) 715–753.

[5] A.V. Balakrishnan, L.W. Taylor, Distributed parameter nonlinear damping models for flight structures, in: Proceedings Daming 89, Flight Dynamics Lab and Air Force Wright Aeronautical Labs, WPAFB, 1989.

[6] J.M. Ball, Initial-boundary value problems for an extensible beam, J. Math. Anal. Appl. 42 (1973) 61–90.

[7] J.M. Ball, Stability theory for an extensible beam, J. Differ. Equ. 14 (1973) 399–418.

[8] A.C. Biazutti, H.R. Crippa, Global attractor and inertial set for the beam equation, Appl. Anal. 55 (1–2) (1994) 61–78.

[9] R. Benavides Guzmán, M. Tucsnak, Energy decay estimates for the damped plate equation with a local degenerated dissipation, Syst. Control Lett. 48 (3–4) (2003) 191–197.

[10] V. Barbu, I. Lasiecka, M.A. Rammaha, Blow-up of generalized solutions to wave equations with nonlinear degenerate damping and source terms, Indiana Univ. Math. J. 56 (3) (2007) 995–1021.

[11] V. Barbu, I. Lasiecka, M.A. Rammaha, Existence and uniqueness of solutions to wave equations with nonlinear degenerate damping and source terms, Control Cybern. 34 (3) (2005) 665–687.

[12] V. Barbu, I. Lasiecka, M.A. Rammaha, Nonlinear wave equations with degenerate damping and source terms, in: Control and Boundary Analysis, in: Lect. Notes Pure Appl. Math., vol. 240, Chapman and Hall/CRC, Boca Raton, FL, 2005, pp. 53–62.

[13] V. Barbu, I. Lasiecka, M.A. Rammaha, On nonlinear wave equations with degenerate damping and source terms, Trans. Am. Math. Soc. 357 (7) (2005) 2571–2611.

[14] H.M. Berger, A new approach to the analysis of large deflections of plates, J. Appl. Mech. 22 (1955) 465–472.

[15] C.A. Bortot, M.M. Cavalcanti, W.J. Corrêa, V.N. Domingos Cavalcanti, Uniform decay rate estimates for Schrödinger and plate equations with nonlinear locally distributed damping, J. Differ. Equ. 254 (9) (2013) 3729–3764.

[16] F. Bucci, D. Toundykov, Finite-dimensional attractor for a composite system of wave/plate equations with localized damping, Nonlinearity 23 (9) (2010) 2271–2306.

[17] M.M. Cavalcanti, V.N. Domingos Cavalcanti, M.A. Jorge Silva, C.M. Webler, Exponential stability for the wave equation with degenerate nonlocal weak damping, Isr. J. Math. 219 (2017) 189–213.

[18] M.M. Cavalcanti, V.N. Domingos Cavalcanti, T.F. Ma, Exponential decay of the viscoelastic Euler-Bernoulli equation with a nonlocal dissipation in general domains, Differ. Integral Equ. 17 (2004) 495–510.

[19] M.M. Cavalcanti, V.N. Domingos Cavalcanti, J.A. Soriano, Global existence and asymptotic stability for the nonlinear and generalized damped extensible plate equation, Commun. Contemp. Math. 6 (5) (2004) 705–731.

[20] R.C. Charão, E. Bisognin, V. Bisognin, A.F. Pazoto, Asymptotic behavior of a Bernoulli-Euler type equation with nonlinear localized damping, in: Contributions to Nonlinear Analysis, in: Progr. Nonlinear Differential Equations Appl., vol. 66, Birkhäuser, Basel, 2006, pp. 67–91.

[21] I. Chueshov, S. Kolbasin, Plate models with state-dependent damping coefficient and their quasi-static limits, Nonlinear Anal. 73 (6) (2010) 1626–1644.

[22] I. Chueshov, S. Kolbasin, Long-time dynamics in plate models with strong nonlinear damping, Commun. Pure Appl. Anal. 11 (2) (2012) 659–674.

[23] I. Chueshov, I. Lasiecka, Long-time behavior of second order evolution equations with nonlinear damping, Mem. Am. Math. Soc. 195 (912) (2008).

[24] I. Chueshov, I. Lasiecka, Von Karman Evolution Equations. Well-Posedness and Long-Time Dynamics, Springer Monographs in Mathematics, Springer, New York, 2010.

[25] M. Coti Zelati, Global and exponential attractors for the singularly perturbed extensible beam, Discrete Contin. Dyn. Syst. 25 (3) (2009) 1041–1060.

[26] B. Dehman, P. Gérard, G. Lebeau, Stabilization and control for the nonlinear Schrödinger equation on a compact surface, Math. Z. 254 (4) (2006) 729–749.

[27] B. Dehman, G. Lebeau, E. Zuazua, Stabilization and control for the subcritical semilinear wave equation, Ann. Sci. Éc. Norm. Supér. 36 (2003) 525–551.

[28] P. Geredeli, J.T. Webster, Decay rates to equilibrium for nonlinear plate equations with degenerate, geometrically-constrained damping, Appl. Math. Optim. 68 (3) (2013) 361–390.

[29] X. Han, M. Wang, Asymptotic behavior for Petrovsky equation with localized damping, Acta Appl. Math. 110 (3) (2010) 1057–1076.

[30] M.A. Jorge Silva, V. Narciso, Long-time behavior for a plate equation with nonlocal weak damping, Differ. Integral Equ. 27 (9–10) (2014) 931–948.

[31] M.A. Jorge Silva, V. Narciso, Attractors and their properties for a class of nonlocal extensible beams, Discrete Contin. Dyn. Syst. 35 (3) (2015) 985–1008.
[32] A.Kh. Khanmamedov, A global attractor for the plate equation with displacement-dependent damping, Nonlinear Anal. 74 (5) (2011) 1607–1615.
[33] S. Kolbasin, Attractors for Kirchhoff's equation with a nonlinear damping coefficient, Nonlinear Anal. 71 (5–6) (2009) 2361–2371.
[34] S. Kouémou Patcheu, On a global solution and asymptotic behaviour for the generalized damped extensible beam equation, J. Differ. Equ. 135 (2) (1997) 299–314.
[35] H. Lange, G. Perla Menzala, Rates of decay of a nonlocal beam equation, Differ. Integral Equ. 10 (1997) 1075–1092.
[36] J.J. Lions, Contrôlabilité Exacte Pertubations et Stabilisation de Systèmes Distribués, Masson, Paris, 1988.
[37] P. Martinez, Precise decay rate estimates for time-dependent dissipative systems, Isr. J. Math. 119 (2000) 291–324.
[38] T.F. Ma, V. Narciso, Global attractor for a model of extensible beam with nonlinear damping and source terms, Nonlinear Anal. 73 (10) (2010) 3402–3412.
[39] N. Pan, P. Pucci, R. Xu, B. Zhang, Degenerate Kirchhoff-type wave problems involving the fractional Laplacian with nonlinear damping and source terms, J. Evol. Equ. 19 (3) (2019) 615–643.
[40] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.
[41] P. Pucci, S. Saldi, Asymptotic stability for nonlinear damped Kirchhoff systems involving the fractional p-Laplacian operator, J. Differ. Equ. 263 (5) (2017) 2375–2418.
[42] P. Pucci, J. Serrin, Asymptotic stability for nonautonomous dissipative wave systems, Commun. Pure Appl. Math. 49 (2) (1996) 177–216.
[43] P. Pucci, J. Serrin, Local asymptotic stability for dissipative wave systems, Isr. J. Math. 104 (1998) 29–50.
[44] P. Pucci, J. Serrin, Precise damping conditions for global asymptotic stability of second order systems, Acta Math. 170 (1993) 275–307.
[45] P. Pucci, J. Serrin, Continuation and limit behavior for damped quasi-variational systems, in: Wei-Ming Ni, L.A. Peletier, J.L. Vazquez (Eds.), Degenerate Diffusions, in: IMA Vol. Math. Appl., vol. 47, Springer-Verlag, New York, 1993, pp. 157–173.
[46] A.F. Pazoto, L. Coelho, R.C. Charão, Uniform stabilization of a plate equation with nonlinear localized dissipation, Proyecciones 23 (3) (2004) 205–234.
[47] S. Simsek, A. Khanmamedov, Exponential decay of solutions for the plate equation with localized damping, Math. Methods Appl. Sci. 38 (9) (2015) 1767–1780.
[48] L. Tebou, Well-posedness and stability of a hinged plate equation with a localized nonlinear structural damping, Nonlinear Anal. 71 (12) (2009) 2288–2297.
[49] M. Tucsnak, Semi-internal stabilization for a non-linear Bernoulli-Euler equation, Math. Methods Appl. Sci. 19 (11) (1996) 897–907.
[50] C.F. Vasconcellos, L.M. Teixeira, Existence, uniqueness and stabilization for a nonlinear plate system with nonlinear damping, Ann. Fac. Sci. Toulouse Math. (6) 8 (1) (1999) 173–193.
[51] D. Wang, J. Zhang, Global attractor for a nonlinear plate equation with supported boundary conditions, J. Math. Anal. Appl. 363 (2) (2010) 468–480.
[52] S. Woinowsky-Krieger, The effect of an axial force on the vibration of hinged bars, J. Appl. Mech. 17 (1950) 35–36.
[53] Z. Wu, Asymptotic behavior for a coupled Petrovsky and wave system with localized damping, Appl. Math. Comput. 224 (2013) 442–449.
[54] Z. Yang, On an extensible beam equation with nonlinear damping and source terms, J. Differ. Equ. 254 (9) (2013) 3903–3927.