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Dynamics of a Cauchy problem related to extensible beams under nonlocal and localized damping effects

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ABSTRACT

This paper is devoted to the study of long-time dynamics of a Cauchy problem related to extensible beam models with dissipative effects coming from the nonlocal Balakrishnan-Taylor and the localized weak damping terms in $\mathbb{R}^n$. Our first main result features a new unique continuation property for models associated with extensible beams. Then, applying such a property and with the fundamental aid of the nonlocal Balakrishnan-Taylor damping term, we state and prove our second main result dealing with the existence, characterization and regularity of a compact global attractor for the corresponding nonlinear dynamical system.

1. Introduction

In the recent approaching by Gomes Tavares et al. [10], supported by the relevant physical deployment in Balakrishnan & Taylor [4], and motivated by the work of Emmrich & Thalhammer [8], the authors have considered the following class of extensible beams with nonlocal Balakrishnan-Taylor and frictional damping in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$,

$$u_{tt} + \Delta^2 u - \left[\beta + \gamma \int_\Omega |\nabla u|^2 dx + \delta \, |\Phi(u, u_t)|^{q-2} \, \Phi(u, u_t)\right] \Delta u + \kappa u_t + f(u) = h \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1.1}$$

with corresponding clamped boundary and initial conditions, where

$$\Phi(u, u_t) := \int_\Omega \nabla u \cdot \nabla u_t \, dx = - \int_\Omega \Delta u \, u_t \, dx. \tag{1.2}$$

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As explained by the authors, when $n = 1$ the model (1.1) corresponds to transversal deflections of an extensible beam with length $2L$, which means to take e.g. $\Omega = [-L, L]$ (cf. [4]). Additionally, the authors prove the existence of global and fractal exponential attractors for the dynamical system corresponding to (1.1) by exploring the global L^q-regularity ($q \geq 2$) in time of the nonlocal Balakrishnan-Taylor damping term coming from $\delta\, |\Phi(u, u_t)|^{q-2}\, \Phi(u, u_t)\Delta u$ with Φ given in (1.2), cf. [10, Section 3]. Among all tools employed in [10], we highlight three important ones when dealing in bounded domains $\Omega \subset \mathbb{R}^n$, say with finite measure $|\Omega| < \infty$,

(a) Technical estimates with upper bounds depending on $|\Omega|^{-1}$ everywhere;
(b) Poincaré's inequality;
(c) Compactness properties when dealing with Sobolev spaces over bounded domains.

In the same spirit in what concerns Balakrishnan-Taylor extensible beam (still combined with viscoelastic Kirchhoff wave) problems in bounded domains $\Omega \subset \mathbb{R}^n$ ($|\Omega| < \infty$), there are several works dealing with stability and long time-behavior of solutions by assuming additional full frictional or viscoelastic damping, see e.g. [4,5,8,9,16,18,22,23,25] just to name a few. Since the additional full damping represents a kind of extra damping to the system, then the asymptotic or/and long-time results for the system in turn are quite expected in the current literature.

In the present paper, we are concerned with a Cauchy problem related to the semilinear extensible beam model (1.1) under localized frictional effects, namely, we consider the following Cauchy problem in $\mathbb{R}^n$:

$$u_{tt} + \Delta^2 u + \lambda u - \left[\beta + \gamma \int\limits_{\mathbb{R}^n} |\nabla u|^2 dx + \delta \left|\int\limits_{\mathbb{R}^n} \nabla u \cdot \nabla u_t \, dx\right|^{q-2} \int\limits_{\mathbb{R}^n} \nabla u \cdot \nabla u_t \, dx\right] \Delta u \qquad (1.3)$$
$$+ \alpha(x) u_t + f(u) = h(x) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^n,$$

subject to initial data

$$u(0, x) = u_0(x), \;\; u_t(0, x) = u_1(x), \;\; x \in \mathbb{R}^n, \qquad (1.4)$$

where $\beta \in \mathbb{R}$, $\lambda, \gamma, \delta > 0$, $q \geq 2$, $h(x)$ is an external force, $\alpha(x)$ is a localized function in $\mathbb{R}^n$, and $f(u)$ is a nonlinear source term of lower order, whose assumptions will be properly given in Section 2. In order to simplify the notation as previously set in (1.1)-(1.2), we define the following representation to be used throughout this paper

$$\Upsilon(u, u_t) := \int\limits_{\mathbb{R}^n} \nabla u \cdot \nabla u_t \, dx = -\int\limits_{\mathbb{R}^n} \Delta u \, u_t \, dx, \qquad (1.5)$$

where the second equality in (1.5) is formally considered. However, it holds true for strong solutions as presented in the existence theorem (Section 2).

To our best knowledge, there is no approach on the dynamics concerning (1.3)-(1.4). It is worth mentioning that the results presented in [10] are no longer valid here for several reasons. Among them, we notice that one looses some useful compactness properties for functional spaces over the whole space $\mathbb{R}^n$ and stress that conditions the (a)-(c) aforementioned can not be employed in the present framework, even if we consider full frictional damping in (1.3) i.e. $\alpha(x) \equiv \kappa > 0$ in $\mathbb{R}^n$, which is not the case to be considered (see Assumption 2.1). Therefore, our main goal is to address the long-time behavior of the dynamical system corresponding to (1.3)-(1.4). To this purpose, we rely on a similar approach as given in [1,2,21], and provide a new unique continuation property for extensible beams that has been essential to ensure the gradient and asymptotic smoothness properties for the related dynamical system. These statements are new for the Cauchy problem related to Balakrishnan-Taylor extensible beam models, namely, (1.3)-(1.4) which in turn can be considered as mathematical extension of (1.1) placed in the whole Euclidean domain $\mathbb{R}^n$.

In what follows, we are going to deliver the state of the art as well as the main novelties of the present work when compared to previous literature related to (1.3)-(1.4).

We start with models in the absence of extensible parameters, mainly in the case $\beta = \gamma = \delta = 0$ in (1.3), which in turn becomes to the simplified semilinear plate model:

$$u_{tt} + \Delta^2 u + \lambda u + \alpha(x)u_t + f(u) = h(x) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^n. \tag{1.6}$$

With respect to asymptotic and long-time behavior of (fourth order) plate equations like (1.6) (and similar models) approached in unbounded domains with localized and nonlocal damping terms, we refer e.g. [1,2,11,12,14,15,17,21,24] and references therein. For the sake of brevity, we do not mention (second order) wave models with localized damping in unbounded domains. As well as for wave problems, for these sort of plate models with localized weak damping (1.6) posed in $\mathbb{R}^n$, there is also a considerable stability theory under proper assumptions on both the nonlinear source $f(u)$ and localized damping coefficient $\alpha(x)$. See, for instance, [2,12,14,21], among others.

To clarify the ideas, one sees for example in [2,21] that an important property in the study of asymptotic and long-time behavior of solution for (1.6) is the property so-called Unique Continuation Property (UCP for short). More precisely, in [21] the authors prove the exponential stability of energy for (1.6) in the homogeneous case $h \equiv 0$. Due to the lack of UCP for weak solutions of (1.6) with non-smooth coefficients, the authors appealed to a combination of techniques from [7,12,26] by using the sequentially limit transition argument, point dissipativity property for the semilinear plate equation and suitable energy estimates. More recently, in [2] the authors prove the UCP for the weak solution of the plate equation with the low regular coefficient. As a consequence, they handle with global attractor for a semilinear plate model with localized damping encompassing (1.6). Nevertheless, the UCP proved in [2] can not be applied to the Balakrishnan-Taylor model (1.3) due to its nonlinear extensible terms ($\gamma, \delta > 0$). Therefore, to overcome this situation, our first main objective is to prove the UCP for an extensible beam model and, with the crucial help of the Balakrishnan-Taylor damping term, we apply the UCP to reach the properties for the associated dynamical system and, consequently, to prove the existence of a global attractor as well. As far as we know, there are no results in this direction for the Balakrishnan-Taylor extensible beam model (1.3)-(1.4) posed in unbounded domains. In Sections 3 and 4 we present the precise results and details on these statements. Moreover, as we also explain in Appendix A, the Balakrishnan-Taylor damping term $-\delta\,|\Upsilon(u,u_t)|^{q-2}\,\Upsilon(u,u_t)\Delta u$ alone ($\alpha \equiv 0$) is not enough to produce the desired results on stability nor long-time behavior. Hence, the additional localized damping term $\alpha(x)u_t$ ($\alpha(x) \geq \alpha_0 > 0$ only a portion of $\mathbb{R}^n$ (see Assumption 2.1) is regarded as a *minimum amount* of localized damping in order to study the long-time dynamics of the regarded model (1.3)-(1.4). On the other hand, in the lack of the Balakrishnan-Taylor damping term ($\delta = 0$), equation (1.3) is handled in [1] by assuming that the $\alpha(x) \geq \alpha_0 > 0$ for all $x \in \mathbb{R}^n$. Hence, when the weak damping is efficient only in a portion of $\mathbb{R}^n$ and $\delta = 0$ in equation (1.3), existence of the global attractor is still an open problem. In this paper, we weaken the strict positivity assumption of [1] imposed on $\alpha(x)$ with the help of the Balakrishnan-Taylor damping term.

Based on the above statements on plate equations in unbounded domains as well as on extensible models in bounded domains, we highlight that our main results (Proposition 3.1 and Theorem 4.1, respectively, in Sections 3 and 4) constitute a generalization/extension of the results presented in [2,10,12,14,21] to the setting of long-time dynamics for the Balakrishnan-Taylor beam model with localized weak damping in $\mathbb{R}^n$, which has not been approached in the literature so far. For the sake of completeness, we also consider the Hadamard well-posedness of problem (1.3)-(1.4) in Section 2 and conclude this work with some remarks in Appendix A.

2. Well-posedness

In this section we discuss the well-posedness of problem (1.3)-(1.4). The existence and uniqueness of global solution is based on theory of C_0-semigroups of linear operators, see e.g. Cazenave [6] and Pazy [19]. We start by fixing some notations and assumptions that shall be used throughout this paper. The spaces $L^p(\mathbb{R}^n)$ stand for p-Lebesgue integrable functions with norm

$$\|u\|_{L^p(\mathbb{R}^n)}^p = \int\limits_{\mathbb{R}^n} |u(x)|^p dx, \quad u \in L^p(\mathbb{R}^n),$$

and $W^{m,p}(\mathbb{R}^n)$ denote well-known Sobolev spaces. In particular, if $p = 2$, then $L^2(\mathbb{R}^n)$ is a Hilbert space and $W^{m,2}(\mathbb{R}^n) := H^m(\mathbb{R}^n)$. We set the following Hilbert phase space to the solution trajectories

$$\mathcal{H} = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n),$$

equipped with the following inner product

$$\langle (u,v), (\widetilde{u}, \widetilde{v}) \rangle_{\mathcal{H}\times\mathcal{H}} = \int\limits_{\mathbb{R}^n} \Delta u \Delta \widetilde{u} dx + \lambda \int\limits_{\mathbb{R}^n} u\widetilde{u} dx + \int\limits_{\mathbb{R}^n} v\widetilde{v} dx$$

and norm

$$\|(u,v)\|_{\mathcal{H}}^2 = \|\Delta u\|_{L^2(\mathbb{R}^n)}^2 + \lambda \|u\|_{L^2(\mathbb{R}^n)}^2 + \|v\|_{L^2(\mathbb{R}^n)}^2,$$

for all (u,v), $(\widetilde{u}, \widetilde{v}) \in \mathcal{H}$. We also set the energy function $E(t) := E(u(t), u_t(t))$ associated with problem (1.3)-(1.4), namely,

$$E(t) = \frac{1}{2}\|(u(t), u_t(t))\|_{\mathcal{H}}^2 + \frac{\beta}{2}\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}^2 + \frac{\gamma}{4}\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}^4 + \int\limits_{\mathbb{R}^n} [\, F(u(t)) - hu(t) \,]\, dx, \tag{2.1}$$

where $F(s) = \int\limits_0^s f(\tau)\, d\tau$ is the primitive of f.

Assumption 2.1. With respect to the coefficients $\lambda, \beta, \delta, \gamma$, the exponent q and the functions $\alpha, h : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ considered in (1.3)-(1.4), we assume the following hypotheses:

$$\beta \in \mathbb{R},\ \lambda, \delta, \gamma > 0,\ q \geq 2,\ h \in L^2(\mathbb{R}^n), \tag{2.2}$$

$$\alpha \in L^\infty(\mathbb{R}^n),\ \alpha(x) \geq 0 \ \text{ a.e. } x \in \mathbb{R}^n, \tag{2.3}$$

$$\alpha(x) \geq \alpha_0 > 0 \text{ a.e. in } \{x \in \mathbb{R}^n : |x| \geq r_0\},\ \text{for some } r_0 > 0, \tag{2.4}$$

$f \in C^1(\mathbb{R})$, $f(0) = 0$ and there exist constants $C, C_0 > 0$ and $C_1 \in (0, \lambda)$ such that

$$C_0|s|^{p-1} - C_1 \leq f'(s) \leq C\big(1 + |s|^{p-1}\big), \quad \forall\, s \in \mathbb{R}, \quad \text{where } \begin{cases} p \geq 1 \ \text{ if } \ 1 \leq n \leq 4, \\ 1 \leq p \leq \frac{n}{n-4} \ \text{ if } \ n \geq 5. \end{cases} \tag{2.5}$$

Remark 2.1. The geometric idea of condition (2.4) in the bi-dimensional case is presented in Fig. 1. Additionally, from (2.5) one can derive the following additional inequalities

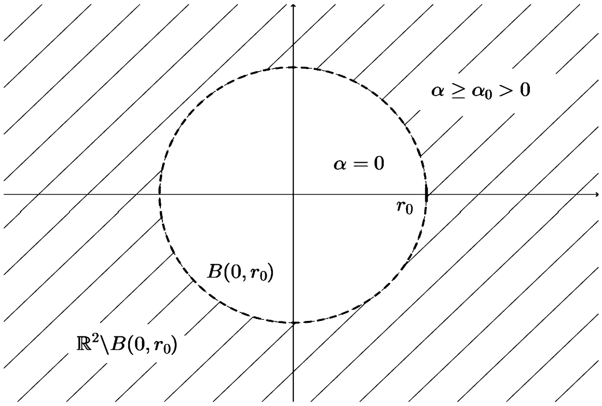

Fig. 1. Idea of assumption (2.4) in $\mathbb{R}^2$, where α can vanish in the ball $B(0, r_0) := \{x \in \mathbb{R}^2 : |x| < r_0\}$ for some radius $r_0 > 0$ that can be large enough.

$$\frac{C_0}{p}|s|^{p+1} - C_1 s^2 \leq f(s)s \leq C\left(s^2 + |s|^{p+1}\right), \quad \forall\, s \in \mathbb{R}, \tag{2.6}$$

$$\frac{C_0}{p(p+1)}|s|^{p+1} - \frac{C_1}{2}s^2 \leq F(s) \leq C(s^2 + |s|^{p+1}), \quad \forall\, s \in \mathbb{R}, \tag{2.7}$$

$$f(s)s - F(s) \leq -\frac{C_1}{2}s^2, \quad \forall\, s \in \mathbb{R}. \tag{2.8}$$

2.1. Nonlinear semigroup setting

Let us rewrite problem (1.3) − (1.4) as a first order equation. Denoting the vector-valued function $U = (u, v)$ with $v = u_t$, the problem (1.3)-(1.4) can be rewritten as the following abstract initial value problem

$$\begin{cases} \dfrac{d}{dt}U(t) = AU(t) + \Phi(U(t)), \quad t > 0, \\ U(0) = (u\,(0)\,, u_t\,(0)) = (u_0, u_1) := U_0, \end{cases} \tag{2.9}$$

where $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator

$$A\,(U) = \begin{bmatrix} v \\ -\Delta^2 u - \lambda u \end{bmatrix}, \quad U \in D(A) = H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n), \tag{2.10}$$

and $\Phi : \mathcal{H} \to \mathcal{H}$ is the nonlinear operator

$$\Phi\,(U) = \begin{bmatrix} 0 \\ h - f\,(u) + \beta\Delta u + \gamma\,\|\nabla u\|^2_{L^2(\mathbb{R}^n)}\,\Delta u + \Xi\,(U)\,\Delta u - \alpha\,(x)\,v \end{bmatrix} \tag{2.11}$$

with

$$\Xi\,(U) = \delta\left|\int_{\mathbb{R}^n} \Delta uvdx\right|^{q-2} \int_{\mathbb{R}^n} \Delta uvdx = \delta|\Upsilon(U)|^{q-2}\Upsilon(U),$$

for $U = (u, v) \in \mathcal{H}$, where the notation $\Upsilon(U)$ is introduced in (1.5).

Under the above assumptions and notations, we have the following Hadamard well-posedness result.

Theorem 2.1. *Let us consider Assumption 2.1 into account. Thus, we have:*

(*i*) *If* $U_0 = (u_0, u_1) \in \mathcal{H}$, *then there exists* $T_{\max} > 0$ *such that problem* (2.9) *has a unique mild (weak) solution* $U \in C([0, T_{\max}]; \mathcal{H})$ *given by*

$$U(t) = e^{At}U_0 + \int_0^t e^{A(t-s)}\Phi(U(s))ds, \quad t \in [0, T_{\max}). \tag{2.12}$$

(*ii*) *If* $U_0 = (u_0, u_1) \in D(A)$, *then* U *is a regular (strong) solution of* (2.9), *with* $U \in C([0, T_{\max}]; D(A))$ *and* $u_{tt} \in C\left([0, T_{\max}]; L^2(\mathbb{R}^n)\right)$.

(*iii*) *In both cases, we have that* $T_{\max} = +\infty$.

(*iv*) *If* $U^1 = (u^1, v^1), U^2 = (u^2, v^2) \in C([0, \infty); \mathcal{H})$ *are two (weak or strong) solutions of* (2.9) *with initial data* $U_0^1 = (u_0^1, v_0^1), U_0^2 = (u_0^2, v_0^2) \in \mathcal{H}$, *respectively, then*

$$\left\|U^1(t) - U^2(t)\right\|_{\mathcal{H}} \le L \left\|U_0^1 - U_0^2\right\|_{\mathcal{H}}, \quad t \in \mathbb{R}^+, \tag{2.13}$$

where $L = L(\|U_0^1\|_{\mathcal{H}}, \|U_0^2\|_{\mathcal{H}})$.

Proof. $(i) - (ii)$ We first prove that $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ given in (2.10) is the infinitesimal generator of a C_0-semigroup of contractions e^{At}. In fact, taking arbitrary $U = (u, v) \in D(A)$, we have

$$Re\,\langle AU, U\rangle_{\mathcal{H}\times\mathcal{H}} = \int_{\mathbb{R}^n} \Delta v \Delta u dx + \lambda \int_{\mathbb{R}^n} vu dx + \int_{\mathbb{R}^n} (-\Delta^2 u - \lambda u)v dx = 0.$$

Which shows that A is dissipative. On the other hand, it is also easy to see that

$$R(I - A) = \mathcal{H},$$

where $R(I - A)$ stands for range of the operator $I - A$. Therefore, from Lummer-Phillips Theorem (see e.g. [19, Chapter 1]), A is the infinitesimal generator of a C_0-semigroup of contractions e^{At} in $\mathcal{H}$. In addition $\Phi : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz continuous in $\mathcal{H}$. Indeed, let us first define

$$\Pi(U) = h - f(u) + \beta\Delta u + \gamma \|\nabla u\|^2_{L^2(\mathbb{R}^n)} \Delta u + \Xi(U)\Delta u - \alpha(x)v.$$

Let $R > 0$ and $U^1 = (u^1, v^1)$, $U^2 = (u^2, v^2)$ such that $||U^1||_{\mathcal{H}}$, $||U^2||_{\mathcal{H}} \le R$. Then, from (2.11), we infer

$$||\Phi(U^1) - \Phi(U^2)||_{\mathcal{H}} = \|\Pi(U^1) - \Pi(U^2)\|_{L^2(\mathbb{R}^n)} = \sup_{\|w\|_{L^2(\mathbb{R}^n)} \le 1} \left| \int_{\mathbb{R}^n} [\,\Pi(U^1) - \Pi(U^2)\,] w dx \right|. \tag{2.14}$$

In what follows, we shall give proper estimates on the right hand side of (2.14). Given $w \in L^2(\mathbb{R}^n)$, adding and subtracting the terms $\gamma\|\nabla u^1\|^2_{L^2(\mathbb{R}^n)}\Delta u^2$ and $\Xi(U^1)\Delta u^2$ in the expression $\Pi(U^1) - \Pi(U^2)$, we denote

$$\left| \int_{\mathbb{R}^n} [\,\Pi(U^1) - \Pi(U^2)\,] w dx \right| = \left| \sum_{i=1}^6 \mathcal{I}_i \right|, \tag{2.15}$$

where

$$\mathcal{I}_1 = \int_{\mathbb{R}^n} \left[\beta + \gamma\|\nabla u^1\|^2_{L^2(\mathbb{R}^n)}\right] (\Delta u^1 - \Delta u^2) w dx,$$

$$\mathcal{I}_2 = \gamma \int\limits_{\mathbb{R}^n} \left[\|\nabla u^1\|^2_{L^2(\mathbb{R}^n)} - \|\nabla u^2\|^2_{L^2(\mathbb{R}^n)} \right] \Delta u^2 w dx,$$

$$\mathcal{I}_3 = \delta \int\limits_{\mathbb{R}^n} \Xi(U^1) \left(\Delta u^1 - \Delta u^2 \right) w dx,$$

$$\mathcal{I}_4 = \delta \int\limits_{\mathbb{R}^n} \left[\Xi(U^1) - \Xi(U^2) \right] \Delta u^2 w dx,$$

$$\mathcal{I}_5 = \int\limits_{\mathbb{R}^n} \left[f(u^2) - f(u^1) \right] w dx,$$

$$\mathcal{I}_6 = \int\limits_{\mathbb{R}^n} \alpha(x)(v^2 - v^1) w dx.$$

Thus, it remains to estimate the terms $\mathcal{I}_1, \ldots, \mathcal{I}_6$. Firstly, using interpolation theorem, there exists $\varrho > 0$ such that

$$\|\nabla u^i\|^2_{L^2(\mathbb{R}^n)} \leq \varrho \left[\|\Delta u^i\|^2_{L^2(\mathbb{R}^n)} + \lambda \|u^i\|^2_{L^2(\mathbb{R}^n)} \right] \leq \varrho \|U^i\|^2_{\mathcal{H}}, \quad \text{for} \quad i = 1, 2.$$

In order, the term $\mathcal{I}_1$ can be estimated as follows

$$\begin{aligned} |\mathcal{I}_1| &\leq \left[|\beta| + \gamma \|\nabla u^1\|^2_{L^2(\mathbb{R}^n)} \right] \|\Delta u^1 - \Delta u^2\|_{L^2(\mathbb{R}^n)} \|w\|_{L^2(\mathbb{R}^n)} \\ &\leq \left[\, |\beta| + \gamma \varrho \|U^1\|^2_{\mathcal{H}} \right] ||U^1 - U^2||_{\mathcal{H}} \|w\|_{L^2(\mathbb{R}^n)} \\ &\leq \left[|\beta| + \gamma \varrho R^2 \right] ||U^1 - U^2||_{\mathcal{H}} \|w\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Using that $a^2 - b^2 = (a - b)(a + b)$ we can estimate $\mathcal{I}_2$ as follows

$$\begin{aligned} |\mathcal{I}_2| &\leq \gamma \left| \|\nabla u^1\|^2_{L^2(\mathbb{R}^n)} - \|\nabla u^2\|^2_{L^2(\mathbb{R}^n)} \right| \|\Delta u^2\|_{L^2(\mathbb{R}^n)} \|w\|_{L^2(\mathbb{R}^n)} \\ &\leq \gamma \left[\|\nabla u^1\|_{L^2(\mathbb{R}^n)} + \|\nabla u^2\|_{L^2(\mathbb{R}^n)} \right] \|\nabla u^1 - \nabla u^2\|_{L^2(\mathbb{R}^n)} \|\Delta u^2\|_{L^2(\mathbb{R}^n)} \|w\|_{L^2(\mathbb{R}^n)} \\ &\leq \gamma \varrho \left[\|U^1\|_{\mathcal{H}} + \|U^2\|_{\mathcal{H}} \right] \|U^1 - U^2\|_{\mathcal{H}} \|U^2\|_{\mathcal{H}} \|w\|_{L^2(\mathbb{R}^n)} \\ &\leq 2\gamma \varrho R^2 \|U^1 - U^2\|_{\mathcal{H}} \|w\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

The term $\mathcal{I}_3$ is estimated by

$$\begin{aligned} |\mathcal{I}_3| \leq \; & \delta \left[\|\Delta u^1\|_{L^2(\mathbb{R}^n)} \|v^1\|_{L^2(\mathbb{R}^n)} \right]^{q-1} \|\Delta u^1 - \Delta u^2\|_{L^2(\mathbb{R}^n)} \|w\|_{L^2(\mathbb{R}^n)} \\ \leq \; & \delta R^{2(q-1)} ||U^1 - U^2||_{\mathcal{H}} \|w\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Now, let $F \in C^1(\mathbb{R})$ be given by $F(s) = |s|^{q-2} s$. From the Mean Value Theorem, one can easily prove that

$$|F(\vartheta_1) - F(\vartheta_2)| \leq 2^{2(q-2)}(q-1) \left[|\vartheta_1|^{q-2} + |\vartheta_2|^{q-2} \right] |\vartheta_1 - \vartheta_2|, \quad \vartheta_1, \vartheta_2 \in \mathbb{R}.$$

Then, taking $\vartheta_i = \Upsilon(U^i) = -\int\limits_{\mathbb{R}^n} \Delta u^i v^i dx$, $i = 1, 2$, we have

$$|\mathcal{I}_4| \leq 2^{2(q-2)}(q-1)\delta \left[\left| \Upsilon(U^1) \right|^{q-2} + \left| \Upsilon(U^2) \right|^{q-2} \right] \left| \Upsilon(U^1) - \Upsilon(U^2) \right| \|\Delta u^2\|_{L^2(\mathbb{R}^n)} \|w\|_{L^2(\mathbb{R}^n)}$$

where

$$\begin{aligned}\left|\Upsilon(U^1)\right|^{q-2}+\left|\Upsilon(U^2)\right|^{q-2} &\le [\ \|\Delta u^1\|_{L^2(\mathbb{R}^n)}\|v^1\|_{L^2(\mathbb{R}^n)}\ ]^{q-2}+[\ \|\Delta u^2\|_{L^2(\mathbb{R}^n)}\|v^2\|_{L^2(\mathbb{R}^n)}\ ]^{q-2}\\ &\le 2R^{2(q-2)}.\end{aligned}$$

Using that $\Upsilon(U^1)-\Upsilon(U^2)=-\int\limits_{\mathbb{R}^n}\left[\,(\Delta u^1-\Delta u^2)v^1-\Delta u^2\left(v^1-v^2\right)\right]dx$, we have

$$\begin{aligned}\left|\Upsilon(U^1)-\Upsilon(U^2)\right| &\le \|\Delta u^1-\Delta u^2\|_{L^2(\mathbb{R}^n)}\|v^1\|_{L^2(\mathbb{R}^n)}+\|\Delta u^2\|_{L^2(\mathbb{R}^n)}\|v^1-v^2\|_{L^2(\mathbb{R}^n)}\\ &\le 2R\|U^1-U^2\|_{\mathcal{H}}.\end{aligned}$$

Thus, the term $\mathcal{I}_4$ can be estimated by

$$|\mathcal{I}_4|\le\left[\sqrt{2}R\right]^{2(q-1)}(q-1)\delta||U^1-U^2||_{\mathcal{H}}\|w\|_{L^2(\mathbb{R}^n)}.$$

From Mean Value Theorem, assumption (2.5), Hölder's inequality with $\frac{p-1}{2p}+\frac{1}{2p}+\frac{1}{2}=1$ and embedding $H^2(\mathbb{R}^n)\hookrightarrow L^{2p}(\mathbb{R}^n)$, we get

$$\begin{aligned}|\mathcal{I}_5| &= \left|\int\limits_{\mathbb{R}^n}[\ f(u^1)-f(u^2)\,]\,w\,dx\right|\\ &\le C\int\limits_{\mathbb{R}^n}\left[\ 1+2^{p-1}[|u^1|^{p-1}+|u^2|^{p-1}]\ \right]|u^1-u^2|\,|w|\,dx\\ &= C\int\limits_{\mathbb{R}^n}|u^1-u^2|\,|w|\,dx+C2^{p-1}\int\limits_{\mathbb{R}^n}[\ |u^1|^{p-1}+|u^2|^{p-1}\ ]|u^1-u^2|\,|w|\,dx\\ &\le C\|u^1-u^2\|_{L^2(\mathbb{R}^n)}\|w\|_{L^2(\mathbb{R}^n)}\\ &\quad +C2^{p-1}\left[\ \|u^1\|^{p-1}_{L^{2p}(\mathbb{R}^n)}+\|u^2\|^{p-1}_{L^{2p}(\mathbb{R}^n)}\ \right]\|u^1-u^2\|_{L^{2p}(\mathbb{R}^n)}\|w\|_{L^2(\mathbb{R}^n)}\\ &\le \frac{C}{\lambda^{1/2}}\|U^1-U^2\|_{\mathcal{H}}\|w\|_{L^2(\mathbb{R}^n)}\\ &\quad +C2^pC_p^{p-1}\left[\ \|u^1\|^{p-1}_{H^2(\mathbb{R}^n)}+\|u^2\|^{p-1}_{H^2(\mathbb{R}^n)}\ \right]\|u^1-u^2\|_{H^2(\mathbb{R}^n)}\|w\|_{L^2(\mathbb{R}^n)}\\ &\le \frac{C}{\lambda^{1/2}}\|U^1-U^2\|_{\mathcal{H}}\|w\|_{L^2(\mathbb{R}^n)}+2^{p+1}C_p^{p-1}R^{p-1}C\|U^1-U^2\|_{\mathcal{H}}\|w\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

where $C_p>0$ is the constant coming from the embedding inequality $\|\cdot\|_{2p}\le C_\rho\|\cdot\|_{H^2(\mathbb{R}^n)}$. Last, using assumption (2.4), we have

$$|\mathcal{I}_6|\le\|\alpha\|_{L^\infty(\mathbb{R}^n)}\,\|v^1-v^2\|_{L^2(\mathbb{R}^n)}\|w\|_{L^2(\mathbb{R}^n)}\le\|\alpha\|_{L^\infty(\mathbb{R}^n)}\,\|U^1-U^2||_{\mathcal{H}}\|w\|_{L^2(\mathbb{R}^n)}.$$

Going back to (2.15) we obtain

$$\left|\int\limits_{\mathbb{R}^n}[\,\Pi(U^1)-\Pi(U^2)\,]w dx\right|\le L_R||U^1-U^2||_{\mathcal{H}}\|w\|_{L^2(\mathbb{R}^n)},\tag{2.16}$$

where $L_R>0$ is given by

$$\begin{aligned}L_R &= |\beta|+3\gamma\varrho R^2+\delta R^{2(q-1)}+\left[\sqrt{2}R\right]^{2(q-1)}(q-1)\delta\\ &\quad +\frac{C}{\lambda^{1/2}}+2^{p+1}C_p^{p-1}R^{p-1}C+\|\alpha\|_{L^\infty(\mathbb{R}^n)},\end{aligned}$$

and replacing (2.16) in (2.14), we arrive at the desired locally Lipschitz condition

$$\|\Phi(U^1) - \Phi(U^2)\|_{\mathcal{H}} \le L_R \|U^1 - U^2\|_{\mathcal{H}}. \tag{2.17}$$

Hence, applying the semigroup theory the existence and uniqueness of mild (respect. regular) solution $U = (u, u_t) \in C([0, T_{\max}); \mathcal{H})$, for some $T_{\max} > 0$, (respect. $U = (u, u_t) \in C\left([0, T_{\max}); H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)\right)$) for (2.9) on $[0, T_{\max})$ follows from Theorems 1.4 and 1.6 in Pazy's book [19, Chapter 6]. This proves (i) and (ii).

Now let's prove (iii), that is, that $t_{\max} = +\infty$. In fact, if $T_{\max} < \infty$, it well-known that

$$\lim_{t \to T_{\max}^-} \|(u(t), u_t(t))\|_{\mathcal{H}} = +\infty. \tag{2.18}$$

Multiplying (1.3) by u_t and integrating over $(0,t) \times \mathbb{R}^n,\ t > 0$, we obtain

$$E(t) + \int_0^t \int_{\mathbb{R}^n} \alpha(x) \left| u_t(\tau, x) \right|^2 dx d\tau + \delta \int_0^t \left| \frac{1}{2} \frac{d}{dt} \|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^n)} \right|^q d\tau = E(0). \tag{2.19}$$

From Young's inequality, (2.2) and (2.7), we get

$$\frac{\beta}{2} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^n)} \ge -\frac{\beta^2}{4\gamma} - \frac{\gamma}{4} \|\nabla u(t)\|^4_{L^2(\mathbb{R}^n)}, \tag{2.20}$$

$$\int_{\mathbb{R}^n} F(u(t)) dx \ge -\frac{C_1}{2} \|u(t)\|^2_{L^2(\mathbb{R}^n)} \tag{2.21}$$

and

$$-\int_{\mathbb{R}^n} h u(t) dx \ge -\|h\|_{L^2(\mathbb{R})} \|u(t)\|_{L^2(\mathbb{R}^n)} \ge -\frac{1}{2\vartheta} \|h\|^2_{L^2(\mathbb{R}^n)} - \frac{\vartheta}{2} \|u(t)\|^2_{L^2(\mathbb{R}^n)}, \tag{2.22}$$

where $\vartheta := \frac{\lambda - C_1}{2} > 0$. Thus, from (2.20)-(2.22), we have

$$\begin{aligned} E(t) &\ge \frac{1}{2} \|u_t(t)\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{2} \|\Delta u(t)\|^2_{L^2(\mathbb{R}^n)} + \frac{\vartheta}{2} \|u(t)\|^2_{L^2(\mathbb{R}^n)} - \frac{\beta^2}{4\lambda} - \frac{1}{2\vartheta} \|h\|^2_{L^2(\mathbb{R}^n)} \\ &\ge \vartheta_0 \|(u(t), u_t(t))\|^2_{\mathcal{H}} - \frac{\beta^2}{4\lambda} - \frac{1}{2\vartheta} \|h\|^2_{L^2(\mathbb{R}^n)}, \quad \text{where} \quad \vartheta_0 = \min\{1, \vartheta\}. \end{aligned} \tag{2.23}$$

Combining (2.19) and (2.23), we obtain

$$\|(u(t), u_t(t))\|^2_{\mathcal{H}} \le \frac{1}{\vartheta_0} \left[E(0) + \frac{\beta^2}{4\lambda} + \frac{1}{2\vartheta} \|h\|^2_{L^2(\mathbb{R}^n)} \right], \quad \forall t \in [0, T_{\max}), \tag{2.24}$$

which is a contradiction with (2.18) for $T_{\max} < +\infty$. Therefore, $T_{\max} = +\infty$. This completes the proof of (iii).

Remains proof (iv). We take two mild or strong solutions $U^1(t) = (u^1(t), v^1(t))$ and $U^2(t) = (u^2(t), v^2(t))$ with initial data $U_0^1 = \left(u_0^1, v_0^2\right)$ and $U_0^2 = \left(u_0^2, v_0^1\right)$, respectively, and set $W = U^1 - U^2$ and $W_0 = U_0^1 - U_0^2$. Then, from (2.12) we have

$$W(t) = e^{At}W_0 + \int_0^t e^{t-s}\left[\Phi(U^1(s)) - \Phi(U^2(s))\right] ds, \quad \forall t \in \mathbb{R}^+.$$

Using (2.17) there exists a constant $L = L(\|U_0^1\|_{\mathcal{H}}, \|U_0^2\|_{\mathcal{H}})$ such that

$$\|W(t)\|_{\mathcal{H}} \leq \|W_0\|_{\mathcal{H}} + L\int_0^t e^{t-s}\|W(s)\|_{\mathcal{H}} ds, \quad \forall t \in \mathbb{R}^+. \tag{2.25}$$

Therefore (2.13) is obtained after applying Gronwall inequality. This completes the proof of Theorem 2.1. □

As a consequence, Theorem 2.1 ensures that problem (1.3)-(1.4) generates a nonlinear C_0-semigroup $\{S(t)\}_{t\geq 0}$ in $\mathcal{H} := H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ through the formula

$$\mathcal{H} \ni (u_0, u_1) \longmapsto S(t)(u_0, u_1) = (u(t), u_t(t)), \tag{2.26}$$

where u is the weak solution of (1.3)-(1.4). Therefore, the pair $(\mathcal{H}, S(t))$ designates the dynamical system corresponding to solutions of the Cauchy problem (1.3)-(1.4). In what follows, the main goal is to analyze the long-time behavior of solutions through the dynamical system $(\mathcal{H}, S(t))$.

3. A unique continuation result

In this section we prove an important result, see Proposition 3.1 below, that provides a unique continuation property for a n-dimensional problem related to (1.3)-(1.4). It extends somehow the result on unique continuation proved in [2] to the case of extensible beams. Then, we shall see that Proposition 3.1 is crucial to conclude that the dynamical system $(\mathcal{H}, S(t))$ is gradient.

Proposition 3.1. *Assume that $q \in L^\infty\left(\mathbb{R}; L^2_{loc}(\mathbb{R}^n)\right)$ and $v \in C\left(\mathbb{R}; H^2(\mathbb{R}^n)\right) \cap C^1\left(\mathbb{R}; L^2(\mathbb{R}^n)\right)$ is a weak solution of the following equation:*

$$v_{tt}(t,x) + \Delta^2 v(t,x) - \Delta v(t,x) + q(t,x)v(t,x) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n. \tag{3.1}$$

If for some $r > 0$, we have that

$$v(t,x) = 0, \quad t \in \mathbb{R}, \;\; x \in \mathbb{R}^n \setminus C(r), \tag{3.2}$$

where $C(r) = \{x = (x_1,\ldots,x_n) \in \mathbb{R}^n : |x_i| < r \text{ for all } i = 1,\ldots,n\}$, then

$$v(t,\cdot) = 0 \;\; a.e. \text{ in } \;\mathbb{R}^n, \quad \text{for all } \; t\in\mathbb{R}.$$

Proof. Testing the equation (3.1) with $e^{\frac{i\pi k}{2}}\prod_{j=1}^n e^{(\varpi+\frac{i\pi k}{2r})x_j}$ in $\mathbb{R}^n$, where $\varpi > 0$ and $k \in \mathbb{Z}^+$, and taking (3.2) into account, we get

$$\frac{d^2}{dt^2}\int_{C(r)} v(t,x)e^{\frac{i\pi k}{2}}\prod_{j=1}^n e^{(\varpi+\frac{i\pi k}{2r})x_j}dx + n^2\left(\varpi+\frac{i\pi k}{2r}\right)^4\int_{C(r)} v(t,x)e^{\frac{i\pi k}{2}}\prod_{j=1}^n e^{(\varpi+\frac{i\pi k}{2r})x_j}dx$$
$$+ \, n\left(\varpi+\frac{i\pi k}{2r}\right)^2(\beta+\gamma\theta)\int_{C(r)} v(t,x)e^{\frac{i\pi k}{2}}\prod_{j=1}^n e^{(\varpi+\frac{i\pi k}{2r})x_j}dx$$

$$+ \int\limits_{C(r)} q(t,x)\, v(t,x)\, e^{\frac{i\pi k}{2}} \prod_{j=1}^{n} e^{(\varpi + \frac{i\pi k}{2r})x_j}\, dx = 0.$$

Now, denoting by

$$\Upsilon_{k,\varpi}(t) := \int\limits_{C(r)} v(t,x)\, e^{\frac{i\pi k}{2}} \prod_{j=1}^{n} e^{(\varpi + \frac{i\pi k}{2r})x_j}\, dx,$$

we have

$$\begin{aligned} &\Upsilon''_{k,\varpi}(t) + \left[n^2 \left(\varpi + \frac{i\pi k}{2r} \right)^4 + n \left(\varpi + \frac{i\pi k}{2r} \right)^2 (\beta + \gamma\theta) \right] \Upsilon_{k,\varpi}(t) \\ &= \int\limits_{C(r)} q(t,x)\, v(t,x)\, e^{\frac{i\pi k}{2}} \prod_{j=1}^{n} e^{(\varpi + \frac{i\pi k}{2r})x_j}\, dx. \end{aligned} \tag{3.3}$$

Addition, we define

$$A_{k,\varpi} := n \left(\varpi + \frac{i\pi k}{2r} \right)^2 (\beta + \gamma\theta),$$

$$B_{k,\varpi}(t) = \int\limits_{C(r)} q(t,x)\, v(t,x)\, e^{\frac{i\pi k}{2}} \prod_{j=1}^{n} e^{(\varpi + \frac{i\pi k}{2r})x_j}\, dx,$$

and

$$c_{k,\varpi} := -in \left(\varpi + \frac{i\pi k}{2r} \right)^2,$$

which satisfy the following conditions:

$$\begin{cases} \operatorname{Re}(c_{k,\varpi}) > 0, \\ |A_{k,\varpi}| < |c| \operatorname{Re}(c), \\ B \in L^\infty(\mathbb{R}; \mathbb{C}). \end{cases} \tag{3.4}$$

Hence, considering the above definitions in (3.3), we obtain the following ODE:

$$\Upsilon''_{k,\varpi}(t) + \left(A_{k,\varpi} - c_{k,\varpi}^2 \right) \Upsilon_{k,\varpi}(t) = B_{k,\varpi}(t), \quad t \in \mathbb{R}. \tag{3.5}$$

Then, one can easily see that there exists $\varpi_0 > 0$ such that $A_{k,\varpi}(t)$, $B_{k,\varpi}(t)$ and $c_{k,\varpi}$ satisfy the conditions of [2, Lemma 2.1] for all $\varpi \geq \varpi_0$ and $k \in \mathbb{Z}^+$. Hence, applying [2, Lemma 2.1] to (3.5), we obtain

$$\begin{aligned} |\Upsilon_{k,\varpi}(t)| &\leq \frac{1}{|c_{k,\varpi}| \operatorname{Re}(c_{k,\varpi}) - \|A_{k,\varpi}\|_{L^\infty(\mathbb{R};\mathbb{C})}} \|B_{k,\varpi}\|_{L^\infty(\mathbb{R};\mathbb{C})} \\ &\leq \frac{\|B_{k,\varpi}\|_{L^\infty(\mathbb{R};\mathbb{C})}}{\frac{n^2 \pi k \varpi}{r} (\varpi^2 + \frac{\pi^2 k^2}{4r^2}) - n(\beta + \gamma\theta)(\varpi^2 + \frac{\pi^2 k^2}{4r^2})} \\ &\leq \frac{\widetilde{c}_1}{\varpi k^3} \left[\operatorname*{ess\,sup}_{t \in \mathbb{R}} \int\limits_{C(r)} |q(t,x)\, v(t,x)| \prod_{j=1}^{n} e^{\varpi x_j}\, dx \right], \end{aligned}$$

for all $t \in \mathbb{R}$, $\lambda \geq \lambda_0$, $k \in \mathbb{Z}^+$. This gives

$$\sum_{k=1}^{\infty} k^4 \left| \Upsilon_{k,\varpi}(t) \right|^2 \leq \frac{\widetilde{c}_2}{\varpi^2} \left[\operatorname*{ess\,sup}_{t\in\mathbb{R}} \int\limits_{C(r)} |q(t,x)\, v(t,x)| \prod_{j=1}^{n} e^{\varpi x_j} dx \right]^2 < \infty,$$

for all $t \in \mathbb{R}$, $\lambda \geq \lambda_0$. Thus, by the definition of $\Upsilon_{k,\varpi}(t)$, we find

$$\sum_{k=1}^{\infty} k^4 \left| \int\limits_{-r}^{r} v(t,x) \prod_{j=1}^{n} e^{\varpi x_j} \prod_{j=1}^{n} \sin\left(\frac{\pi}{2r}(x_j + r)\, k\right) dx \right|^2$$
$$\leq \frac{\widetilde{c}_2}{\varpi^2} \left[\operatorname*{ess\,sup}_{t\in\mathbb{R}} \int\limits_{C(r)} |q(t,x)\, v(t,x)| \prod_{j=1}^{n} e^{\varpi x_j} dx \right]^2 < \infty,$$

for all $t \in \mathbb{R}$ and $\varpi \geq \varpi_0$. Since $\left\{ \prod\limits_{j=1}^{n} \frac{1}{\sqrt{r}} \sin\left(\frac{\pi}{2r}(x_j + r)\, k\right) \right\}_{k=1}^{\infty}$ is an orthonormal basis in $L^2(C(r))$ consisting of the eigenfunctions of the operator $-\Delta$ in $L^2(C(r))$ with the domain $H^2(C(r)) \cap H_0^1(C(r))$, by the last inequality, we find that $v \in L^\infty\left(\mathbb{R}; H^2(C(r)) \cap H_0^1(C(r))\right)$, and

$$\int\limits_{C(r)} \left| \Delta(v(t,x) \prod_{j=1}^{n} e^{\varpi x_j}) \right|^2 dx \leq \frac{\widetilde{c}_3}{\varpi^2} \left[\operatorname*{ess\,sup}_{t\in\mathbb{R}} \int\limits_{C(r)} |q(t,x)\, u(t,x)| \prod_{j=1}^{n} e^{\varpi x_j} dx \right]^2, \tag{3.6}$$

for all $t \in \mathbb{R}$ and $\varpi \geq \varpi_0$. Since, for any bounded set $\Omega \subset \mathbb{R}^n$, we have that

$$M_1 \|\Delta u\|_{L^2(\Omega)} \leq \|u\|_{H^2(\Omega)} \leq M_2 \|\Delta u\|_{L^2(\Omega)}, \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega),$$

for some $M_2 \geq M_1 > 0$, then (3.6) implies

$$\|\widehat{v}(t)\|_{H^2(C(r))} \leq \frac{\widetilde{c}_4}{\varpi} \|\widehat{v}\|_{L^\infty(\mathbb{R}, L^2(C(r)))}, \quad \forall t \in \mathbb{R}, \ \forall \varpi \geq \varpi_0,$$

where $\widehat{v}(t,x) = v(t,x) \prod\limits_{j=1}^{n} e^{\varpi x_j}$, from where we obtain

$$\|\widehat{v}(t)\|_{L^2(C(r))} \leq \frac{\widetilde{c}_4}{\varpi} \|\widehat{v}\|_{L^\infty(\mathbb{R}, L^2(C(r)))}, \quad \forall t \in \mathbb{R}, \ \forall \varpi \geq \varpi_0.$$

Choosing ϖ large enough in the above inequality, we obtain

$$v(t,x) = 0, \quad \text{a.e. in } \ x \in \mathbb{R}^n,$$

for all $t \in \mathbb{R}$, which proves Proposition 3.1. $\square$

3.1. Application: the gradient property

Corollary 3.2. *Under the assumptions of Theorem 2.1, then the semigroup $\{S(t)\}_{t\geq 0}$ defined in (2.26) has a strict Lyapunov functional. In other words, $(\mathcal{H}, S(t))$ is a gradient dynamical system.*

Proof. We start by observing that, since the operator A is also maximal dissipative (A defined in (2.10)), one can extend the semigroup $\{S(t)\}_{t\geq 0}$ set by the relation (2.26) to a group $\{S(t)\}_{t\in\mathbb{R}}$. In what follows, we consider $\varphi \in \mathcal{H}$ and set $(u(t), u_t(t)) = S(t)\varphi$, for all $t \in \mathbb{R}$. From (2.19), we have

$$E(u(t), u_t(t)) \leq E(u(0), u_t(0)), \quad \forall t > 0,$$

which means that the function $t \mapsto E(S(t)\varphi)$ is a non-increasing function, for any $\varphi \in \mathcal{H}$. In order to show that the Lyapunov function $E(\varphi)$ is strict, let us consider $E(S(t)\varphi) = E(\varphi)$ for all $t > 0$ and for some φ. Then, from (2.19), it follows that

$$\int_0^t \int_{\mathbb{R}^n} \alpha(x) |u_t(\tau, x)|^2 \, dx d\tau + \int_0^t \left| \frac{d}{dt} \|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^n)} \right|^q d\tau = 0, \quad \forall t \in \mathbb{R}. \tag{3.7}$$

Thus, from (2.3) and (2.4), identity (3.7) leads to

$$u_t(t, x) = 0, \text{ a.e. in } \mathbb{R} \times (\mathbb{R}^n \backslash B(0, r_0)), \tag{3.8}$$

and

$$\frac{d}{dt} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^n)} = 0, \quad \text{a.e. in } (0, \infty), \tag{3.9}$$

which gives

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^n)} = \theta, \text{ a.e. in } (0, \infty), \tag{3.10}$$

where θ is a constant. In this way, defining $v(t, x) := u_t(t, x)$ and taking into account (3.8)-(3.10), we note that v is a solution of the following problem

$$\begin{cases} v_{tt} + \Delta^2 v - (\beta + \gamma\theta)\Delta v + (f'(u) + \lambda) v = 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^n, \\ v(\tau, x) = 0, \quad \text{a.e. in } \mathbb{R} \times (\mathbb{R}^n \backslash B(0, r_0)). \end{cases}$$

Since $B(0, r_0) \subset C(r_0)$, then applying Proposition 3.1, we deduce that

$$v(t, x) = 0, \quad \text{a.e. in } \mathbb{R}^n,$$

for all $t \in \mathbb{R}$, which implies that $u_t(t) \equiv 0$ in $L^2(\mathbb{R}^n)$. Therefore, $S(t)\varphi = (u(t), 0)$, from where one can prove that it is a stationary solution

$$S(t)\varphi = \varphi,$$

and, consequently, $E(\varphi)$ is a strict Lyapunov functional. □

4. Long-time dynamics

Now we are in position to state and prove our main result on the existence and regularity of an attractor to the dynamical system given by (2.26). More precisely, we have:

Theorem 4.1. *Under the assumptions of Theorem 2.1, we have:*

I. ***Global Attractor***. *The semigroup* $\{S(t)\}_{t\geq 0}$ *defined in* (2.26) *possesses a compact global attractor* $\mathcal{A} \subset \mathcal{H} = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

II. ***Characterization***. *The global attractor* $\mathcal{A}$ *is precisely the unstable manifold* $\mathcal{A} = \mathrm{M}^u(\mathcal{N})$ *emanating from the set of stationary solutions* $\mathcal{N}$. *In addition,* $\mathcal{A}$ *consists of full trajectories* $\Gamma = \{U(t) = (u(t), u_t(t)) : t \in \mathbb{R}\}$ *such that*

$$\lim_{t\to-\infty} dist_{\mathcal{H}}(U(t), \mathcal{N}) = 0 \quad \text{and} \quad \lim_{t\to+\infty} dist_{\mathcal{H}}(U(t), \mathcal{N}) = 0. \tag{4.1}$$

III. ***Regularity***. *The global attractor* $\mathcal{A}$ *is bounded in* $H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$.

The proof of Theorem 4.1 will be completed at the end of this section as a consequence of an asymptotic smoothness result (see Theorem 4.6 and Corollary 4.7 below) in combination with Corollary 3.2. However, to obtain such an asymptotic smoothness property for $\{S(t)\}_{t\geq 0}$, we are going to prove some auxiliary technical lemmas as follows.

4.1. Technical lemmas

Lemma 4.2. *Let the sequence* $\{v_m\}_{m=1}^{\infty}$ *be bounded in* $L^\infty\left(0,T;H^2(\mathbb{R}^n)\right) \cap W^{1,\infty}\left(0,T;L^2(\mathbb{R}^n)\right)$ *and the sequence* $\left\{\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)}\right\}_{m=1}^{\infty}$ *be convergent, for all* $t \in [0,T]$. *We define* $z_{m,l} := v_m - v_l$ *and*

$$\widetilde{E}(z_{m,l}(t)) := \|(z_{m,l})_t(t)\|^2_{L^2(\mathbb{R}^n)} + \|\Delta z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n)} + \lambda\|z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n)} + \beta\|\nabla z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n)}.$$

Then, for every $\mu > 0$, *there exists* $c_\mu > 0$ *such that*

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} t\left[\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\Delta v_m(t,x) - \|\nabla v_l(t)\|^2_{L^2(\mathbb{R}^n)}\Delta v_l(t,x)\right](z_{m,l})_t(t,x)\,dxdt \\
&\qquad\leq \mu\int_0^T t\widetilde{E}(z_{m,l}(t))\,dt + C\int_0^T \widetilde{E}(z_{m,l}(t))\,dt \\
&\qquad\quad + c_\mu\int_0^T t\left|\frac{d}{dt}\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\right|^q \widetilde{E}(z_{m,l}(t))\,dt + \Pi^{m,l}(T), \quad \forall T\geq 0,
\end{aligned} \tag{4.2}$$

where $C > 0$, $\Pi^{m,l} \in C[0,\infty)$ *and* $\limsup_{m\to\infty}\limsup_{l\to\infty}\left|\Pi^{m,l}(T)\right| = 0$, *for all* $T \geq 0$.

Proof. Firstly, we have

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} t\left[\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\Delta v_m(t,x) - \|\nabla v_l(t)\|^2_{L^2(\mathbb{R}^n)}\Delta v_l(t,x)\right]((z_{m,l})_t(t,x))\,dxdt \\
&\qquad = -\frac{1}{2}\int_0^T t\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\frac{d}{dt}\|\nabla z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n)}\,dt + \Pi^{m,l}(T),
\end{aligned} \tag{4.3}$$

where

$$\Pi^{m,l}(T) := \int_0^T t \left[\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)} - \|\nabla v_l(t)\|^2_{L^2(\mathbb{R}^n)} \right] \int_{\mathbb{R}^n} \Delta v_l(t,x) \left((z_{m,l})_t(t,x) \right) dxdt.$$

By the assumption of the lemma, we get

$$\limsup_{m\to\infty} \limsup_{l\to\infty} \left| \Pi^{m,l}(T) \right| = 0. \tag{4.4}$$

Now, let us estimate the first term on the right hand side of (4.3). Using integration by parts, it follows that

$$\begin{aligned} &-\frac{1}{2}\int_0^T t \|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)} \frac{d}{dt} \|\nabla z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n)} dt + \frac{T}{2} \|\nabla v_m(T)\|^2_{L^2(\mathbb{R}^n)} \|\nabla z_{m,l}(T)\|^2_{L^2(\mathbb{R}^n)} \\ &= \frac{1}{2}\int_0^T \|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)} \|\nabla z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n)} dt + \frac{1}{2}\int_0^T t \frac{d}{dt} \|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)} \|\nabla z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n)} dt. \end{aligned}$$

Recalling again the assumption of the lemma and with the help of the Young inequality with $\mu > 0$, we obtain from the last equality that

$$\begin{aligned} &-\frac{1}{2}\int_0^T t \|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)} \frac{d}{dt} \|\nabla z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n)} dt \\ &\le C\int_0^T \widetilde{E}(z_{m,l}(t)) dt + \mu \int_0^T t\widetilde{E}(z_{m,l}(t)) dt + c_\mu \int_0^T t \left| \frac{d}{dt} \|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)} \right|^q \widetilde{E}(z_{m,l}(t)) dt. \end{aligned} \tag{4.5}$$

Hence, replacing (4.4) and (4.5) in (4.3), we conclude that the desired estimate (4.2) holds true, which proves Lemma 4.2. □

Lemma 4.3. *Let the sequence $\{v_m\}_{m=1}^\infty$ be bounded in $L^\infty\left(0,T;H^2(\mathbb{R}^n)\right) \cap W^{1,\infty}\left(0,T;L^2(\mathbb{R}^n)\right)$ and the sequence $\left\{\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)}\right\}_{m=1}^\infty$ be convergent, for all $t \in [0,T]$. Then, for every $\phi \in W^{1,\infty}(\mathbb{R}^n)$ such that $\phi(x) \ge 0$ for all $x \in \mathbb{R}^n$ and $\sup(\phi_{x_i}) \subset (B(0,2r) \backslash B(0,r))$ for $i = 1,\dots,n$, the following limit holds*

$$\begin{aligned} \limsup_{m\to\infty} \limsup_{l\to\infty} \int_0^T \int_{\mathbb{R}^n} t \left[\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)} \Delta v_m(t,x) - \|\nabla v_l(t)\|^2_{L^2(\mathbb{R}^n)} \Delta v_l(t,x) \right] \\ \times \phi(x) (z_{m,l}(t,x)) dxdt = 0, \end{aligned}$$

where $z_{m,l} = v_m - v_l$ and $B(0,r) = \{x \in \mathbb{R}^n : \ |x| < r\}$.

Proof. First of all, using integration by parts, we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} t \left[\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)} \Delta v_m(t,x) - \|\nabla v_l(t)\|^2_{L^2(\mathbb{R}^n)} \Delta v_l(t,x) \right] \phi(x) (z_{m,l}(t,x)) dxdt \\ &= \widetilde{\Pi}^{m,l}(T) - \int_0^T t \|\nabla v_l(t)\|^2_{L^2(\mathbb{R}^n)} \left\| \sqrt{\phi} (\nabla z_{m,l}(t)) \right\|^2_{L^2(\mathbb{R}^n)} dt \end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{n} \int_0^T \int_{B(0,2r)\setminus B(0,r)} t \left\| \nabla v_l (t) \right\|_{L^2(\mathbb{R}^n)}^2 \left(z_{m,l} (t,x) \right)_{x_i} \phi_{x_i} (x) \left(z_{m,l} (t,x) \right) dxdt \\
&\leq \widetilde{\Pi}^{m,l} (T) + C\, T^2 \left\| v_m - v_l \right\|_{C([0,T];H^1(B(0,2r)\setminus B(0,r)))} \qquad (4.6)
\end{aligned}
$$

where

$$
\widetilde{\Pi}^{m,l} (T) := \int_0^T \int_{\mathbb{R}^n} t \left[\left\| \nabla v_m (t) \right\|_{L^2(\mathbb{R}^n)}^2 - \left\| \nabla v_l (t) \right\|_{L^2(\mathbb{R}^n)}^2 \right] \Delta v_m(t,x) \phi (x) \left(z_{m,l} (t,x) \right) dxdt.
$$

In addition, under the assumptions of the lemma, it is easy to see that

$$
\sup_{m,l} \left\| \widetilde{\Pi}^{m,l} \right\|_{C[0,T]} < \infty \text{ and } \limsup_{m\to\infty} \limsup_{l\to\infty} \left| \widetilde{\Pi}^{m,l} (T) \right| = 0, \ \forall T \geq 0. \qquad (4.7)
$$

Furthermore, exploiting the compactness results in [20], we obtain that the sequence $\{v_m\}_{m=1}^{\infty}$ is relatively compact in $C\left([0,T]; H^{2-\varepsilon}\left(B(0,r)\right)\right)$ for every $\varepsilon > 0$, $T > 0$ and $r > 0$. This gives the convergence

$$
v_m \to v \text{ strongly in } C\left([0,T]; H^{2-\varepsilon}\left(B(0,r)\right)\right), \qquad (4.8)
$$

for some $v \in C\left([0,T]; H^{2-\varepsilon}\left(B(0,r)\right)\right)$. With the help of (4.7) and (4.8), passing to the limit in (4.6), we complete the proof of Lemma 4.3. □

Lemma 4.4. *Under the condition (2.5) and assuming that the sequence $\{v_m\}_{m=1}^{\infty}$ is weakly star convergent in $L^{\infty}\left(0,T; H^2(\mathbb{R}^n)\right) \cap W^{1,\infty}\left(0,T; L^2(\mathbb{R}^n)\right)$, then there exists a constant $c > 0$ such that*

$$
\begin{aligned}
&\limsup_{m\to\infty} \limsup_{l\to\infty} \int_0^T \int_{\mathbb{R}^n} t \left[f(v_l (t,x)) - f(v_m (t,x)) \right] \left[(z_{m,l})_t (t,x) \right] dxdt \\
&\leq c \limsup_{m\to\infty} \int_0^T \left\| v_m (t) \right\|_{H^2(\mathbb{R}^n \setminus B(0,r))}^2 dt, \ \forall T \geq 0 \text{ and } \forall r \geq 0. \qquad (4.9)
\end{aligned}
$$

Proof. Using integration by parts, we have

$$
\begin{aligned}
&\limsup_{m\to\infty} \limsup_{l\to\infty} \int_0^T \int_{\mathbb{R}^n} t \left[f(v_l (t,x)) - f(v_m (t,x)) \right] \left[(z_{m,l})_t (t,x) \right] dxdt \\
&\leq T \limsup_{m\to\infty} \limsup_{l\to\infty} \left[- \int_{\mathbb{R}^n} \left(F(v_m (T,x)) + F(v_l (T,x)) - 2F(v (T,x)) \right) dx \right] \\
&\quad + \limsup_{m\to\infty} \limsup_{l\to\infty} \int_0^T \int_{\mathbb{R}^n} \left(F(v_m (t,x)) + F(v_m (t,x)) - 2F(v (t,x)) \right) dxdt \\
&= - T \liminf_{m\to\infty} \liminf_{l\to\infty} \int_{\mathbb{R}^n} \left(F(v_m (T,x)) + F(v_l (T,x)) - 2F(v (T,x)) \right) dx \qquad (4.10) \\
&\quad + \limsup_{m\to\infty} \limsup_{l\to\infty} \int_0^T \int_{B(0,r)} \left(F(v_m (t,x)) + F(v_l (t,x)) - 2F(v (t,x)) \right) dxdt
\end{aligned}
$$

$$+\limsup_{m\to\infty}\limsup_{l\to\infty}\int_0^T\int_{\mathbb{R}^n\setminus B(0,r)}\left(F(v_m(t,x))+F(v_l(t,x))-2F(v(t,x))\right)dxdt.$$

Furthermore, under the conditions of the lemma and (2.5), one sees that

$$F(v_m(t,x))\to F(v(t,x)) \text{ a.e. in } (0,T)\times B(0,r)\,,\ \forall r>0.$$

On the other hand, since $\{F(v_m)\}_{m=1}^{\infty}$ is bounded in $W^{1,1}((0,T)\times\mathbb{R}^n)$, the following limits happen

$$\begin{cases} F(v_m)\to F(v) \text{ strongly in } L^1((0,T)\times B(0,r))\,,\ \forall T>0,\ \forall r>0,\\ F(v_m)\to F(v) \text{ weakly in } L^{\frac{n+1}{n}}((0,T)\times\mathbb{R}^n)\,. \end{cases} \tag{4.11}$$

Thus, taking into account (4.11) in (4.10) and recalling the conditions of the lemma, we obtain

$$\begin{aligned}&\limsup_{m\to\infty}\limsup_{l\to\infty}\int_0^T\int_{\mathbb{R}^n} t\left(f(v_l(t,x))-f(v_m(t,x))\right)\left((z_{m,l})_t(t,x)\right)dxdt\\ &\quad\le c\limsup_{m\to\infty}\int_0^T\|v_m(t)\|^2_{H^2(\mathbb{R}^n\setminus B(0,r))}\,dt,\ \forall\ T\ge 0 \text{ and } \forall\ r\ge 0,\end{aligned}$$

as desired in (4.9). This concludes the proof of Lemma 4.4. □

Lemma 4.5. *Let us consider (2.5) in force and assume that the sequence $\{v_m\}_{m=1}^{\infty}$ is weakly star convergent in $L^\infty\left(0,T;H^2(\mathbb{R}^n)\right)\cap W^{1,\infty}\left(0,T;L^2(\mathbb{R}^n)\right)$. Then, for every $\varphi\in C(\mathbb{R}^n)$ such that $\varphi(x)\ge 0$ for all $x\in\mathbb{R}^n$, there hold the following inequality*

$$\limsup_{m\to\infty}\limsup_{l\to\infty}\int_0^T\int_{\mathbb{R}^n} t\varphi(x)\left[f(v_l(t,x))-f(v_m(t,x))\right]\left[v_m(t,x)-v(t,x)\right]dxdt\le 0. \tag{4.12}$$

Proof. First, we readily get

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n} t\varphi(x)\left[f(v_l)(t,x)-f(v_m(t,x))\right]\left[z_{m,l}(t,x)\right]dxdt\\ &\quad=\int_0^T\int_{\mathbb{R}^n} t\varphi(x)\left[f(v_l(t,x))\,v_m(t,x)+f(v_m(t,x))\,v_l(t,x)\right.\\ &\qquad\left.-f(v_m(t,x))\,v_m(t,x)-f(v_l(t,x))\,v_l(t,x)\right]dxdt.\end{aligned} \tag{4.13}$$

Then, recalling the assumptions of the lemma, for any $r>0$ and $T>0$, we have

$$v_m\to v \text{ strongly in } L^2((0,T)\times B(0,r))\,.$$

Thus, there exists a subsequence of $\{v_m\}_{m=1}^{\infty}$, still denoted by $\{v_m\}_{m=1}^{\infty}$, which is convergent to v almost everywhere in $(0,T)\times B(0,r)$ for all $r>0$. Next, since $f\in C^1(\mathbb{R})$, we obtain

$$f(v_m(t,x))v_m(t,x)\to f(v(t,x))v(t,x) \text{ a.e. in } (0,T)\times B(0,r) \text{ for all } r>0.$$

Moreover, by (2.5), the sequence $\{f(v_m)v_m\}_{k=1}^{\infty}$ is bounded in $L^{\frac{2p}{p+1}}((0,\infty)\times\mathbb{R}^n)$. Hence, since $r>0$ is arbitrary, we infer that

$$f(v_m)v_m \to f(v)v \text{ weakly in } L^{\frac{2p}{p+1}}\left((0,T)\times\mathbb{R}^n\right). \tag{4.14}$$

Therefore, passing to the limit in (4.13), considering (4.14) and taking into account the assumptions of the lemma, we deduce

$$\begin{aligned}
&\limsup_{m\to\infty}\limsup_{l\to\infty}\int_0^T\int_{\mathbb{R}^n} t\varphi(x)\left[f(v_l(t,x))-f(v_m(t,x))\right]\left[z_{m,l}(t,x)\right]dxdt\\
&=\limsup_{m\to\infty}\limsup_{l\to\infty}\int_0^T\int_{\mathbb{R}^n} t\varphi(x)\left[2f(v(t,x))v(t,x)-\sum_{j=m,l}f(v_j(t,x))v_j(t,x)\right]dxdt\\
&=-\liminf_{m\to\infty}\liminf_{l\to\infty}\int_0^T\int_{\mathbb{R}^n} t\varphi(x)\left[\sum_{j=m,l}f(v_j(t,x))v_j(t,x)-2f(v(t,x))v(t,x)\right]dxdt\le 0,
\end{aligned}$$

which proves (4.12) and, therefore, Lemma 4.5. □

4.2. The asymptotic smoothness property

The next result plays a key role in the existence of the global attractor. As a matter of fact, it provides the Ladyzhenskaya condition for the semigroup $\{S(t)\}_{t\ge 0}$ set in (2.26) and, consequently, the asymptotic smoothness property by using the general theory in dynamical systems, see for instance [7, Chapter 7].

Theorem 4.6. *Let us assume that conditions (2.2)-(2.5) hold, with $\beta\ge-\zeta$, where $\zeta>0$ a small constant, and let B be a bounded subset of $\mathcal{H}$. Then, every sequence of the form $\{S(t_k)\varphi_k\}_{k=1}^{\infty}$, with $\{\varphi_k\}_{k=1}^{\infty}\subset B$ and $t_k\to\infty$, has a subsequence that converges in $\mathcal{H}$.*

Proof. We will prove the theorem by establishing the following sequential limit

$$\liminf_{k\to\infty}\liminf_{m\to\infty}\left\|S(t_k)\varphi_k-S(t_m)\varphi_m\right\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}=0, \tag{4.15}$$

for every $\{\varphi_k\}_{k=1}^{\infty}\subset B$ and $t_k\to\infty$. Therefore, combining (4.15) with the argument presented in [13, Lemma 4.5], we infer that the statement of Theorem 4.6 follows.

From now on, our goal is to prove (4.15). We start by observing that by (2.24) we have

$$\sup_{t\ge 0}\sup_{\varphi\in B}\left\|S(t)\varphi\right\|_{\mathcal{H}}<\infty. \tag{4.16}$$

Since $\{\varphi_k\}_{k=1}^{\infty}$ is bounded in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, by (4.16), the sequence $\{S(.)\varphi_k\}_{k=1}^{\infty}$ is bounded in $C_b\left(0,\infty;H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\right)$, where $C_b\left(0,\infty;H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\right)$ is the space of continuously bounded functions from $[0,\infty)$ to $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Then for any $T_0\ge 0$ there exists a subsequence $\{k_m\}_{m=1}^{\infty}$ such that $t_{k_m}\ge T_0$, and

$$\begin{cases}
v_m\to v \text{ weakly star in } L^{\infty}\left(0,\infty;H^2(\mathbb{R}^n)\right),\\
v_{mt}\to v_t \text{ weakly star in } L^{\infty}\left(0,\infty;L^2(\mathbb{R}^n)\right),\\
\left\|\nabla v_m(t)\right\|^2_{L^2(\mathbb{R}^n)}\to q(t) \text{ weakly star in } W^{1,\infty}(0,\infty),\\
v_m(t)\to v(t) \text{ weakly in } H^2(\mathbb{R}^n),\ \forall t\ge 0,
\end{cases} \tag{4.17}$$

for some $q \in W^{1,\infty}(0,\infty)$ and $v \in L^{\infty}\left(0,\infty;H^2(\mathbb{R}^n)\right) \cap W^{1,\infty}\left(0,\infty;L^2(\mathbb{R}^n)\right)$, where we have $(v_m(t), v_{mt}(t)) = S(t+t_{k_m}-T_0)\varphi_{k_m}$. By (1.3), we also have

$$\begin{aligned}
&(z_{m,l})_{tt}(t,x) + \Delta^2 z_{m,l}(t,x) + \lambda z_{m,l}(t,x) - \beta\Delta z_{m,l}(t,x) \\
&- \gamma\left[\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\Delta v_m(t,x) - \|\nabla v_l(t)\|^2_{L^2(\mathbb{R}^n)}\Delta v_l(t,x)\right] + \alpha(x)(z_{m,l})_t(t,x) \\
&+ \left[f(v_m(t,x)) - f(v_l(t,x))\right] - \delta\left[\Xi(v_m(t))\Delta v_m(t,x) - \Xi(v_l(t))\Delta v_l(t,x)\right] = 0.
\end{aligned} \tag{4.18}$$

In what follows, we are going to establish the following estimates for the smooth solutions of (1.3)-(1.4) with the initial data in $H^4(\mathbb{R}^n)\times H^2(\mathbb{R}^n)$. By using standard density arguments, these estimates can be extended to the weak solutions with the initial data in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Firstly, considering (2.3)-(2.5) in (2.19), we obtain that

$$\int_0^T \|v_{mt}(t)\|^2_{L^2(\mathbb{R}^n\setminus B(0,r_0))}\,dt \le C,\ \forall T\ge 0, \tag{4.19}$$

$$\int_0^T \left|\frac{d}{dt}\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\right|^q dt \le C,\ \forall T\ge 0. \tag{4.20}$$

Now, putting v_m instead of v in (1.3), we have

$$\begin{aligned}
v_{mtt}(t,x) + \Delta^2 v_m(t,x) - \beta\Delta v_m + \lambda v_m(t,x) - \gamma\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\Delta v_m \\
+ \alpha(x)v_{mt}(t,x) + f(v_m) - \delta\,\Xi(\nabla v_m(t,x))\Delta v_m = h(x).
\end{aligned}$$

In the next computations, we are going to use the following cut-off function $\eta_r(x) = \eta\left(\frac{x}{r}\right)$, where

$$\eta\in C^{\infty}(\mathbb{R}^n),\quad 0\le\eta(x)\le 1,\quad \eta(x)=\begin{cases}0, & |x|\le 1,\\ 1, & |x|\ge 2.\end{cases} \tag{4.21}$$

Multiplying the previous equation by $\eta_r^2 v_m$ and integrating over $(0,T)\times\mathbb{R}^n$, we get

$$\begin{aligned}
&\int_0^T \left[\|\eta_r\Delta v_m(t)\|^2_{L^2(\mathbb{R}^n)} + \lambda\|\eta_r v_m(t)\|^2_{L^2(\mathbb{R}^n)} + \beta\|\eta_r\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\right]dt \\
&\quad+\gamma\int_0^T \|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\|\eta_r\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\,dt + \int_0^T\int_{\mathbb{R}^n} f(v_m(t,x))\,\eta_r^2 v_m(t,x)\,dxdt \\
&=\int_0^T \|\eta_r v_{mt}(t)\|^2_{L^2(\mathbb{R}^n)}\,dt - \left[\int_{\mathbb{R}^n}\eta_r^2(x)\,v_{mt}(t,x)\,v_m(t,x)dx\right]_0^T \\
&\quad-\frac{4}{r}\sum_{i=1}^n\int_0^T \eta_r(x)\,\eta_{x_i}\left(\frac{x}{r}\right)\Delta v_m(t,x)v_{mx_i}(t,x)dxdt - \int_0^T\int_{\mathbb{R}^n}\Delta\left(\eta_r^2(x)\right)\Delta v_m(t,x)v_m(t,x)dxdt \\
&\quad-\frac{2\beta}{r}\sum_{i=1}^n\int_0^T \eta_r(x)\,\eta_{x_i}\left(\frac{x}{r}\right)v_m(t,x)v_{mx_i}(t,x)dxdt - \frac{1}{2}\left[\int_{\mathbb{R}^n}\eta_r^2(x)\,\alpha(x)\,(v_m(t,x))^2\,dx\right]_0^T
\end{aligned}$$

$$-\delta\int_0^T \Xi\left(v_m\left(t\right)\right)\left\|\eta_r\nabla v_m(t)\right\|^2_{L^2(\mathbb{R}^n)}dt+\int_0^T\int_{\mathbb{R}^n} h\left(x\right)\eta_r^2\left(x\right)v_m\left(t,x\right)dxdt,$$

where $\Xi\left(v\left(t\right)\right)=\left|\frac{d}{dt}\left\|\nabla v\left(t\right)\right\|^2_{L^2(\mathbb{R}^n)}\right|^{q-2}\frac{d}{dt}\left\|\nabla v\left(t\right)\right\|^2_{L^2(\mathbb{R}^n)}$.

From assumption (2.6), we have

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n} f(v_m(t,x))\eta_r^2 v_m(t,x)dxdt &\geq -C_1\int_0^T\|\eta_r v_m(t)\|^2_{L^2(\mathbb{R}^n)}dt\\ &=-\frac{C_1}{\lambda}\int_0^T\lambda\|\eta_r v_m(t)\|^2_{L^2(\mathbb{R}^n)}dt\\ &\geq -\frac{C_1}{\lambda}\int_0^T\left[\lambda\|\eta_r v_m(t)\|^2_{L^2(\mathbb{R}^n)}+\|\eta_r\Delta v_m(t)\|^2_{L^2(\mathbb{R}^n)}\right].\end{aligned}\tag{4.22}$$

Using that $\beta\geq-\zeta$, interpolation theorem and Young inequality, we obtain

$$\begin{aligned}\beta\|\eta_r\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)} &\geq -\zeta\left[\left\|\nabla\left(\eta_r v_m(t)\right)\right\|^2_{L^2(\mathbb{R}^n)}+\left\|\nabla\left(\eta_r\right)v_m(t)\right\|^2_{L^2(\mathbb{R}^n)}\right]\\ &\geq -\zeta\varrho\left[\lambda\|\eta_r v_m(t)\|^2_{L^2(\mathbb{R}^n)}+\left\|\Delta\left(\eta_r v_m(t)\right)\right\|^2_{L^2(\mathbb{R}^n)}\right]-\zeta\left\|\nabla\left(\eta_r\right)v_m(t)\right\|^2_{L^2(\mathbb{R}^n)}\\ &\geq -\zeta\varrho\left[\lambda\|\eta_r v_m(t)\|^2_{L^2(\mathbb{R}^n)}+\|\eta_r\Delta v_m(t)\|^2_{L^2(\mathbb{R}^n)}\right]-\frac{C}{r}\end{aligned}\tag{4.23}$$

where $\varrho>0$ comes from the inequality

$$\|\nabla z\|^2_{L^2(\mathbb{R}^n)}\leq\varrho\left[\lambda\|z\|^2_{L^2(\mathbb{R}^n)}+\|\Delta z\|^2_{L^2(\mathbb{R}^n)}\right].$$

Thus, taking into account (4.16), (4.19), (4.20), (4.22) and (4.23) with $\zeta<\frac{\vartheta}{\lambda\varrho}$, we obtain

$$\begin{aligned}&\int_0^T\left[\left\|\Delta v_m(t)\right\|^2_{L^2(\mathbb{R}^n\backslash B(0,2r))}+\lambda\left\|v_m(t)\right\|^2_{L^2(\mathbb{R}^n\backslash B(0,2r))}\right]dt\\ &\quad+\gamma\int_0^T\left\|\nabla v_m\left(t\right)\right\|^2_{L^2(\mathbb{R}^n)}\left\|\nabla v_m(t)\right\|^2_{L^2(\mathbb{R}^n\backslash B(0,2r))}dt\\ &\leq C\left[1+\frac{T}{r}+T\left\|h\right\|_{L^2(\mathbb{R}^n\backslash B(0,r))}\right],\ \forall T\geq0\text{ and }\forall r\geq r_0.\end{aligned}\tag{4.24}$$

Now, multiplying (4.18) by $\sum_{i=1}^n x_i\left(1-\eta_{2r}\right)\left(v_m-v_l\right)_{x_i}+\frac{1}{2}\left(n-1\right)\left(1-\eta_{2r}\right)\left(v_m-v_l\right)$, and integrating over $(0,T)\times\mathbb{R}^n$, we obtain

$$\begin{aligned}&\frac{3}{2}\int_0^T\|\Delta z_{m,l}(t)\|^2_{L^2(B(0,2r))}dt+\frac{1}{2}\int_0^T\|(z_{m,l})_t(t)\|^2_{L^2(B(0,2r))}dt\\ &\leq\sum\nolimits_{i=1}^n\int_{B(0,4r)}\left|\left(1-\eta_{2r}\left(x\right)\right)x_i\left(z_{m,l}\right)_{x_i}(T,x)\right)\left(z_{m,l}\right)_t(T,x)\right|dx\end{aligned}$$

$$+ \sum_{i=1}^{n} \int_{B(0,4r)} \left| (1-\eta_{2r}(x))\, x_i\, (z_{m,l})_{x_i}(0,x)\right) (z_{m,l})_t(0,x) \,|\, dx$$

$$+ \frac{1}{2}(n-1) \int_{B(0,4r)} \left| (1-\eta_{2r}(x))\left((z_{m,l})_t(T,x)\right)(z_{m,l}(T,x)) \right| dx$$

$$+ \frac{1}{2}(n-1) \int_{B(0,4r)} \left| (1-\eta_{2r}(x))\left((z_{m,l})_t(0,x)\right)(z_{m,l}(0,x)) \right| dx$$

$$+ \frac{1}{4r}\sum_{i=1}^{n} \int_0^T \int_{B(0,4r)\setminus B(0,2r)} \left| \eta_{x_i}\left(\frac{x}{2r}\right) x_i \left[(z_{m,l})_t(t,x) \right]^2 \right| dxdt$$

$$+ \frac{1}{4r}\sum_{i=1}^{n} \int_0^T \int_{B(0,4r)\setminus B(0,2r)} \left| \eta_{x_i}\left(\frac{x}{2r}\right) x_i \left[\Delta z_{m,l}(t,x) \right]^2 \right| dxdt$$

$$+ \sum_{i=1}^{n} \int_0^T \int_{B(0,4r)} \left| \Delta\left((1-\eta_{2r}(x))\, x_i\right) (z_{m,l}(t,x))_{x_i}\, \Delta(z_{m,l}(t,x)) \right| dxdt$$

$$+ \frac{1}{r}\sum_{i,j=1}^{n} \int_0^T \int_{B(0,4r)\setminus B(0,2r)} \left| \eta_{x_j}\left(\frac{x}{2r}\right) x_i\, (z_{m,l}(t,x))_{x_i x_j}\, \Delta(z_{m,l}(t,x)) \right| dxdt$$

$$+ \frac{1}{2}(n-1) \int_0^T \int_{B(0,4r)\setminus B(0,2r)} \left| \Delta\left((1-\eta_{2r}(x))\right)(z_{m,l}(t,x))\, \Delta(z_{m,l}(t,x)) \right| dxdt$$

$$+ \frac{1}{2r}(n-1)\sum_{i=1}^{n} \int_0^T \int_{B(0,4r)\setminus B(0,2r)} \left| \eta_{x_i}\left(\frac{x}{2r}\right)(z_{m,l}(t,x))_{x_i}\, \Delta(z_{m,l}(t,x)) \right| dxdt$$

$$+ \beta\sum_{i=1}^{n} \int_0^T \int_{B(0,4r)} \left| (1-\eta_{2r}(x))\, x_i\, (z_{m,l}(t,x))_{x_i}\, \Delta(z_{m,l}(t,x)) \right| dxdt$$

$$+ \frac{\beta}{2}(n-1) \int_0^T \int_{B(0,4r)} \left| (1-\eta_{2r})\,(z_{m,l}(t,x))\, \Delta(z_{m,l}(t,x)) \right| dxdt$$

$$+ \lambda\sum_{i=1}^{n} \int_0^T \int_{B(0,4r)} \left| (1-\eta_{2r}(x))\, x_i\, (z_{m,l}(t,x))_{x_i}\, (z_{m,l}(t,x)) \right| dxdt$$

$$+ \gamma\sum_{i=1}^{n} \int_0^T \int_{B(0,4r)} \Big| \Big[\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\, \Delta v_m(t,x) - \|\nabla v_l(t)\|^2_{L^2(\mathbb{R}^n)}\, \Delta v_l(t,x) \Big]$$

$$\times (1-\eta_{2r}(x))\, x_i\, (z_{m,l}(t,x))_{x_i} \Big| dxdt$$

$$+ \frac{\gamma}{2}(n-1) \int_0^T \int_{B(0,4r)} \Big| \Big[\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\, \Delta v_m(t,x) - \|\nabla v_l(t)\|^2_{L^2(\mathbb{R}^n)}\, \Delta v_l(t,x) \Big]$$

$$\times \left(1-\eta_{2r}\right)\left(v_m - v_l\right)\big|\, dxdt$$

$$+ \sum\nolimits_{i=1}^{n} \int_0^T \int_{B(0,4r)} \left| \left(1-\eta_{2r}(x)\right) x_i \left(z_{m,l}(t,x)\right)_{x_i} a(x) \left(\left(z_{m,l}\right)_t(t,x)\right) \right| dxdt$$

$$+ \frac{1}{2}(n-1) \int_0^T \int_{B(0,4r)} \left| \left(1-\eta_{2r}(x)\right)\left(z_{m,l}(t,x)\right) a(x) \left(\left(z_{m,l}\right)_t(t,x)\right) \right| dxdt$$

$$+ \delta \sum\nolimits_{i=1}^{n} \int_0^T \int_{B(0,4r)} \big| \left(\Xi\left(v_m(t)\right) \Delta v_m(t,x) - \Xi\left(v_l(t)\right) \Delta v_l(t,x)\right)$$

$$\times \left(1-\eta_{2r}(x)\right) x_i \left(z_{m,l}(t,x)\right)_{x_i} \Big| \, dxdt$$

$$+ \frac{\delta}{2}(n-1) \Bigg| \sum\nolimits_{i=1}^{n} \int_0^T \int_{B(0,4r)} \left(\Xi\left(v_m(t)\right) \Delta v_m(t,x) - \Xi\left(v_l(t)\right) \Delta v_l(t,x)\right)$$

$$\times \left(1-\eta_{2r}\right)\left(z_{m,l}(t,x)\right) dxdt\big|$$

$$+ \sum\nolimits_{i=1}^{n} \int_0^T \int_{B(0,4r)} \left| \left(1-\eta_{2r}(x)\right) x_i \left(z_{m,l}(t,x)\right)_{x_i} \left(f\left(v_m(t,x)\right) - f\left(v_l(t,x)\right)\right) \right| dxdt$$

$$+ \frac{1}{2}(n-1) \int_0^T \int_{B(0,4r)} \left| \left(1-\eta_{2r}(x)\right)\left(z_{m,l}(t,x)\right)\left(f\left(v_m(t,x)\right) - f\left(v_l(t,x)\right)\right) \right| dxdt.$$

Now we note that by (4.16),

$$\begin{aligned}
&\left\| \Xi\left(v_m(t)\right) \Delta v_m(t) - \Xi\left(v_l(t)\right) \Delta v_l(t) \right\|_{L^2(B(0,4r))} \\
&\le \sum_{j=m,l} \left| \frac{d}{dt} \left\| \nabla v_j(t) \right\|^2_{L^2(\mathbb{R}^n)} \right|^{q-1} \left\| \Delta v_j(t) \right\|_{L^2(B(0,4r))} \\
&\le 2 \sum_{j=m,l} \left[\left\| \Delta v_j(t) \right\|_{L^2(\mathbb{R}^n)} \left\| v_{jt}(t) \right\|_{L^2(\mathbb{R}^n)} \right]^{q-1} \left\| \Delta v_j(t) \right\|_{L^2(B(0,4r))} \le \overline{C},
\end{aligned}$$

from where, along with the previous estimate, it follows that

$$\begin{aligned}
&\frac{3}{2} \int_0^T \left\| \Delta z_{m,l}(t) \right\|^2_{L^2(B(0,2r))} dt + \frac{1}{2} \int_0^T \left\| \left(z_{m,l}\right)_t(t) \right\|^2_{L^2(B(0,2r))} dt \\
&\le C\, r \left[\left\| \nabla z_{m,l}(T) \right\|_{L^2(B(0,4r))} + \left\| \nabla z_{m,l}(0) \right\|_{L^2(B(0,4r))} \right] \\
&\quad \times C\ \left\| \left(z_{m,l}\right)_t \right\|^2_{L^2(0,T;L^2(B(0,4r)\setminus B(0,2r)))} + C\ \left\| z_{m,l} \right\|^2_{L^2(0,T;H^2(B(0,4r)\setminus B(0,2r)))} \\
&\quad + C\, \sqrt{T} \left\| \nabla z_{m,l} \right\|_{L^2((0,T)\times B(0,4r))}.
\end{aligned} \tag{4.25}$$

Since the sequence $\{v_m\}_{m=1}^{\infty}$ is bounded in $C\left([0,T]; H^2(\mathbb{R}^n)\right)$ and the sequence $\{v_{mt}\}_{m=1}^{\infty}$ is bounded in $C\left([0,T]; L^2(\mathbb{R}^n)\right)$, by the generalized Arzela-Ascoli Theorem, the sequence $\{v_m\}_{m=1}^{\infty}$ is relatively compact in $C\left([0,T]; H^1(B(0,r))\right)$ for every $r > 0$. So, according to (4.17),

$$v_m \to v \text{ strongly in } C\left([0,T]\,;H^1\left(B\left(0,r\right)\right)\right). \tag{4.26}$$

Then, using (4.24) and (4.26) in (4.25), we get

$$\limsup_{m\to\infty}\limsup_{l\to\infty}\int\limits_0^T \left[\,\|\Delta z_{m,l}\,(t)\|^2_{L^2(B(0,2r))}+\|(z_{m,l})_t\,(t)\|^2_{L^2(B(0,2r))}\right]dt$$
$$\leq C\left[1+\frac{T}{r}+T\,\|h\|_{L^2(\mathbb{R}^n\backslash B(0,r))}\right],\quad \forall T\geq 0,\ \forall r\geq r_0.$$

Now, using (4.19), (4.24) and the last inequality, we obtain

$$\limsup_{m\to\infty}\,\limsup_{l\to\infty}\int\limits_0^T \left[\,\|z_{m,l}\,(t)\|^2_{H^2(\mathbb{R}^n)}+\|(z_{m,l})_t\,(t)\|^2_{L^2(\mathbb{R}^n)}\right]dt$$
$$\leq C\left[1+\frac{T}{r}+T\,\|h\|_{L^2(\mathbb{R}^n\backslash B(0,r))}\right],\quad \forall T\geq 0,\ \forall r\geq r_0.$$

After passing the last inequality to the limit as $r\to\infty$, we deduce

$$\limsup_{m\to\infty}\,\limsup_{l\to\infty}\int\limits_0^T \widetilde{E}\left(z_{m,l}(t)\right)dt\leq C,\quad \forall T\geq 0. \tag{4.27}$$

Now, multiplying (4.18) by $2t(z_{m,l})_t=2t\left(v_{mt}-v_{lt}\right)$, integrating over $(0,T)\times\mathbb{R}^n$, using integration by parts and considering (2.4), we find

$$\begin{aligned}
&T\widetilde{E}(z_{m,l}(T))+\alpha_0\int\limits_0^T t\,\|(z_{m,l})_t(t)\|^2_{L^2(\mathbb{R}^n\backslash B(0,r))}\,dt\ \leq\ \int\limits_0^T \widetilde{E}(z_{m,l}(t))dt\\
&+2\gamma\int\limits_0^T\int\limits_{\mathbb{R}^n}\left[\,\|\nabla v_m\,(t)\|^2_{L^2(\mathbb{R}^n)}\,\Delta v_m\,(t,x)-\|\nabla v_l\,(t)\|^2_{L^2(\mathbb{R}^n)}\,\Delta v_l\,(t,x)\right]t(z_{m,l})_t(t,x)dxdt\\
&+C\int\limits_0^T t\left[\sum_{j=m,l}\left|\frac{d}{dt}\,\|\nabla v_j\,(t)\|^2_{L^2(\mathbb{R}^n)}\right|^{q-2}+\left|\frac{d}{dt}\,\|\nabla v_l\,(t)\|^2_{L^2(\mathbb{R}^n)}\right|^{q-1}\right]\widetilde{E}(z_{m,l}(t))dt\\
&+2\int\limits_0^T\int\limits_{\mathbb{R}^n} t\left(f(v_l\,(t,x))-f(v_m\,(t,x))\right)(z_{m,l})_t(t,x)dxdt.
\end{aligned} \tag{4.28}$$

Multiplying (4.18) by $\varepsilon t\eta_r z_{m,l}$, integrating over $(0,T)\times\mathbb{R}^n$ and using integration by parts, we get

$$\varepsilon\int\limits_0^T t\left[\,\|\Delta z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n\backslash B(0,2r))}+\lambda\,\|z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n\backslash B(0,2r))}+\beta\,\|\nabla z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n\backslash B(0,2r))}\right]dt$$
$$\leq\varepsilon C\ T\ \widetilde{E}\left(z_{m,l}(T)\right)+\varepsilon\int\limits_0^T\int\limits_{\mathbb{R}^n\backslash B(0,r)} t\left(z_{m,l}(t)\right)^2_t dxdt+\varepsilon C\int\limits_0^T\widetilde{E}\left(z_{m,l}(t)\right)dt$$

$$+\ \varepsilon\gamma\int_0^T\int_{\mathbb{R}^n}\Big[\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\Delta v_m(t,x)-\|\nabla v_l(t)\|^2_{L^2(\mathbb{R}^n)}\Delta v_l(t,x)\Big]t\eta_r(x)(z_{m,l}(t,x))\,dxdt$$

$$+\ \varepsilon C\int_0^T t\left[\sum_{j=m,l}\left|\frac{d}{dt}\|\nabla v_j(t)\|^2_{L^2(\mathbb{R}^n)}\right|^{q-2}+\left|\frac{d}{dt}\|\nabla v_l(t)\|^2_{L^2(\mathbb{R}^n)}\right|^{q-1}\right]\widetilde{E}(z_{m,l}(t))\,dt$$

$$+\ \varepsilon C\frac{T^2}{r}\|z_{m,l}\|_{C([0,T];H^1(B(0,2r)\setminus B(0,r)))}$$

$$+\ \varepsilon\int_0^T\int_{\mathbb{R}^n}t\left(f(v_l(t,x))-f(v_m(t,x))\right)\eta_r(x)(z_{m,l}(t,x))\,dxdt, \tag{4.29}$$

for all $T\geq 0$ and $r\geq r_0$.

Now, multiplying (4.18) by $\varepsilon^2\sum_{i=1}^n tx_i(1-\eta_{2r})(z_{m,l})_{x_i}+\varepsilon^2\frac{t}{2}(n-1)(1-\eta_{2r})z_{m,l}$, and integrating over $(0,T)\times\mathbb{R}^n$, we obtain

$$\frac{\varepsilon^2}{2}\int_0^T t\Big[3\|\Delta z_{m,l}(t)\|^2_{L^2(B(0,2r))}dt+\|(z_{m,l})_t(t)\|^2_{L^2(B(0,2r))}\Big]dt$$

$$\leq\varepsilon^2 C\ r\left(T+T^2\right)\|z_{m,l}\|_{C([0,T];H^1(B(0,4r)))}+\varepsilon^2 C\int_0^T t\|(z_{m,l})_t(t)\|^2_{L^2(B(0,4r)\setminus B(0,2r))}dt$$

$$+\ \varepsilon^2\ C\int_0^T t\|z_{m,l}(t)\|^2_{H^2(B(0,4r)\setminus B(0,2r))}dt. \tag{4.30}$$

Summing (4.28)-(4.30), choosing ε sufficiently small, applying Young inequality and recalling Lemma 4.2, we infer

$$T\widetilde{E}(z_{m,l}(T))+\frac{\varepsilon^2}{2}\int_0^T t\Big[3\|\Delta z_{m,l}(t)\|^2_{L^2(B(0,2r))}dt+\|(z_{m,l})_t(t)\|^2_{L^2(B(0,2r))}\Big]dt$$

$$+\ \varepsilon\int_0^T t\Big[\|\Delta z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n\setminus B(0,2r))}+\lambda\|z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n\setminus B(0,2r))}+\beta\|\nabla z_{m,l}(t)\|^2_{L^2(\mathbb{R}^n\setminus B(0,2r))}\Big]dt$$

$$+\ \frac{\varepsilon^2}{2}\int_0^T t\Big[3\|\Delta z_{m,l}(t)\|^2_{L^2(B(0,2r))}dt+\|(z_{m,l})_t(t)\|^2_{L^2(B(0,2r))}\Big]dt$$

$$\leq C\int_0^T(1+\mu t)\widetilde{E}(z_{m,l}(t))\,dt+\Pi^{m,l}(T)+C\int_0^T t\left[\sum_{j=m,l}\left|\frac{d}{dt}\|\nabla v_j(t)\|^2_{L^2(\mathbb{R}^n)}\right|^q\right]\widetilde{E}(z_{m,l}(t))\,dt$$

$$+\ C\int_0^T\int_{\mathbb{R}^n}t\left(f(v_l(t,x))-f(v_m(t,x))\right)((z_{m,l})_t(t,x))\,dxdt$$

$$+\ \varepsilon C\frac{T^2}{r}\|z_{m,l}\|_{C([0,T];H^1(B(0,2r)\setminus B(0,r)))}+\varepsilon^2 Cr\left(T+T^2\right)\|z_{m,l}\|_{C([0,T];H^1(B(0,4r)))}$$

$$+\varepsilon\gamma\int_0^T\int_{\mathbb{R}^n}\left(\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\Delta v_m(t,x)-\|\nabla v_l(t)\|^2_{L^2(\mathbb{R}^n)}\Delta v_l(t,x)\right)t\ \eta_r(x)\left(z_{m,l}(t,x)\right)dxdt$$

$$+\varepsilon C\int_0^T\int_{\mathbb{R}^n}t\left(f(v_l)-f(v_m)\right)\eta_r(x)\left(z_{m,l}(t,x)\right)dxdt,\quad \forall T\geq 0,\ \forall r\geq r_0$$

At this point, let us denote $\kappa_{m,l}(T)=T\widetilde{E}\left(z_{m,l}(T)\right)$ and

$$\begin{aligned}\Lambda_{m,l}(T)=&\int_0^T\widetilde{E}\left(z_{m,l}(t)\right)dt+\Pi^{m,l}(T)\\&+\int_0^T\int_{\mathbb{R}^n}t\left(f(v_l(t,x))-f(v_m(t,x))\right)\left((z_{m,l})_t(t,x)\right)dxdt\\&+\left(r+\frac{1}{r}\right)\left(T+T^2\right)\|v_{m,l}\|_{C([0,T];H^1(B(0,4r)))}\\&+\varepsilon\int_0^T\int_{\mathbb{R}^n}t\left(f(v_l)-f(v_m)\right)\eta_r(x)\left(z_{m,l}(t,x)\right)dxdt\\&+\ \varepsilon\int_0^T\int_{\mathbb{R}^n}\left[\|\nabla v_m(t)\|^2_{L^2(\mathbb{R}^n)}\Delta v_m(t,x)-\|\nabla v_l(t)\|^2_{L^2(\mathbb{R}^n)}\Delta v_l(t,x)\right]\\&\quad\times t\ \eta_r(x)\left(z_{m,l}(t,x)\right)dxdt.\end{aligned}$$

Then, we obtain

$$\kappa_{m,l}(T)\leq C\Lambda_{m,l}(T)+C\int_0^T\left[\sum_{j=m,l}\left|\frac{d}{dt}\|\nabla v_j(t)\|^2_{L^2(\mathbb{R}^n)}\right|^q\right]\kappa_{m,l}(t)\,dxdt,$$

and applying Gronwall inequality, it yields

$$\kappa_{m,l}(T)\leq C\Lambda_{m,l}(T)\,e^{C\int_0^T\left[\sum_{j=m,l}\left|\frac{d}{dt}\|\nabla v_j(t)\|^2_{L^2(\mathbb{R}^n)}\right|^q\right]dt}.$$

Therefore, using Lemma 4.3, Lemma 4.4, Lemma 4.5, (4.4), (4.20), (4.26) and (4.27), there holds the estimate

$$\limsup_{m\to\infty}\limsup_{l\to\infty}\kappa_{m,l}(T)\leq C\left[1+\limsup_{m\to\infty}\int_0^T\|v_m(t)\|^2_{H^2(\mathbb{R}^n\setminus B(0,r))}dt\right].$$

Recalling (4.24) and the definition of the $\kappa_{m,l}(T)$, from the last estimate, it follows that

$$\limsup_{m\to\infty}\limsup_{l\to\infty}T\widetilde{E}\left(z_{m,l}(T)\right)\leq C\left[1+\frac{T}{r}+T\|h\|_{L^2(\mathbb{R}^n\setminus B(0,r))}\right],\quad\forall T\geq 0,\ \forall r\geq r_0.$$

Hence, taking the limit as $r\to\infty$ in the above inequality, there holds the estimate

$$\limsup_{m\to\infty}\limsup_{l\to\infty} T\widetilde{E}\left(z_{m,l}\left(T\right)\right) \le C, \quad \forall T \ge 0,$$

which yields

$$\limsup_{m\to\infty}\limsup_{l\to\infty} \|S(T+t_{k_m}-T_0)\varphi_{k_m} - S(t+t_{k_l}-T_0)\varphi_{k_l}\|_{\mathcal{H}} \le \frac{C}{\sqrt{T}}, \ \forall\, T>0.$$

Picking $T=T_0$ in the last estimate, we deduce that

$$\limsup_{m\to\infty}\limsup_{l\to\infty} \|S(t_{k_m})\varphi_{k_m} - S(t_{k_l})\varphi_{k_l}\|_{\mathcal{H}} \le \frac{C}{\sqrt{T_0}}, \ \forall\, T_0 \ge 1,$$

which proves (4.15), as desired. This completes the proof of Theorem 4.6. □

Corollary 4.7. *Under the assumptions of Theorem 4.6, the dynamical system $(\mathcal{H}, S(t))$ is asymptotically smooth.*

Proof. It is an immediate consequence of Theorem 4.6 and [7, Proposition 7.1.6]. □

We are finally in position to conclude the proof of Theorem 4.1.

4.3. Proof of Theorem 4.1: items I and II

From Corollaries 3.2 and 4.7 we have obtained that $(\mathcal{H}, S(t))$ is a gradient asymptotically smooth dynamical system. Moreover, it is not so difficult to prove the following additional properties:

(i) the Lyapunov function $E(\varphi)$ is bounded from above on every bounded subset of $\mathcal{H}$;

(ii) the set $E_R = \{\varphi :\ E(\varphi) \le R\}$ is bounded for every R;

(iii) the set of stationary solutions $\mathcal{N}$ is bounded.

Therefore, applying [7, Corollary 7.5.7], we can conclude that $\{S\left(t\right)\}_{t\ge 0}$ defined in (2.26) has a compact global attractor $\mathcal{A} = \mathrm{M}^u(\mathcal{N}) \subset \mathcal{H}$. In addition, the property (4.1) is a direct consequence of [7, Theorem 7.5.6]. □

4.4. Proof of Theorem 4.1: item III

Let us consider $(u_0, u_1) \in \mathcal{A}$. By the invariance of $\mathcal{A}$, it follows that (see for instance [3, p. 159]) there exists an invariant trajectory $\Gamma = \{(u\left(t\right), u_t\left(t\right)) : t\in\mathbb{R}\} \subset \mathcal{A}$ such that $(u\left(0\right), u_t\left(0\right)) = (u_0, u_1)$. According to [3], an invariant trajectory is a curve $\Gamma = \{(u\left(t\right), u_t\left(t\right)) : t\in\mathbb{R}\}$ such that $S\left(t\right)\left(u\left(\tau\right), u_t\left(\tau\right)\right) = \left(u\left(t+\tau\right), u_t\left(t+\tau\right)\right)$ for all $t\ge 0$ and $\tau\in\mathbb{R}$. As in (2.26), we denote $S(t)\left(u_0, u_1\right) = \left(u\left(t\right), u_t\left(t\right)\right)$, and we also define

$$v\left(t,x\right) := \frac{u\left(t+\tau,x\right)-u\left(t,x\right)}{\tau}, \quad \tau>0.$$

Thus, using equation (1.3) and denoting, for simplicity, $\Upsilon(u)$ instead $\Upsilon(u,u_t)$, we get

$$v_{tt}(t,x) + \Delta^2 v(t,x) + \lambda v(t,x) - \beta\Delta v(t,x) + \alpha(x)v_t(t,x) - \gamma\left\|\nabla u\left(t\right)\right\|^2_{L^2(\mathbb{R}^n)}\Delta v\left(t,x\right)$$
$$-\ \gamma\frac{\left\|\nabla u\left(t+\tau\right)\right\|^2_{L^2(\mathbb{R}^n)} - \left\|\nabla u\left(t\right)\right\|^2_{L^2(\mathbb{R}^n)}}{\tau}\Delta u(t+\tau,x) - \delta\left|\Upsilon\left(u\left(t\right)\right)\right|^{q-2}\Upsilon\left(u\left(t\right)\right)\Delta v\left(t,x\right)$$

$$- \delta \frac{|\Upsilon(u(t+\tau))|^{q-2}\Upsilon(u(t+\tau)) - |\Upsilon(u(t))|^{q-2}\Upsilon(u(t))}{\tau}\Delta u(t+\tau,x)$$
$$+ \frac{f(u(t+\tau,x)) - f(u(t,x))}{\tau} = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n. \tag{4.31}$$

From the definition of v and (2.24), we firstly observe that

$$\|v(s)\|_{L^2(\mathbb{R}^n)} = \left\| \frac{u(s+\tau,x) - u(s,x)}{\tau} \right\|_{L^2(\mathbb{R}^n)} \le \sup_{0\le s<\infty} \|u_t(s)\|_{L^2(\mathbb{R}^n)} < \widehat{C}. \tag{4.32}$$

Additionally, there hold the following inequalities

$$\begin{aligned}
&\frac{|\Upsilon(u(t+\tau))|^{q-2}\Upsilon(u(t+\tau)) - |\Upsilon(u(t))|^{q-2}\Upsilon(u(t))}{\tau} \\
&\quad \le C\frac{|\Upsilon(u(t+\tau)) - \Upsilon(u(t))|}{\tau}\left[|\Upsilon(u(t+\tau))|^{q-2} + |\Upsilon(u(t))|^{q-2}\right] \\
&\quad = C\frac{1}{\tau}\left|\int_{\mathbb{R}^n} \Delta(u(t+\tau,x))\, u_t(t+\tau,x)\,dx - \int_{\mathbb{R}^n} \Delta(u(t,x))\, u_t(t,x)\,dx\right| \\
&\qquad \times \left[|\Upsilon(u(t+\tau))|^{q-2} + |\Upsilon(u(t))|^{q-2}\right] \\
&\quad \le C\left|\int_{\mathbb{R}^n} \Delta(u(t+\tau,x))\, v_t(t,x)\,dx\right|\left[|\Upsilon(u(t+\tau))|^{q-2} + |\Upsilon(u(t))|^{q-2}\right] \\
&\qquad + \left|\int_{\mathbb{R}^n} \Delta v(t,x)\, u_t(t,x)\,dx\right|\left[|\Upsilon(u(t+\tau))|^{q-2} + |\Upsilon(u(t))|^{q-2}\right] \\
&\quad \le C\left[\|v_t(t)\|_{L^2(\mathbb{R}^n)} + \|\Delta v(t)\|_{L^2(\mathbb{R}^n)}\right]\left[|\Upsilon(u(t+\tau))|^{q-2} + |\Upsilon(u(t))|^{q-2}\right].
\end{aligned} \tag{4.33}$$

Then, multiplying (4.31) by $2v_t$, integrating over $\mathbb{R}^n$ and exploiting (4.32)-(4.33), we find

$$\begin{aligned}
&\frac{d}{dt}\left(\overline{E}(v(t))\right) + 2\alpha_0 \|v_t(t)\|^2_{L^2(\mathbb{R}^n\setminus B(0,r_0))} \\
&\quad \le C|\Upsilon(u(t))|\,\overline{E}(v(t)) + C\|\nabla v(t)\|_{L^2(\mathbb{R}^n)}\|v_t(t)\|_{L^2(\mathbb{R}^n)} + C|\Upsilon(u(t))|^{q-1}\,\overline{E}(v(t)) \\
&\qquad + C\left[|\Upsilon(u(t))|^{q-2} + |\Upsilon(u(t+\tau))|^{q-2}\right]\overline{E}(v(t)) + C\|v_t(t)\|_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{4.34}$$

where

$$\begin{aligned}
\overline{E}(v(t)) = {} & \|v_t(t)\|^2_{L^2(\mathbb{R}^n)} + \|\Delta(v(t))\|^2_{L^2(\mathbb{R}^n)} + \beta\|\nabla v(t)\|^2_{L^2(\mathbb{R}^n)} + \lambda\|v(t)\|^2_{L^2(\mathbb{R}^n)} \\
& + \gamma\|\nabla u(t)\|^2_{L^2(\mathbb{R}^n)}\|\nabla v(t)\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Now, let η be the cut-off function as defined in (4.21). Multiplying the equation (4.31) by $\sum_{i=1}^n \varepsilon^2 x_i(1-\eta_{2r})v_{x_i} + \frac{\varepsilon^2}{2}(n-1)(1-\eta_{2r})v$, integrating over $\mathbb{R}^n$, and exploiting (4.32)-(4.33), and using (2.5) and (2.24), it follows that

$$\frac{3\varepsilon^2}{2}\|\Delta v(t)\|^2_{L^2(B(0,2r))} + \frac{\varepsilon^2}{2}\|v_t(t)\|^2_{L^2(B(0,2r))} + \frac{\varepsilon^2}{2}\beta\|\nabla v(t)\|^2_{L^2(B(0,2r))} + \frac{\varepsilon^2}{2}\lambda\|v(t)\|^2_{L^2(B(0,2r))}$$

$$
\begin{aligned}
&+ \frac{\varepsilon^2}{2}\gamma \left\|\nabla u(t)\right\|^2_{L^2(\mathbb{R}^n)} \left\|\nabla v(t)\right\|^2_{L^2(B(0,2r))} + \varepsilon^2 \frac{(n-1)}{4}\frac{d}{dt}\left[\int\limits_{\mathbb{R}^n} (1-\eta_{2r}(x))\, a(x)\, |v(t,x)|^2\, dx\right] \\
&+ \varepsilon^2 \frac{d}{dt}\left[\sum_{i=1}^{n}\int\limits_{\mathbb{R}^n} x_i (1-\eta_{2r}(x))\, v_{x_i}(t,x)\, v_t(t,x)\, dx + \frac{1}{2}(n-1)\int\limits_{\mathbb{R}^n}(1-\eta_{2r}(x))\, v_t(t,x)\, v(t,x)\, dx\right] \\
&+ \varepsilon^2 \frac{(n-1)}{4}\frac{d}{dt}\left[\int\limits_{\mathbb{R}^n} (1-\eta_{2r}(x))\, a(x)\, |v(t,x)|^2\, dx\right] \\
&\qquad \le \varepsilon^2 C \left\|v_t(t)\right\|^2_{L^2(B(0,4r)\backslash B(0,2r))} ds + \varepsilon^2 C \left\|\Delta v(t)\right\|^2_{L^2(B(0,4r)\backslash B(0,2r))} \\
&\qquad\quad + \varepsilon^2 C \left\|\nabla v(t)\right\|^2_{L^2(B(0,4r)\backslash B(0,2r))} ds + \varepsilon^2 C \left\|v(t)\right\|^2_{L^2(B(0,4r)\backslash B(0,2r))} \\
&\qquad\quad + \varepsilon^2 C\left[1 + r\left\|v(t)\right\|_{H^2(\mathbb{R}^n)} + r\left\|v(t)\right\|^{1/2}_{H^2(\mathbb{R}^n)}\right] + \varepsilon^2 C(r+1)\left|\Upsilon(u(t))\right|^{q-1}\overline{E}(v(t)) \\
&\qquad\quad + \varepsilon^2 C(r+1)\left[\left|\Upsilon(u(t))\right|^{q-2} + \left|\Upsilon(u(t+\tau))\right|^{q-2}\right]\overline{E}(v(t)), \quad \forall t \ge 0,\ \forall r \ge r_0.
\end{aligned}
$$

Next, a further multiplication of (4.31) by $\varepsilon\eta_r^2 v$, with the help of (4.32) and (4.33), leads to

$$
\begin{aligned}
&\varepsilon\left\|\Delta v(t)\right\|^2_{L^2(\mathbb{R}^n\backslash B(0,r))} + \varepsilon\beta\left\|\nabla v(t)\right\|^2_{L^2(\mathbb{R}^n\backslash B(0,r))} + \varepsilon\lambda\left\|v(t)\right\|^2_{L^2(\mathbb{R}^n\backslash B(0,r))} \\
&\quad + \varepsilon\gamma\left\|\nabla u(t)\right\|^2_{L^2(\mathbb{R}^n)}\left\|\nabla v(t)\right\|^2_{L^2(\mathbb{R}^n\backslash B(0,r))} \\
&\quad + \varepsilon\frac{d}{dt}\left[\int\limits_{\mathbb{R}^n}\eta_r^2(x)\, v_t(t,x)\, v(t,x) dx + \frac{1}{2}\int\limits_{\mathbb{R}^n}\eta_r^2(x)\,\alpha(x)\,(v(t,x))^2\, dx\right] \\
&\le C\varepsilon\left[1 + \left\|\eta_r v_t(t)\right\|^2_{L^2(\mathbb{R}^n)} + \left\|v(t)\right\|^{3/2}_{H^2(\mathbb{R}^n\backslash B(0,r))} + \left\|v(t)\right\|_{H^2(\mathbb{R}^n\backslash B(0,r))} + \left\|v(t)\right\|^{1/2}_{H^2(\mathbb{R}^n)}\right] \\
&\quad + \varepsilon C\left[\left|\Upsilon(u(t))\right|^{q-2} + \left|\Upsilon(u(t+\tau))\right|^{q-2}\right]\overline{E}(v(t)), \quad \forall t\ge 0,\ \forall r \ge r_0.
\end{aligned}
\tag{4.35}
$$

Adding the inequalities (4.34)-(4.35) and applying Young's inequality with $\varepsilon > 0$ sufficiently small, we deduce

$$
\begin{aligned}
&\frac{d}{dt}\left(\overline{E}(v(t))\right) + C\overline{E}(v(t) \\
&+ \varepsilon^2\frac{d}{dt}\left[\sum_{i=1}^{n}\int\limits_{\mathbb{R}^n} x_i(1-\eta_{2r}(x))\, v_{x_i}(t,x)\, v_t(t,x)\, dx + \frac{1}{2}(n-1)\int\limits_{\mathbb{R}^n}(1-\eta_{2r}(x))\, v_t(t,x)\, v(t,x)\, dx\right] \\
&+ \varepsilon^2\frac{(n-1)}{4}\frac{d}{dt}\left[\int\limits_{\mathbb{R}^n}(1-\eta_{2r}(x))\, a(x)\, |v(t,x)|^2\, dx\right] \\
&+ \varepsilon\frac{d}{dt}\left[\int\limits_{\mathbb{R}^n}\eta_r^2(x)\, v_t(t,x)\, v(t,x) dx + \frac{1}{2}\int\limits_{\mathbb{R}^n}\eta_r^2(x)\,\alpha(x)\,(v(t,x))^2\, dx\right] \\
&\le C + C\left|\Upsilon(u(t))\right|^q\overline{E}(v(t)), \quad \forall t \ge 0.
\end{aligned}
\tag{4.36}
$$

Now, let us define

$$
\begin{aligned}
\Theta(t) = {} & \overline{E}(v(t)) + \varepsilon^2 \int_{\mathbb{R}^n} (1-\eta_{2r}(x))\, a(x) \left|v(t,x)\right|^2 dx \\
& + \varepsilon^2 \sum_{i=1}^{n} \int_{\mathbb{R}^n} x_i (1-\eta_{2r}(x))\, v_{x_i}(t,x)\, v_t(t,x)\, dx \\
& + \frac{\varepsilon^2}{2}(n-1) \int_{\mathbb{R}^n} (1-\eta_{2r}(x))\, v_t(t,x)\, v(t,x)\, dx \\
& + \varepsilon \left[\int_{\mathbb{R}^n} \eta_r^2(x)\, v_t(t,x)\, v(t,x) dx + \frac{1}{2} \int_{\mathbb{R}^n} \eta_r^2(x)\, \alpha(x) \left(v(t,x)\right)^2 dx \right].
\end{aligned}
$$

Thus, from (4.36), it follows that

$$
\frac{d}{dt}\Theta(t) + C\overline{E}(v(t)) \le C + C\left|\Upsilon(u(t))\right|^q \overline{E}(v(t)), \quad \forall t \ge 0. \tag{4.37}
$$

It is easy to see that, choosing ε small enough, there exist constants $c > 0$, $\widetilde{c} > 0$ such that

$$
c\overline{E}(v(t)) \le \Theta(t) \le \widetilde{c}\overline{E}(v(t)). \tag{4.38}
$$

Then, by (4.37) and (4.38), we readily get

$$
\frac{d}{dt}\Theta(t) + C\Theta(t) \le C + C\left|\Upsilon(u(t))\right|^q \Theta(t),
$$

which yields

$$
\Theta(t) \le C + C\int_0^t \left|\Upsilon(u(s))\right|^q \Theta(s)\, ds, \quad \forall t \in [0,T],
$$

for every $T > 0$ and some $C > 0$. Applying Gronwall's inequality in the last estimate, we obtain

$$
\Theta(t) \le Ce^{C\int_0^t \left|\Upsilon(u(s))\right|^q ds}, \quad \forall t \in [0,T], \tag{4.39}
$$

for every $T \ge 0$ and some $C > 0$. On the other hand, recalling (2.19), we have the estimate

$$
\int_0^\infty \left|\Upsilon(u(s))\right|^q ds \le \widehat{c},
$$

for some constant $\widehat{c} > 0$. Therefore, taking into account the last estimate in (4.39) and using once more (4.38), we infer that

$$
\overline{E}(v(t)) \le C, \quad \forall t \in [0,T], \tag{4.40}
$$

for every $T \ge 0$ and some constant $C > 0$. Taking the limit as $\tau \to 0$ in (4.40) and regarding the definition of v, we eventually obtain that

$$
\overline{E}(u_t(t)) \le C, \quad \forall t \ge 0,
$$

for some constant $C > 0$. From this last estimate and (1.3), we also deduce that

$$\|u(t)\|_{H^4(\mathbb{R}^n)} \le C,$$

for some constant $C > 0$. Therefore, the last two estimates are enough to conclude that

$$\|(u(t), u_t(t))\|_{H^4(\mathbb{R}^n)\times H^2(\mathbb{R}^n)} \le C,$$

for some constant $C > 0$. This finishes the proof of Theorem 4.1. □

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Appendix A. Final remarks

We finally highlight two interesting points to conclude the paper, namely, on the Balakrishnan-Taylor damping term (as a not so good stabilizer alone) and also on the exponential stabilization in the homogeneous framework.

A.1. Weakness of the Balakrishnan-Taylor damping

Although acting as a dissipative structure in the system, the Balakrishnan-Taylor damping seems to be not enough to produce "a good" stability to the homogeneous problem, say the exponential one. Indeed, just to fix the idea, let us formally take problem (1.3)-(1.4) with $\beta = \gamma = 0$, $\alpha, h \equiv 0$ in $\mathbb{R}^n$ and $f \equiv 0$ in $\mathbb{R}$, namely, the next simplified model

$$u_{tt} + \Delta^2 u + \lambda u - \delta \left|\Upsilon(u, u_t)\right|^{q-2} \Upsilon(u, u_t)\Delta u = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^n, \tag{A.1}$$

where we have used the notation (1.5). Thus, the corresponding energy set in (2.1) becomes to the following

$$E(t) = \frac{1}{2}\|\Delta u(t)\|^2_{L^2(\mathbb{R}^n)} + \frac{\lambda}{2}\|u(t)\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{2}\|u_t(t)\|^2_{L^2(\mathbb{R}^n)}, \quad t \ge 0, \tag{A.2}$$

whose derivative formally (and rigorously) satisfies

$$\frac{d}{dt}E(t) \;=\; -\delta \left|\Upsilon(u(t), u_t(t))\right|^q \;=\; -\delta \left|(\Delta u(t), u_t(t))\right|^q \le 0, \quad \forall\, t > 0. \tag{A.3}$$

This shows that E is non-increasing and we can ask about its stability. For example, if $E(t)$ does not go to zero as t goes to infinity, then nothing could be done in terms of stabilization. On the other, even if we have the strong stability $E(t) \to 0$ as $t \to \infty$, then one has

$$|\Upsilon(u(t), u_t(t))| \le E(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{A.4}$$

Accordingly, (A.4) implies that the damping coefficient in (A.1) becomes less and less effective as the parameter t goes to infinity and, consequently, no fast decay rate estimates could be expected. Moreover, from (A.3)-(A.4), one can easily see that

$$E(t) \geq \frac{1}{\left[(q-1)\delta\, t + E^{1-q}(0)\right]^{\frac{1}{q-1}}}, \quad t > 0, \tag{A.5}$$

which clearly means that the Balakrishnan-Taylor damping term $-\delta\, |\Upsilon(u,u_t)|^{q-2}\, \Upsilon(u,u_t)\Delta u$ (alone) is no longer enough to conclude (uniform) exponential stability, once the energy is bounded from below polynomially as in (A.5). In conclusion, the additional localized damping term $\alpha(x)u_t$ seems to be a minimum amount of damping necessary to recover uniform (exponential) decay patterns for (A.1) and, thus, in the study of long-time dynamics for the fuller model (1.3)-(1.4).

A.2. Exponential stabilization in the homogeneous case

By taking $\beta = \gamma = \delta = 0$ and $h \equiv 0$ in (1.3)-(1.4), then it precisely reduces to the problem proposed by [21], see (1.1)-(1.2) therein. In such a case, under suitable assumptions on f, the authors show that the corresponding energy functional is exponentially stable, see [21, Theorem 1.1]. In the present article, if we only assume $h \equiv 0$ in (1.3)-(1.4), then it also consists a sort of more general homogeneous problem, by including extensible beams of Balakrishnan-Taylor type. In such a case, from (2.1) one sees that

$$E_0(t) = \frac{1}{2}\|(u(t),u_t(t))\|_{\mathcal{H}}^2 + \frac{\beta}{2}\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}^2 + \frac{\gamma}{4}\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}^4 + \int\limits_{\mathbb{R}^n} F(u(t))\, dx, \tag{A.6}$$

and from (2.19) it is also non-increasing. Moreover, by strengthening the assumptions on f (if necessary), taking advantage of the unique continuation property in the present setting, and following the same lines of [21], one can prove that $E_0(t)$ given in (A.6) (related to the homogeneous problem) is exponential stable, that is, there exist constants $C, c > 0$ such that

$$E_0(t,u_0,u_1) \leq C E_0(0,u_0,u_1)e^{-ct}, \quad \forall\, t \geq 0, \tag{A.7}$$

for initial data lying in bounded subsets $(u_0,u_1) \in B \subset H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. In other words, the attractor $\mathcal{A}_0 = \{(0,0)\}$ would be the trivial one. To the proof of such statements, only computations on the extensible term

$$-\left(\beta + \gamma\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}^2 + \delta\, |\Upsilon(u,u_t)|^{q-2}\, \Upsilon(u,u_t)\right)\Delta u,$$

are necessary when compared to [21, Section 2], by consisting somehow very similar to those performed in the latter. Hence, we omit the details here.

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