# Long-Time Dynamics of Balakrishnan-Taylor Extensible Beams 

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#### Abstract

This paper is concerned with well-posedness and long-time dynamics for a class extensible beams with nonlocal Balakrishnan-Taylor and frictional damping. The related model describes vibrations in nonlinear extensible beams arising in connection with models of oscillation in pipes and supersonic panel flutter. Our main results feature the study of the nonlinear dynamical system generated by the problem. The main novelty is to explore the global $L^{q}$-regularity ( $q \geq 2$ ) in time of the nonlocal Balakrishnan-Taylor term and show how it generates a dissipative term that plays an important role in the asymptotic behavior of solutions, mainly in what concerns to achieve the useful property of quasi-stability in the theory of infinite-dimensional dynamical systems.


Keywords Extensible beam • Balakrishnan-Taylor damping • Long-time dynamics • Global attractor • Fractal dimension

Mathematics Subject Classification 35B40 • 35B41 • 37L30 • 35L75 • 74H40 • 74K99

## 1 Introduction

In 1989 Balakrishnan and Taylor [2] proposed the following new model for flight structures with viscous and nonlinear nonlocal damping in the one dimensional case

$$
\begin{align*}
& \varrho u_{t t}+E I u_{x x x x}-c u_{x x t} \\
& \quad-\left[H+\frac{E A}{2 L} \int_{0}^{L}\left|u_{x}\right|^{2} d x+\tau\left(\int_{0}^{L} u_{x} u_{x t} d x\right)^{2(N+\eta)+1}\right] u_{x x}=0, \tag{1.1}
\end{align*}
$$

[^0]where $u=u(x, t)$ represents the transversal deflection of an extensible beam with length $2 L>0$ in the rest position, $\varrho>0$ is the mass density, $E$ is the Young's modulus of elasticity, $I$ is the cross-sectional moment of inertia, $H$ is the axial force (either traction or compression), $A$ is the cross-sectional area, $c>0$ is the coefficient of viscous damping, $\tau>0$ is the Balakrishnan-Taylor damping coefficient, $0 \leq \eta<\frac{1}{2}$ and $N \in \mathbb{N}$. We refer to [2, Sect. 4] for the precise modeling of (1.1). See also Bass and Zes [3, Eqs.(14a-c)]. We still encourage the reader to see the reference [5] where a nice tribute to the memory of A.V. Balakrishnan brings up his career and contributions.

In the case $N=\eta=0$, Eq. (1.1) arises in connection with models of oscillation in pipes and supersonic panel flutter, whose derivation is known since 1970s with the works by Dowell [10] and Holmes [15]. In this context, global existence and asymptotic stability under some hypotheses about the aerodynamic pressure were first considered by Marsden et al. [16,17] and You [29]. Later, the asymptotic behavior of solutions was studied by Clark in [9], where the structural (viscous) damping $-c u_{x x t}$ is replaced by the strong damping $\nu u_{x x x x t}, v>0$. Going back to the case $0 \leq \eta<\frac{1}{2}$ and $N \in \mathbb{N}$, the well-posedness and long-time dynamics of solutions for (1.1) were first treated by You [30], where exponential stabilization and inertial manifolds are established.

In 2011 Emmrich and Thalhammer [12] considered a class of integro-differential equations with applications in nonlinear elastodynamics. They proposed a general model for description of nonlinear extensible beams incorporating weak, viscous, strong and Balakrishnan-Taylor damping as follows (see [12, Eq. (1.1)]):

$$
\begin{align*}
& u_{t t}+\alpha \Delta^{2} u+\xi u+\kappa u_{t}-\lambda \Delta u_{t}+\mu \Delta^{2} u_{t} \\
& \quad-\left[\beta+\gamma \int_{\Omega}|\nabla u|^{2} d x+\delta\left|\int_{\Omega} \nabla u \cdot \nabla u_{t} d x\right|^{q-2} \int_{\Omega} \nabla u \cdot \nabla u_{t} d x\right] \Delta u=h \tag{1.2}
\end{align*}
$$

in $\Omega \times(0, T)$, where $\Omega \subset \mathbb{R}^{n}$ is a bounded domain and $T>0$. The constants have the physical meaning: $\alpha>0$ is the elasticity coefficient, $\gamma>0$ is the extensibility coefficient, $\lambda \geq 0$ is the viscous damping coefficient, $\mu \geq 0$ is the strong damping coefficient, $\delta \geq 0$ is the Balakrishnan-Taylor damping coefficient, $\beta \in \mathbb{R}$ is the axial force coefficient ( $\beta>0$ traction or $\beta<0$ compression), $\kappa \in \mathbb{R}$ is the weak damping coefficient (although without sign condition), $\xi \in \mathbb{R}$ is a source coefficient and the exponent $q$ belongs to $[2, \infty)$. We refer to [12, Sect. 1] for a nice survey on references with special cases of Eq. (1.2).

In the presence of the interesting nonlinear Balakrishnan-Taylor term $(\delta>0)$, the existence of a weak solution (in the sense of [12, Def. 2.1]) for (1.2) is only established by the authors (via time discretization) in either cases: in the presence of viscous and strong damping $(\lambda, \mu>0)$ and arbitrary $q \in[2, \infty)$ or else neglecting viscous and strong damping ( $\lambda=\mu=0$ ) with the restriction $q=2$. See Theorem 4.1 in [12] for more details. However, as remarked by the authors in [12] (see the seventh line on page 2526), they were not able to prove the existence if $\lambda=\mu=0$ and $q>2$ in (1.2). Such a case seems to be not addressed up to now in what concerns the existence and stability of solutions. In fact, the case approached by Clark [9] corresponds to analyze Eq. (1.2) when $\kappa=\lambda=0, \mu>0$ (strong damping) and $q=2$. The same case is considered by Tatar and Zaraï [24], where $\xi u$ is replaced by a nonlinear source like $|u|^{p} u$ on the right hand side of (1.2). They also analyzed blow up phenomena in finite time. The case considered by You [30] matches the system (1.2) with $\kappa=\mu=0, \lambda>0$ (viscous damping) and $q>2$. There are also some more recent works dealing with Kirchhoff's wave models ( $\alpha=\mu=0$ in (1.2)) with Balakrishnan-Taylor damping in the case $q=2$ and memory term of second order, see e.g. Tatar and Zaraï [23,25,26], Wu [28], Lee et al. [20] and Park [21]. General stability (depending on the memory kernel)
and blow up in finite time are the main issues approached in these latter. Therefore, we can not compare our long-time dynamics result for extensible beam models to their results for viscoelastic Kirchhoff's wave models. It is worth mentioning that in the presence of viscous damping with $\lambda>0$ in (1.2) ( $c>0$ in (1.1), respectively) the velocity $u_{t}$ has the following regularity $u_{t} \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)$ and the dissipative term $\lambda \int_{\Omega}\left|\nabla u_{t}\right|^{2} d x$ gives a way to control more easily the terms generated by the nonlinear Balakrishnan-Taylor term (mainly in the case $q=2$ ) as in the previous works. The same happens in the presence of strong damping when $\mu>0$ in (1.2). Moreover, even in the presence of these stronger dissipations, the Balakrishnan-Taylor damping seems to be not a locally Lipschitz operator in the pattern weak phase space in the representative case $q>0$. Therefore, the global existence and asymptotic behavior of (1.2) in the intrigued case $\lambda=\mu=0, \kappa>0$ (weak damping) and $q>2$ seems to be a much more delicate case, once we only have the poor regularity $u_{t} \in L^{2}\left(0, T ; L^{2}(\Omega)\right)$ and the dissipative term $\kappa \int_{\Omega}\left|u_{t}\right|^{2} d x$ is not enough to control nonlinear bad terms coming from the Balakrishnan-Taylor component. This requires to take into account some regularity of the own Balakrishnan-Taylor damping in some sense. To the best of our knowledge, this latter case was not approached in the literature so far.

Motivated by the above scenario, the main aim of the present article is to complement and extend the works $[9,12,24,30]$ by proving the existence of a unique mild (and strong) solution for (1.2) when $\lambda=\mu=0, \kappa \in \mathbb{R}$ and $q \geq 2$. Moreover, we also analyze the longtime dynamics of solutions (in the referred case) if $\kappa>0$ and $\beta$ is bounded from below by a negative term. We also consider a more general (standard) nonlinear source than the linear one $\xi u$. To this end, we work with an alternative expression of the following Balakrishnan-Taylor term

$$
\begin{equation*}
\Phi\left(u, u_{t}\right):=\int_{\Omega} \nabla u \cdot \nabla u_{t} d x=-\int_{\Omega} \Delta u u_{t} d x, \tag{1.3}
\end{equation*}
$$

where the last equality in (1.3) is formally obtained. Obviously, it holds true for strong solutions $\left(u, u_{t}\right) \in L^{\infty}\left(0, T ;\left(H^{4}(\Omega) \cap H_{0}^{2}(\Omega)\right) \times H_{0}^{2}(\Omega)\right)$ after a simple integration by parts. In this case, from (1.2) we have the following regularity for the Balakrishnan-Taylor term

$$
\Phi\left(u, u_{t}\right) \in L^{q}(0, T), \quad q \geq 2, T>0 .
$$

Since we shall only consider frictional damping in our model, then the above $L^{q}$-regularity in time of $\Phi\left(u, u_{t}\right)$ will play an important role in the asymptotic behavior of solutions as well as the term $\left|\Phi\left(u, u_{t}\right)\right|^{q}$ will be used to control bad terms generated by the own Balakrishnan-Taylor damping, mainly if $q>2$, which features the main difference from the above mentioned papers. This assertion will be clarified in our technical results presented in Sect. 3.

Now, in light of the aforementioned remarks, we consider our main problem. More precisely, we shall study well-posedness and long-time dynamics to the following class of extensible beams with Balakrishnan-Taylor and frictional damping
$u_{t t}+\Delta^{2} u-\left[\beta+\gamma\|\nabla u\|_{2}^{2}+\delta\left|\Phi\left(u, u_{t}\right)\right|^{q-2} \Phi\left(u, u_{t}\right)\right] \Delta u+\kappa u_{t}+f(u)=h$ in $\Omega \times \mathbb{R}^{+}$,
where $\Omega$ is a bounded domain of $\mathbb{R}^{n}$ with smooth boundary $\Gamma=\partial \Omega$. Hereafter, the notations $(\cdot, \cdot)$ and $\|\cdot\|_{p}$ shall stand for the $L^{2}$-inner product and $L^{p}$-norm, respectively. The precise assumptions on the constants and functions of the system will be given later. We study Eq. (1.4) with clamped boundary condition

$$
\begin{equation*}
u=\frac{\partial u}{\partial v}=0 \quad \text { on } \quad \Gamma \times \mathbb{R}^{+}, \tag{1.5}
\end{equation*}
$$

where $v$ is the unit exterior normal to $\Gamma$, and initial data

$$
\begin{equation*}
u(x, 0)=u_{0}(x) \quad \text { and } \quad u_{t}(x, 0)=u_{1}(x), \quad x \in \Omega . \tag{1.6}
\end{equation*}
$$

In Sect. 2 we introduce the notations and assumptions on the problem (1.4)-(1.6) as well as its Hadamard well-posedness. In Sect. 3 we study the long-time behavior of solutions through the dynamical system associated with problem (1.4)-(1.6). We first provide some important technical results and then our main theorems with respect to the existence of attractors and their properties like quasi-stable systems, finite fractal dimension, geometrical structure and regularity from the attractor. The concept of fractal exponential attractor is also addressed.

## 2 Well-Posedness

The well-posedness of problem (1.4)-(1.6) will be made by means of the general theory on $C_{0}$-semigroups. See, for instance, Pazy's book [22]. We consider initially the Hilbert phase space to the solution trajectories

$$
\mathcal{H}=H_{0}^{2}(\Omega) \times L^{2}(\Omega)
$$

with norm

$$
\|(u, v)\|_{\mathcal{H}}^{2}=\|\Delta u\|_{2}^{2}+\|v\|_{2}^{2}, \quad(u, v) \in \mathcal{H} .
$$

Denoting the vector-valued function $U=(u, v)$, with $v=u_{t}$, let us rewrite the system (1.4)-(1.6) as the following abstract Cauchy problem

$$
\left\{\begin{array}{l}
U_{t}=A U+B(U), \quad t>0  \tag{2.1}\\
U(0)=\left(u_{0}, u_{1}\right):=U_{0}
\end{array}\right.
$$

where $A: D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ is the linear differential operator

$$
A U=\left[\begin{array}{c}
v  \tag{2.2}\\
-\Delta^{2} u
\end{array}\right], \quad U \in D(A)=\left(H^{4}(\Omega) \cap H_{0}^{2}(\Omega)\right) \times H_{0}^{2}(\Omega)
$$

and $B: \mathcal{H} \rightarrow \mathcal{H}$ is the nonlinear operator

$$
B(U)=\left[\begin{array}{c}
0  \tag{2.3}\\
\Pi(U)
\end{array}\right], \quad U=(u, v) \in \mathcal{H}
$$

where

$$
\Pi(U)=\left[\beta+\gamma\|\nabla u\|_{2}^{2}+\delta|\Phi(U)|^{q-2} \Phi(U)\right] \Delta u-f(u)-\kappa v+h
$$

with $\Phi(U)=-(\Delta u, v)$. Under the above notations, the existence and uniqueness result for (1.4)-(1.6) shall be given upon the equivalent problem (2.1). Next, we give the assumptions on the problem.

Assumption 2.1 Let us assume that $h \in L^{2}(\Omega), f \in C^{1}(\mathbb{R})$ and there exist constants $C_{f}, C_{f^{\prime}}>0$ such that

$$
\begin{align*}
& \left|f^{\prime}(u)\right| \leq C_{f^{\prime}}\left(1+|u|^{\rho}\right), \quad \forall u \in \mathbb{R},  \tag{2.4}\\
& -C_{f}-\frac{\alpha}{2} u^{2} \leq \widehat{f}(u):=\int_{0}^{u} f(\tau) d \tau \leq f(u) u+\frac{\alpha}{2} u^{2}, \quad \forall u \in \mathbb{R}, \tag{2.5}
\end{align*}
$$

where $\alpha \in\left[0, \lambda_{1}\right)$ with $\lambda_{1}>0$ denoting the first eigenvalue of the bi-harmonic operator $\Delta^{2}$ with boundary condition (1.5), and growth exponent $\rho$ satisfies

$$
\begin{equation*}
\rho>0 \text { if } 1 \leq n \leq 4 \quad \text { or } \quad 0<\rho \leq \frac{4}{n-4} \text { if } n \geq 5 . \tag{2.6}
\end{equation*}
$$

Remark 2.1 Under the definition of $\lambda_{1}$ and since $\alpha \in\left[0, \lambda_{1}\right)$, one has

$$
\|u\|_{2}^{2} \leq \frac{1}{\lambda_{1}}\|\Delta u\|_{2}^{2}, \quad\|\nabla u\|_{2}^{2} \leq \frac{1}{\lambda_{1}^{1 / 2}}\|\Delta u\|_{2}^{2}, \quad \forall u \in H_{0}^{2}(\Omega),
$$

and $\omega:=1-\frac{\alpha}{\lambda_{1}}>0$. In addition, from (2.6) it holds the standard continuous Sobolev embedding $H_{0}^{2}(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$, which is compact when we take

$$
\begin{equation*}
\rho>0 \text { if } 1 \leq n \leq 4 \quad \text { or } \quad 0<\rho<\frac{4}{n-4} \text { if } n \geq 5 . \tag{2.7}
\end{equation*}
$$

Also, some standard examples on the nonlinear source $f(u)$ can be found in [18].
Theorem 2.1 (Hadamard Well-Posedness). Let us consider $q \geq 2, \gamma, \delta>0, \beta, \kappa \in \mathbb{R}$, and take Assumption 2.1 into account. Thus:
(i) If $U_{0} \in \mathcal{H}$, then there exists $T_{\max }>0$ such that problem (2.1) has a unique mild solution $U \in C\left(\left[0, T_{\max }\right), \mathcal{H}\right)$ given by

$$
\begin{equation*}
U(t)=e^{A t} U_{0}+\int_{0}^{t} e^{A(t-s)} B(U(s)) d s, \quad t \in\left[0, T_{\max }\right) . \tag{2.8}
\end{equation*}
$$

(ii) If $U_{0} \in D(A)$, then the mild solution $U$ is the strong solution of (2.1) on $\left[0, T_{\max }\right)$.
(iii) In both cases, we have that $T_{\max }=+\infty$.
(iv) If $U^{1}=\left(u^{1}, v^{1}\right)$ and $U^{2}=\left(u^{2}, v^{2}\right)$ are two mild (or strong) solutions corresponding to initial data $U_{0}^{1}=\left(u_{0}^{1}, u_{1}^{1}\right)$ and $U_{0}^{2}=\left(u_{0}^{2}, u_{1}^{2}\right)$, respectively, then

$$
\begin{equation*}
\left\|U^{1}(t)-U^{2}(t)\right\|_{\mathcal{H}} \leq e^{C t}\left\|U_{0}^{1}-U_{0}^{2}\right\|_{\mathcal{H}}, \quad \forall t \in[0, \infty), \tag{2.9}
\end{equation*}
$$

for some positive constant $C\left(\left\|U_{0}^{i}\right\|_{\mathcal{H}}\right), i=1,2$.
Proof (i)-(ii) It is very well-known (and easy) to prove that $A: D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ defined in (2.2) is the infinitesimal generator of a $C_{0}$-semigroup of contractions $e^{A t}$ on $\mathcal{H}$, see e.g. [4, Proposition 1]. In addition, we claim that $B: \mathcal{H} \rightarrow \mathcal{H}$ defined in (2.3) is a locally Lipschitz continuous operator. Indeed, let us first take $R>0$ and $U=(u, v), V=(\widetilde{u}, \widetilde{v})$ such that $\|U\|_{\mathcal{H}},\|V\|_{\mathcal{H}} \leq R$. Then, from (2.3), we infer

$$
\begin{equation*}
\|B(U)-B(V)\|_{\mathcal{H}}=\|\Pi(U)-\Pi(V)\|_{2}=\sup _{\|w\|_{2} \leq 1}|(\Pi(U)-\Pi(V), w)| \tag{2.10}
\end{equation*}
$$

In what follows, we shall give proper estimates on the right hand side of (2.10). Given $w \in L^{2}(\Omega)$, adding and subtracting the terms $\gamma\|\nabla u\|_{2}^{2} \Delta \tilde{u}$ and $|\Phi(U)|^{q-2} \Phi(U) \Delta \tilde{u}$ in the expression $\Pi(U)-\Pi(V)$, we denote

$$
\begin{equation*}
|(\Pi(U)-\Pi(V), w)|=\left|\sum_{i=1}^{6} \mathcal{I}_{i}\right| \tag{2.11}
\end{equation*}
$$

where

$$
\begin{aligned}
& \mathcal{I}_{1}=\left(\beta(\Delta u-\Delta \widetilde{u})+\gamma\|\nabla u\|_{2}^{2}(\Delta u-\Delta \widetilde{u}), w\right), \\
& \mathcal{I}_{2}=\left(\gamma\left[\|\nabla u\|_{2}^{2}-\|\nabla \widetilde{u}\|_{2}^{2}\right] \Delta \widetilde{u}, w\right), \\
& \mathcal{I}_{3}=\left(\delta|\Phi(U)|^{q-2} \Phi(U)(\Delta u-\Delta \widetilde{u}), w\right), \\
& \mathcal{I}_{4}=\left(\delta\left[|\Phi(U)|^{q-2} \Phi(U)-|\Phi(V)|^{q-2} \Phi(V)\right] \Delta \widetilde{u}, w\right), \\
& \mathcal{I}_{5}=((f(\widetilde{u})-f(u)), w), \\
& \mathcal{I}_{6}=(\kappa(\widetilde{v}-v), w) .
\end{aligned}
$$

Thus, it remains to estimate the terms $\mathcal{I}_{1}, \ldots, \mathcal{I}_{6}$. Firstly, it is easy to see that

$$
\left|\mathcal{I}_{1}\right| \leq\left(|\beta|+\gamma\|\nabla u\|_{2}^{2}\right)\|\Delta u-\Delta \widetilde{u}\|_{2}\|w\|_{2} \leq\left[|\beta|+\frac{\gamma R^{2}}{\lambda_{1}^{1 / 2}}\right]\|U-V\|_{\mathcal{H}}\|w\|_{2},
$$

and

$$
\begin{aligned}
\left|\mathcal{I}_{2}\right| & \leq \gamma\left|\|\nabla u\|_{2}^{2}-\|\nabla \widetilde{u}\|_{2}^{2}\right|\|\Delta \widetilde{u}\|_{2}\|w\|_{2} \\
& \leq \gamma\left[\|\nabla u\|_{2}+\|\nabla \widetilde{u}\|_{2}\right]\|\nabla u-\nabla \widetilde{u}\|_{2}\|\Delta \widetilde{u}\|_{2}\|w\|_{2} \\
& \leq \frac{\gamma}{\lambda_{1}^{1 / 2}}\left[\|\Delta u\|_{2}+\|\Delta \widetilde{u}\|_{2}\right]\|\Delta u-\Delta \widetilde{u}\|_{2}\|\Delta \widetilde{u}\|_{2}\|w\|_{2} \\
& \leq \frac{2 \gamma R}{\lambda_{1}^{1 / 2}}\|U-V\|_{\mathcal{H}}\|w\|_{2} .
\end{aligned}
$$

Since $(u, v) \in H_{0}^{2}(\Omega) \times L^{2}(\Omega)$, then $\Phi(U)=-(\Delta u, v)$ makes sense and so

$$
\left|\mathcal{I}_{3}\right| \leq \delta\left[\|\Delta u\|_{2}\|v\|_{2}\right]^{q-1}\|\Delta u-\Delta \widetilde{u}\|_{2}\|w\|_{2} \leq \delta R^{2(q-1)}\|U-V\|_{\mathcal{H}}\|w\|_{2} .
$$

Now, let $F \in C^{1}(\mathbb{R})$ be given by $F(s)=|s|^{q-2} s$. From the Mean Value Theorem, one can easily prove that

$$
\left|F\left(\vartheta_{1}\right)-F\left(\vartheta_{2}\right)\right| \leq 2^{2(q-2)}(q-1)\left[\left|\vartheta_{1}\right|^{q-2}+\left|\vartheta_{2}\right|^{q-2}\right]\left|\vartheta_{1}-\vartheta_{2}\right|, \quad \vartheta_{1}, \vartheta_{2} \in \mathbb{R} .
$$

Taking $\vartheta_{1}=\Phi(U)$ and $\vartheta_{2}=\Phi(V)$, we have

$$
\left|\mathcal{I}_{4}\right| \leq 2^{2(q-2)}(q-1) \delta\left[|\Phi(U)|^{q-2}+|\Phi(V)|^{q-2}\right]|\Phi(U)-\Phi(V)|\|\Delta \widetilde{u}\|_{2}\|w\|_{2} .
$$

As above, recalling that $\Phi(U)=-(\Delta u, v)$, it follows that

$$
|\Phi(U)|^{q-2}+|\Phi(V)|^{q-2} \leq\left[\|\Delta u\|_{2}\|v\|_{2}\right]^{q-2}+\left[\|\Delta \widetilde{u}\|_{2}\|\widetilde{v}\|_{2}\right]^{q-2} \leq 2 R^{2(q-2)} .
$$

Similarly, $\Phi(U)-\Phi(V)=-(\Delta u-\Delta \widetilde{u}, v)-(\Delta \widetilde{u}, v-\widetilde{v})$, which implies that

$$
|\Phi(U)-\Phi(V)| \leq\|\Delta u-\Delta \widetilde{u}\|_{2}\|v\|_{2}+\|\Delta \widetilde{u}\|_{2}\|v-\widetilde{v}\|_{2} \leq 2 R\|U-V\|_{\mathcal{H}} .
$$

Thus, the fourth term can be estimated as follows

$$
\left|\mathcal{I}_{4}\right| \leq[\sqrt{2} R]^{2(q-1)}(q-1) \delta\|U-V\|_{\mathcal{H}}\|w\|_{2} .
$$

Using Assumption (2.1)-(2.4), again the Mean Value Theorem, Hölder's inequality with $\frac{\rho}{2(\rho+1)}+\frac{1}{2(\rho+1)}+\frac{1}{2}=1$ and embedding $H_{0}^{2}(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$, we get

$$
\begin{aligned}
\left|\mathcal{I}_{5}\right| & =\left|\int_{\Omega}(f(u)-f(\widetilde{u})) w d x\right| \\
& \leq C_{f^{\prime}} \int_{\Omega}\left[1+2^{\rho}\left[|u|^{\rho}+|\widetilde{u}|^{\rho}\right]\right]|u-\widetilde{u}||w| d x \\
& \leq C_{f^{\prime}}\left[|\Omega|^{\frac{\rho}{2(\rho+1)}}+2^{\rho}\left[\|u\|_{2(\rho+1)}^{\rho}+\|\widetilde{u}\|_{2(\rho+1)}^{\rho}\right]\right]\|u-\widetilde{u}\|_{2(\rho+1)}\|w\|_{2} \\
& \leq C_{f^{\prime}}\left[|\Omega|^{\frac{\rho}{2(\rho+1)}}+2^{\rho+1} C_{\rho}^{\rho} R^{\rho}\right] C_{\rho}\|U-V\|_{\mathcal{H}}\|w\|_{2},
\end{aligned}
$$

where $C_{\rho}>0$ is the constant coming from the embedding inequality $\|\cdot\|_{2(\rho+1)} \leq C_{\rho}\|\Delta \cdot\|_{2}$. Last, we have

$$
\left|\mathcal{I}_{6}\right| \leq|\kappa|\|v-\widetilde{v}\|_{2}\|w\|_{2} \leq|\kappa|\|U-V\|_{\mathcal{H}}\|w\|_{2} .
$$

Going back to (2.11) we obtain

$$
\begin{equation*}
|(\Pi(U)-\Pi(V), w)| \leq L_{R}\|U-V\|_{\mathcal{H}}\|w\|_{2}, \tag{2.12}
\end{equation*}
$$

where $L_{R}>0$ is given by

$$
\begin{aligned}
L_{R}= & |\beta|+\frac{2 \gamma R}{\lambda_{1}^{1 / 2}}+\frac{\gamma R^{2}}{\lambda_{1}^{1 / 2}}+\delta\left[\frac{R^{2}}{\lambda_{1}^{1 / 2}}\right]^{q-1}+[\sqrt{2} R]^{2(q-1)}(q-1) \delta \\
& +C_{f^{\prime}}\left(|\Omega|^{\frac{\rho}{2(\rho+1)}}+2^{\rho+1} C_{\rho}^{\rho} R^{\rho}\right) C_{\rho}+|\kappa|,
\end{aligned}
$$

and replacing (2.12) in (2.10), we arrive at the desired locally Lipschitz condition

$$
\begin{equation*}
\|B(U)-B(V)\|_{\mathcal{H}} \leq L_{R}\|U-V\|_{\mathcal{H}} . \tag{2.13}
\end{equation*}
$$

Hence, the existence and uniqueness of mild (and strong) solution for (2.1) on [0, $T_{\max }$ ) follows from Theorems 1.4 and 1.6 in Pazy's book [22, Chapter 6].
(iii) Remains to prove that $T_{\max }=+\infty$, that is, the mild (and strong) solution of (2.1) is globally defined. In fact, if $T_{\max }<\infty$, then it well-known that

$$
\begin{equation*}
\lim _{t \rightarrow T_{\max }^{-\operatorname{lax}}}\|U(t)\|_{\mathcal{H}}=+\infty \tag{2.14}
\end{equation*}
$$

On the other hand, the energy functional corresponding to system (1.4)-(1.6) is given by

$$
\begin{equation*}
E(t)=\frac{1}{2}\|U(t)\|_{\mathcal{H}}^{2}+\frac{\beta}{2}\|\nabla u(t)\|_{2}^{2}+\frac{\gamma}{4}\|\nabla u(t)\|_{2}^{4}+\int_{\Omega}[\widehat{f}(u(t))-h u(t)] d x, \tag{2.15}
\end{equation*}
$$

where $U(t)=(u(t), v(t))$, with $v=u_{t}$, is the mild (or strong ) solution of problem (2.1). Let us also define the perturbed energy

$$
\begin{equation*}
\widetilde{E}(t):=E(t)+\frac{\beta^{2}}{4 \gamma}+\frac{1}{\omega \lambda_{1}}\|h\|_{2}^{2}+C_{f}|\Omega| . \tag{2.16}
\end{equation*}
$$

From Young's inequality and Assumption (2.1)-(2.5), we get

$$
\begin{align*}
& \frac{\beta}{2}\|\nabla u(t)\|_{2}^{2} \geq-\frac{\beta^{2}}{4 \gamma}-\frac{\gamma}{4}\|\nabla u(t)\|_{2}^{4},  \tag{2.17}\\
& \int_{\Omega} \widehat{f}(u(t)) d x \geq-\frac{\alpha}{2}\|u(t)\|^{2}-C_{f}|\Omega| \geq-\frac{\alpha}{2 \lambda_{1}}\|\Delta u(t)\|_{2}^{2}-C_{f}|\Omega|, \tag{2.18}
\end{align*}
$$

and

$$
\begin{equation*}
-\int_{\Omega} h u d x \geq-\|h\|_{2}\|u(t)\|_{2} \geq-\frac{1}{\omega \lambda_{1}}\|h\|_{2}^{2}-\frac{\omega}{4}\|\Delta u(t)\|_{2}^{2} . \tag{2.19}
\end{equation*}
$$

Thus, combining (2.15)-(2.19), we obtain

$$
\begin{equation*}
\widetilde{E}(t) \geq \frac{1}{2}\left\|u_{t}(t)\right\|_{2}^{2}+\frac{\omega}{4}\|\Delta u(t)\|_{2}^{2} \geq \frac{\omega}{4}\|U(t)\|_{\mathcal{H}}^{2} . \tag{2.20}
\end{equation*}
$$

Now, taking the multiplier $v=u_{t}$ in (1.4), noting that $\frac{d}{d t} E(t)=\frac{d}{d t} \widetilde{E}(t)$ and from (2.20), we infer

$$
\begin{equation*}
\frac{d}{d t} \widetilde{E}(t)+\delta|\Phi(U(t))|^{q}=-\kappa\left\|u_{t}(t)\right\|_{2}^{2} \leq 2|\kappa| \widetilde{E}(t) \tag{2.21}
\end{equation*}
$$

from where we obtain

$$
\widetilde{E}(t) \leq \widetilde{E}(0) e^{2|\kappa| t}, \quad \forall t \in\left[0, T_{\max }\right)
$$

Again from (2.20) we conclude

$$
\begin{equation*}
\|U(t)\|_{\mathcal{H}} \leq \frac{4}{\omega}\left[E(0)+\frac{\beta^{2}}{4 \gamma}+\frac{1}{\omega \lambda_{1}}\|h\|_{2}^{2}+C_{f}|\Omega|\right] e^{2|\kappa| t}, \quad \forall t \in\left[0, T_{\max }\right), \tag{2.22}
\end{equation*}
$$

which is a contradiction with (2.14) for $T_{\max }<+\infty$. Therefore, $T_{\max }=+\infty$.
(iv) The local continuous dependence stated in (2.9) can be proved as a direct consequence of the formula (2.8), the locally Lipschitz property (2.13) and Gronwall's inequality.

Hence, the proof of Theorem 2.1 is complete.
Remark 2.2 In order to work with a simplified presentation, we have restricted to the case of extensible beams with clamped boundary condition (1.5). Nonetheless, our result on wellposedness (and long-time behavior) also holds true in the case of simply supported boundary condition $u=\Delta u=0$ on $\Gamma \times \mathbb{R}^{+}$without additional difficulty. See e.g. [4,18].

Remark 2.3 We also observe that estimate (2.22) predicts that the solution $U=\left(u, u_{t}\right)$ exists globally with respect to $\mathcal{H}$-topology, but it may growth exponentially as time goes to infinity. However, in the case $\kappa>0$, we go back to (2.21) and extract the following global boundedness

$$
\widetilde{E}(t) \leq \widetilde{E}(0), \quad \forall t \geq 0
$$

which allows us to conclude that the solution is globally bounded on the weak phase $\mathcal{H}$. Hence, is this case, we can analyze the asymptotic behavior of mild (and strong) solutions with respect to $\mathcal{H}$-topology. In the next section we shall be restricted to the case $\kappa>0$.

## 3 Long-Time Behavior

### 3.1 Technical Results

We start this section with some technical results on the trajectory solution $U(t)=$ $\left(u(t), u_{t}(t)\right) \in \mathcal{H}$ of problem (1.4)-(1.6) for $t \geq 0$.

Proposition 3.1 Let us take the assumptions of Theorem 2.1 into account with $\kappa>0$. Then there exists a small constant $\epsilon>0$ (which may depend on initial data) such that

$$
\begin{equation*}
\widetilde{E}(t)+\frac{2 \delta}{q} \int_{0}^{t} e^{-\epsilon(t-s)}|\Phi(U(t))|^{q} d s \leq 3 \widetilde{E}(0) e^{-\epsilon t}+\frac{\omega R^{*}}{16}, \quad t \geq 0, \tag{3.1}
\end{equation*}
$$

where $\widetilde{E}(t)$ is set in (2.16) and

$$
\begin{equation*}
R^{*}:=\frac{48}{\omega}\left[\frac{\beta^{2}}{2 \gamma}+\frac{1}{\omega \lambda_{1}}\|h\|_{2}^{2}+C_{f}|\Omega|\right] . \tag{3.2}
\end{equation*}
$$

In addition, it holds the identity

$$
\begin{equation*}
\widetilde{E}(t)+\kappa \int_{0}^{t}\left\|u_{t}(s)\right\|_{2}^{2} d s+\delta \int_{0}^{t}|\Phi(U(s))|^{q} d s=\widetilde{E}(0), \quad t \geq 0 . \tag{3.3}
\end{equation*}
$$

Proof Let us start by defining the perturbed energy

$$
\begin{equation*}
\widetilde{E}_{\varepsilon}(t)=\widetilde{E}(t)+\varepsilon \int_{\Omega} u_{t}(t) u(t) d x \tag{3.4}
\end{equation*}
$$

with $\varepsilon>0$ to be fixed later. Then, from Young's inequality and (2.20),

$$
\left|\widetilde{E}_{\varepsilon}(t)-\widetilde{E}(t)\right| \leq \frac{2 \varepsilon}{\lambda_{1}^{1 / 2} \omega} \widetilde{E}(t)
$$

which implies, after choosing $\varepsilon \leq \frac{\lambda_{1}^{1 / 2} \omega}{4}$, the next equivalence

$$
\begin{equation*}
\frac{1}{2} \widetilde{E}(t) \leq \widetilde{E}_{\varepsilon}(t) \leq \frac{3}{2} \widetilde{E}(t), \quad t \geq 0 . \tag{3.5}
\end{equation*}
$$

On the other hand, from (2.21) one sees

$$
\begin{equation*}
\frac{d}{d t} \widetilde{E}(t)=-\kappa\left\|u_{t}(t)\right\|_{2}^{2}-\delta|\Phi(U(t))|^{q} \tag{3.6}
\end{equation*}
$$

Besides, deriving $\widetilde{E}_{\varepsilon}(t)$ in (3.4), using Eq. (1.4) and substituting (3.6) in the resulting expression, it follows that

$$
\begin{align*}
& \frac{d}{d t} \widetilde{E}_{\varepsilon}(t)+(\kappa-\varepsilon)\left\|u_{t}(t)\right\|_{2}^{2}+\varepsilon\|\Delta u(t)\|_{2}^{2}+\varepsilon \beta\|\nabla u(t)\|_{2}^{2}+\varepsilon \gamma\|\nabla u(t)\|_{2}^{4} \\
& \quad+\varepsilon \int_{\Omega} f(u(t)) u(t)-\varepsilon \int_{\Omega} h u(t) d x+\delta|\Phi(U(t))|^{q} \\
& \quad \leq-\varepsilon \kappa \int_{\Omega} u_{t}(t) u(t) d x+\varepsilon \delta|\Phi(U(t))|^{q-2} \Phi(U(t))\|\nabla u(t)\|_{2}^{2} . \tag{3.7}
\end{align*}
$$

From condition (2.5), we get

$$
\begin{aligned}
\varepsilon \int_{\Omega} f(u(t)) u(t) d x & \geq \varepsilon \int_{\Omega} \widehat{f}(u(t)) d x-\frac{\varepsilon \alpha}{2}\|u(t)\|_{2}^{2} \\
& \geq \varepsilon \int_{\Omega} \widehat{f}(u(t)) d x-\frac{\varepsilon \alpha}{2 \lambda_{1}}\|\Delta u(t)\|_{2}^{2},
\end{aligned}
$$

and using Hölder and Young's inequalities we have

$$
\left|\varepsilon \kappa \int_{\Omega} u_{t}(t) u(t) d x\right| \leq \frac{\varepsilon \kappa}{\lambda_{1}^{1 / 2}}\left\|u_{t}(t)\right\|_{2}\|\Delta u(t)\|_{2} \leq \frac{\kappa}{2}\left\|u_{t}(t)\right\|_{2}^{2}+\frac{\varepsilon^{2} \kappa}{2 \lambda_{1}}\|\Delta u(t)\|_{2}^{2} .
$$

Now, taking $\varepsilon>0$ so that $\varepsilon \leq \min \left\{\frac{\lambda_{1} \omega}{2 \kappa}, \frac{\kappa}{3}\right\}$ and returning to (3.7), we infer

$$
\begin{aligned}
& \frac{d}{d t} \widetilde{E}_{\varepsilon}(t)+\varepsilon E(t)+\frac{\varepsilon \beta}{2}\|\nabla u(t)\|_{2}^{2}+\frac{3 \varepsilon \gamma}{4}\|\nabla u(t)\|_{2}^{4}+\delta|\Phi(U(t))|^{q} \\
& \quad \leq \varepsilon \delta|\Phi(U(t))|^{q-2} \Phi(U(t))\|\nabla u(t)\|_{2}^{2} .
\end{aligned}
$$

Moreover, since

$$
\frac{\varepsilon \beta}{2}\|\nabla u(t)\|_{2}^{2}+\frac{3 \varepsilon \gamma}{4}\|\nabla u(t)\|_{2}^{4} \geq-\frac{\varepsilon \beta^{2}}{4 \gamma}+\frac{\varepsilon \gamma}{4}\|\nabla u(t)\|_{2}^{4}
$$

we obtain

$$
\begin{align*}
& \frac{d}{d t} \widetilde{E}_{\varepsilon}(t)+\varepsilon E(t)-\frac{\beta^{2}}{4 \gamma}+\frac{\varepsilon \gamma}{4}\|\nabla u(t)\|_{2}^{4}+\delta|\Phi(U(t))|^{q} \\
& \quad \leq \varepsilon \delta|\Phi(U(t))|^{q-2} \Phi(U(t))\|\nabla u(t)\|_{2}^{2} . \tag{3.8}
\end{align*}
$$

Adding the term $\varepsilon\left[\frac{\beta^{2}}{4 \gamma}+\frac{1}{\omega \lambda_{1}}\|h\|_{2}^{2}+C_{f}|\Omega|\right]$ in both sides of (3.8) we deduce

$$
\begin{align*}
& \frac{d}{d t} \widetilde{E}_{\varepsilon}(t)+\varepsilon \widetilde{E}(t)+\frac{\varepsilon \gamma}{4}\|\nabla u(t)\|_{2}^{4}+\delta|\Phi(U(t))|^{q} \\
& \leq \underbrace{\varepsilon \delta|\Phi(U(t))|^{q-2} \Phi(U(t))\|\nabla u(t)\|_{2}^{2}}_{:=\mathcal{J}}+\varepsilon \frac{\omega R^{*}}{48}, \tag{3.9}
\end{align*}
$$

where $R^{*}$ is given in (3.2). From Young's inequality with $\frac{q-1}{q}+\frac{1}{q}=1$, yields

$$
\begin{aligned}
\mathcal{J} & \leq \delta^{\frac{q-1}{q}}|\Phi(U(t))|^{q-1} \varepsilon \delta^{\frac{1}{q}}\|\nabla u(t)\|_{2}^{2} \\
& \leq \frac{q-1}{q} \delta|\Phi(U(t))|^{q}+\frac{\varepsilon^{q} \delta}{q}\|\nabla u(t)\|_{2}^{2 q} .
\end{aligned}
$$

Using (2.20) and (3.6), since $q \geq 2$ and considering $C_{0}=C_{0}\left(\|U(0)\|_{\mathcal{H}}\right)>0$ such that $\widetilde{E}(0) \leq C_{0}$, we get

$$
\begin{aligned}
\frac{\varepsilon^{q} \delta}{q}\|\nabla u(t)\|_{2}^{2 q} & =\frac{\varepsilon^{q} \delta}{q}\|\nabla u(t)\|_{2}^{2(q-2)}\|\nabla u(t)\|_{2}^{4} \\
& \leq \frac{\varepsilon^{q} \delta}{q}\left[\frac{4}{\omega \lambda_{1}^{\frac{1}{2}}} \widetilde{E}(0)\right]^{(q-2)}\|\nabla u(t)\|_{2}^{4} \\
& \leq \frac{\varepsilon^{q} \delta}{q}\left[\frac{4}{\omega \lambda_{1}^{\frac{1}{2}}} C_{0}\right]^{(q-2)}\|\nabla u(t)\|_{2}^{4},
\end{aligned}
$$

and so

$$
\begin{equation*}
\mathcal{J} \leq \frac{q-1}{q} \delta|\Phi(U(t))|^{q}+\frac{\varepsilon^{q} \delta}{q}\left[\frac{4 C_{0}}{\omega \lambda_{1}^{1 / 2}}\right]^{q-2}\|\nabla u(t)\|_{2}^{4} . \tag{3.10}
\end{equation*}
$$

Replacing (3.10) in (3.9) and choosing $\varepsilon>0$ such that $\varepsilon \leq\left(\frac{q \gamma}{4 \delta}\right)^{\frac{1}{q-1}}\left[\frac{\omega \lambda_{1}^{1 / 2}}{4 C_{0}}\right]^{\frac{q-2}{q-1}}$, we obtain

$$
\begin{equation*}
\frac{d}{d t} \widetilde{E}_{\varepsilon}(t)+\varepsilon \widetilde{E}(t)+\frac{\delta}{q}|\Phi(U(t))|^{q} \leq \varepsilon \frac{\omega R^{*}}{48} . \tag{3.11}
\end{equation*}
$$

Therefore, from the above choices on $\varepsilon>0$, the estimates (3.5) and (3.11) hold true. Combining them, we arrive at

$$
\begin{equation*}
\frac{d}{d t} \widetilde{E}_{\varepsilon}(t)+\frac{2 \varepsilon}{3} \widetilde{E}_{\varepsilon}(t)+\frac{\delta}{q}|\Phi(U(t))|^{q} \leq \varepsilon \frac{\omega R^{*}}{48} . \tag{3.12}
\end{equation*}
$$

Multiplying (3.12) by $e^{\frac{2 \varepsilon}{3} t}$ and integrating from 0 to $t$, then a straightforward computation leads to

$$
\widetilde{E}_{\varepsilon}(t)+\frac{\delta}{q} \int_{0}^{t} e^{-\frac{2 \varepsilon}{3}(t-s)}|\Phi(U(t))|^{q} d s \leq \widetilde{E}_{\varepsilon}(0) e^{-\frac{2 \varepsilon}{3} t}+\frac{\omega R^{*}}{32} .
$$

Hence, using (3.5) and denoting $\epsilon=\frac{2 \varepsilon}{3}$, we achieve inequality (3.1). In addition, identity (3.3) follows readily by integrating (3.6). This completes the proof of Proposition 3.1

It follows immediately from Proposition 3.1 the next two consequences.
Corollary 3.2 (Exponential Stabiliby). Under the assumptions of Proposition 3.1 with $\beta=$ $C_{f}=0$ and $h \equiv 0$, the system (1.4)-(1.6) is exponentially stable. More precisely, there exists a constant $\epsilon>0$ (which may depend on initial data) such that the energy $E(t)$ defined in (2.15) satisfies

$$
E(t) \leq 3 E(0) e^{-\epsilon t}, \quad t \geq 0 .
$$

Proof It is enough to combine (2.16) and (3.1)-(3.2), where $\widetilde{E}=E$ and $R^{*}=0$.
Corollary 3.3 (Global Boundedness). Let assumptions of Proposition 3.1 be in force. Then, for every $R>0$ with $\left\|\left(u_{0}, u_{1}\right)\right\|_{\mathcal{H}} \leq R$, there exists a time $t_{R}>0$ such that

$$
\begin{equation*}
\|U(t)\|_{\mathcal{H}} \leq R^{*}, \quad \forall t \geq t_{R}, \tag{3.13}
\end{equation*}
$$

where $U(t)=\left(u(t), u_{t}(t)\right)$ is the unique (mild) solution of (1.4)-(1.6) with $U(0)=\left(u_{0}, u_{1}\right)$ and $R^{*}>0$ is given in (3.2). In particular, every trajectory solution with bounded initial data is itself globally bounded. More precisely, there exists a constant $C_{R}>0$ such that

$$
\begin{equation*}
\|U(t)\|_{\mathcal{H}}^{2}+\int_{0}^{t}|\Phi(U(s))|^{q} d s \leq C_{R}, \quad \forall t \geq 0 \tag{3.14}
\end{equation*}
$$

Proof From (2.20), (3.1) and denoting by $C_{R}>0$ such that $\widetilde{E}(0) \leq C_{R}$, then

$$
\|U(t)\|_{\mathcal{H}}^{2} \leq \frac{12 C_{R}}{\omega} e^{-\epsilon t}+\frac{R^{*}}{4}, \quad t \geq 0
$$

Thus, taking $t_{R}>0$ large enough such that $t_{R} \geq \epsilon^{-1} \ln \left[\frac{16 C_{R}}{\omega R^{*}}\right]$, we conclude (3.13). The global estimate (3.14) follows promptly from (2.20) and (3.3).

Proposition 3.4 (Stabilizability Inequality). Let us take the assumptions of Theorem 2.1 into account with $\kappa>0$ and $\beta \geq-\varpi$, where $\varpi \in\left[0, \lambda_{1}\right)$. Given a bounded set $B \subset \mathcal{H}$, let $U^{i}=\left(u^{i}, u_{t}^{i}\right), i=1,2$, be two (mild) solutions of problem (1.4)-(1.6) such that $U^{i}(0)=$ $\left(u_{0}^{i}, u_{1}^{i}\right) \in B$. Then, there exist a uniform constant $\sigma>0$ and constants $\widetilde{\varrho}_{B}, \varrho_{B}>0$ depending on $B$ such that

$$
\begin{align*}
\left\|U^{1}(t)-U^{2}(t)\right\|_{\mathcal{H}}^{2} \leq & \widetilde{\varrho}_{B}\left\|U^{1}(0)-U^{2}(0)\right\|_{\mathcal{H}}^{2} e^{-\sigma t} \\
& +\varrho_{B} \int_{0}^{t} e^{-\sigma(t-s)}\left[\|\nabla w(s)\|_{2}^{2}+\|w(s)\|_{2(\rho+1)}^{2}\right] d s, \tag{3.15}
\end{align*}
$$

for all $t>0$, where $w=u^{1}-u^{2}$.

Proof The proof is done for strong solutions and using standard density arguments it holds true for mild solutions. We start by noting that function $\left(w, w_{t}\right)=U^{1}-U^{2}$ is the mild (or strong) solution of problem

$$
\begin{align*}
& w_{t t}+\Delta^{2} w+\kappa w_{t}-\beta \Delta w-\gamma\left\|\nabla u^{1}\right\|_{2}^{2} \Delta w-\gamma\left[\left\|\nabla u^{1}\right\|_{2}^{2}-\left\|\nabla u^{2}\right\|_{2}^{2}\right] \Delta u^{2} \\
& \quad-\delta\left|\Phi\left(U^{1}(t)\right)\right|^{q-2} \Phi\left(U^{1}(t)\right) \Delta w-\delta \Delta_{\Phi} \Delta u^{2}+f\left(u^{1}\right)-f\left(u^{2}\right)=0 \tag{3.16}
\end{align*}
$$

with initial condition

$$
\left(w(0), w_{t}(0)\right)=U^{1}(0)-U^{2}(0)
$$

where we denote

$$
\Delta_{\Phi}:=\left|\Phi\left(U^{1}(t)\right)\right|^{q-2} \Phi\left(U^{1}(t)\right)-\left|\Phi\left(U^{2}(t)\right)\right|^{q-2} \Phi\left(U^{2}(t)\right) .
$$

Also, the corresponding energy functional to (3.16) is given by

$$
\mathcal{E}(t):=\left\|w_{t}(t)\right\|_{2}^{2}+\|\Delta w(t)\|_{2}^{2}+\beta\|\nabla w(t)\|_{2}^{2}+\gamma\left\|\nabla u^{1}(t)\right\|_{2}^{2}\|\nabla w(t)\|_{2}^{2} .
$$

Inequality (3.14) in Corollary 3.3 implies that there exists a constant $K_{B}>0$ depending on $B$ such that

$$
\begin{equation*}
\left\|U^{i}(t)\right\|_{\mathcal{H}}^{2}+\int_{0}^{t}\left|\Phi\left(U^{i}(s)\right)\right|^{q} d s \leq K_{B}, \quad \forall t \geq 0, \quad i=1,2 . \tag{3.17}
\end{equation*}
$$

Then, defining $\omega_{1}=1-\frac{\sigma}{\lambda_{1}}>0$ and denoting $\widetilde{K}_{B}=1+\frac{|\beta|}{\lambda_{1}^{1 / 2}}+\frac{\gamma}{\lambda_{1}} K_{B}$, we have

$$
\begin{equation*}
\omega_{1}\left\|U^{1}(t)-U^{2}(t)\right\|_{\mathcal{H}}^{2} \leq \mathcal{E}(t) \leq \widetilde{K}_{B}\left\|U^{1}(t)-U^{2}(t)\right\|_{\mathcal{H}}^{2} . \tag{3.18}
\end{equation*}
$$

On the other hand, taking the multiplier $w_{t}$ in (3.16), we get

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t} \mathcal{E}(t) \leq-\kappa\left\|w_{t}(t)\right\|_{2}^{2}+\sum_{i=1}^{5} \mathcal{J}_{i} \tag{3.19}
\end{equation*}
$$

where

$$
\begin{aligned}
\mathcal{J}_{1} & =\gamma \Phi\left(U^{1}(t)\right)\|\nabla w(t)\|_{2}^{2}, \\
\mathcal{J}_{2} & =\gamma\left[\left\|\nabla u^{1}(t)\right\|_{2}^{2}-\left\|\nabla u^{2}(t)\right\|_{2}^{2}\right] \int_{\Omega} \Delta u^{2}(t) w_{t}(t) d x, \\
\mathcal{J}_{3} & =\delta\left|\Phi\left(U^{1}(t)\right)\right|^{q-2} \Phi\left(U^{1}(t)\right) \int_{\Omega} \Delta w(t) w_{t}(t) d x, \\
\mathcal{J}_{4} & =\delta \Delta_{\Phi} \int_{\Omega} \Delta u^{2}(t) w_{t}(t) d x, \\
\mathcal{J}_{5} & =\int_{\Omega}\left[f\left(u^{2}(t)\right)-f\left(u^{1}(t)\right)\right] w_{t}(t) d x .
\end{aligned}
$$

Let us estimate the terms $\mathcal{J}_{1}, \ldots, \mathcal{J}_{5}$ of (3.19). For simplicity, the same constant $K_{B}>0$ will be used to denote several different constants depending on $B$ in the next estimates. Initially, since $\Phi(U(t))=-\left(\Delta u^{1}(t), u_{t}^{1}(t)\right)$ and from (3.17), we observe that

$$
\left|\mathcal{J}_{1}\right| \leq \gamma\left\|\Delta u^{1}(t)\right\|_{2}\left\|u_{t}^{1}(t)\right\|_{2}\|\nabla w(t)\|_{2}^{2} \leq K_{B}\|\nabla w(t)\|_{2}^{2}
$$

and

$$
\begin{aligned}
\left|\mathcal{J}_{2}\right| & \leq \gamma\left[\left\|\nabla u^{1}(t)\right\|+\left\|\nabla u^{2}(t)\right\|_{2}\right]\|\nabla w(t)\|_{2}\left\|\Delta u^{2}(t)\right\|_{2}\left\|w_{t}(t)\right\|_{2} \\
& \leq \frac{\kappa}{4}\left\|w_{t}(t)\right\|_{2}^{2}+K_{B}\|\nabla w(t)\|_{2}^{2} .
\end{aligned}
$$

From definition of $\mathcal{E}(t)$,

$$
\left|\mathcal{J}_{3}\right| \leq \delta|\Phi(U(t))|^{q-1}\|\Delta w(t)\|_{2}\left\|w_{t}(t)\right\|_{2} \leq \delta|\Phi(U(t))|^{q-1} \mathcal{E}(t) .
$$

Also, using (3.17), the Mean Value Theorem (for some $\theta \in(0,1))$ and the identity $\Phi\left(U^{1}\right)-$ $\Phi\left(U^{2}\right)=-\left[\left(\Delta u^{1}, w_{t}\right)+\left(\Delta w, u_{t}^{2}\right)\right]$, then

$$
\begin{aligned}
\left|\mathcal{J}_{4}\right| & \leq K_{B}\left|\theta \Phi\left(U^{1}(t)\right)+(1-\theta) \Phi\left(U^{2}(t)\right)\right|^{q-2}\left[\Phi\left(U^{1}(t)\right)-\Phi\left(U^{2}(t)\right)\right]\left\|w_{t}(t)\right\|_{2} \\
& \leq K_{B}\left[\left|\Phi\left(U^{1}(t)\right)\right|^{q-2}+\left|\Phi\left(U^{2}(t)\right)\right|^{q-2}\right]\left|\left(\Delta u^{1}(t), w_{t}(t)\right)+\left(\Delta w(t), u_{t}^{2}(t)\right)\right| \mathcal{E}(t)^{1 / 2} \\
& \leq K_{B}\left[\left|\Phi\left(U^{1}(t)\right)\right|^{q-2}+\left|\Phi\left(U^{2}(t)\right)\right|^{q-2}\right] \mathcal{E}(t) .
\end{aligned}
$$

Finally, using Assumption (2.1)-(2.4), again the Mean Value Theorem, generalized Hölder and Young's inequality and (3.17), yields

$$
\left|\mathcal{J}_{5}\right| \leq \frac{\kappa}{4}\left\|w_{t}(t)\right\|_{2}^{2}+K_{B}\|w(t)\|_{2(\rho+1)}^{2} .
$$

Replacing these last five estimates in (3.19), we obtain

$$
\begin{equation*}
\frac{d}{d t} \mathcal{E}(t) \leq-\kappa\left\|w_{t}(t)\right\|_{2}^{2}+K_{B} \Theta(t) \mathcal{E}(t)+K_{B}\left[\|\nabla w(t)\|_{2}^{2}+\|w(t)\|_{2(\rho+1)}^{2}\right], \tag{3.20}
\end{equation*}
$$

for some constant $K_{B}>0$, where we denote

$$
\Theta(t):=\left[\left|\Phi\left(U^{1}(t)\right)\right|^{q-1}+\sum_{i=1}^{2}\left|\Phi\left(U^{i}(t)\right)\right|^{q-2}\right]
$$

Now, for $\mu>0$ to be determined later, let us consider the perturbed functional

$$
\begin{equation*}
\mathcal{E}_{\mu}(t)=\mathcal{E}(t)+\mu \int_{\Omega} w_{t}(t) w(t) d x \tag{3.21}
\end{equation*}
$$

Using Young's inequality and choosing $\mu \leq \frac{\lambda_{1}^{1 / 2}}{2}$, it is easy to see that

$$
\begin{equation*}
\frac{1}{2} \mathcal{E}(t) \leq \mathcal{E}_{\mu}(t) \leq \frac{3}{2} \mathcal{E}(t), \quad t \geq 0 \tag{3.22}
\end{equation*}
$$

Additionally, deriving (3.21) and taking Eq. (3.16) into account, it follows that

$$
\begin{align*}
\frac{d}{d t} \mathcal{E}_{\mu}(t)= & \frac{d}{d t} \mathcal{E}(t)+\mu\left\|w_{t}(t)\right\|_{2}^{2}-\mu\|\Delta w(t)\|_{2}^{2}-\mu \beta\|\nabla w(t)\|_{2}^{2} \\
& -\mu \gamma\left\|\nabla u^{1}(t)\right\|_{2}^{2}\|\nabla w(t)\|_{2}^{2}+\sum_{i=1}^{5} \mathcal{L}_{i}, \tag{3.23}
\end{align*}
$$

where

$$
\begin{aligned}
& \mathcal{L}_{1}=\mu \gamma\left[\left\|\nabla u^{1}(t)\right\|_{2}^{2}-\left\|\nabla u^{2}(t)\right\|_{2}^{2}\right] \int_{\Omega} \Delta u^{2}(t) w(t) d x \\
& \mathcal{L}_{2}=-\mu \delta|\Phi(U(t))|^{q-2} \Phi(U(t))\|\nabla w(t)\|_{2}^{2}, \\
& \mathcal{L}_{3}=\mu \delta \Delta_{\Phi} \int_{\Omega} \Delta u^{2}(t) w(t) d x, \\
& \mathcal{L}_{4}=\mu \int_{\Omega}\left[f\left(u^{2}(t)\right)-f\left(u^{1}(t)\right)\right] w(t) d x, \\
& \mathcal{L}_{5}=-\mu \kappa \int_{\Omega} w_{t}(t) w(t) d x .
\end{aligned}
$$

The estimates for $\mathcal{L}_{1}, \ldots, \mathcal{L}_{5}$ are truly similar to those made for $\mathcal{J}_{1}, \ldots, \mathcal{J}_{5}$. In the same spirit, we infer that there exists a constant $K_{B}>0$ such that

$$
\begin{align*}
& \left|\mathcal{L}_{1}\right| \leq K_{B}\|\nabla w(t)\|_{2}^{2} \\
& \left|\mathcal{L}_{2}\right| \leq K_{B}\|\nabla w(t)\|_{2}^{2}, \\
& \left|\mathcal{L}_{3}\right| \leq K_{B}\left[\sum_{i=1}^{2}\left|\Phi\left(U^{i}(t)\right)\right|^{q-2}\right] \mathcal{E}(t), \\
& \left|\mathcal{L}_{4}\right| \leq K_{B}\|w(t)\|_{2(\rho+1)}^{2}, \\
& \left|\mathcal{L}_{5}\right| \leq \frac{\kappa}{4}\left\|w_{t}(t)\right\|_{2}^{2}+\frac{\mu^{2} \kappa}{\lambda_{1}}\|\Delta w(t)\|_{2}^{2}, \tag{3.24}
\end{align*}
$$

Replacing (3.20) and (3.24) in (3.23), taking $\mu \leq \min \left\{\frac{\kappa}{4}, \frac{\lambda_{1}}{2 \kappa}\right\}$ and neglecting nonnegative terms, we get

$$
\begin{equation*}
\frac{d}{d t} \mathcal{E}_{\mu}(t)+\frac{\mu}{2} \mathcal{E}(t)-K_{B} \Theta(t) \mathcal{E}(t) \leq K_{B}\left[\|\nabla w(t)\|_{2}^{2}+\|w(t)\|_{2(\rho+1)}^{2}\right] \tag{3.25}
\end{equation*}
$$

for some constant $K_{B}>0$. Combining (3.22) and (3.25) we have

$$
\frac{d}{d t} \mathcal{E}_{\mu}(t)+\Theta_{\mu}(t) \mathcal{E}_{\mu}(t) \leq K_{B}\left[\|\nabla w(t)\|_{2}^{2}+\|w(t)\|_{2(\rho+1)}^{2}\right]
$$

where

$$
\Theta_{\mu}(t)=\frac{\mu}{3}-K_{B} \Theta(t), \quad K_{B}>0
$$

Applying Gronwall's inequality we deduce

$$
\begin{equation*}
\mathcal{E}_{\mu}(t) \leq e^{-\int_{0}^{t} \Theta_{\mu}(s) d s} \mathcal{E}_{\mu}(0)+K_{B} \int_{0}^{t} e^{-\int_{s}^{t} \Theta_{\mu}(\tau) d \tau}\left[\|\nabla w(s)\|_{2}^{2}+\|w(s)\|_{2(\rho+1)}^{2}\right] d \tag{3.26}
\end{equation*}
$$

Now, this is the precise (and crucial) moment we use the $L^{q}$-regularity of the BalakrishnanTaylor term $\Phi(U)$ in (3.17). Indeed, from this and in view of Young's inequality with $0<$ $\mu \leq \min \left\{\frac{\lambda_{1}^{1 / 2}}{2}, \frac{\kappa}{4}, \frac{\lambda_{1}}{2 \kappa}\right\}$ and exponents $\frac{q-1}{q}+\frac{1}{q}=1, \frac{q-2}{q}+\frac{2}{q}=1$, it results

$$
\begin{aligned}
\int_{s}^{t}\left[K_{B} \Theta(s)\right] d s & \leq \frac{\mu}{6}(t-s)+c_{\mu} K_{B} \sum_{i=1}^{2} \int_{s}^{t}\left|\Phi\left(U^{i}(s)\right)\right|^{q} d s \\
& \leq \frac{\mu}{6}(t-s)+K_{B}, \quad \forall t>s \geq 0,
\end{aligned}
$$

for some constant $K_{B}>0$. Then,

$$
\begin{equation*}
e^{-\int_{s}^{t} \Theta_{\mu}(s) d s} \leq K_{B} e^{-\frac{\mu}{6}(t-s)}, \quad \forall t>s \geq 0, \tag{3.27}
\end{equation*}
$$

for some constant $K_{B}>0$. Replacing (3.27) in (3.26) we obtain

$$
\mathcal{E}_{\mu}(t) \leq K_{B} \mathcal{E}_{\mu}(0) e^{-\frac{\mu}{6} t}+K_{B} \int_{0}^{t} e^{-\frac{\mu}{6}(t-s)}\left[\|\nabla w(s)\|_{2}^{2}+\|w(s)\|_{2(\rho+1)}^{2}\right] d s
$$

for all $t>0$ and some constant $K_{B}>0$. Using again (3.22) we arrive at

$$
\begin{equation*}
\mathcal{E}(t) \leq 3 K_{B} \mathcal{E}(0) e^{-\frac{\mu}{6} t}+2 K_{B} \int_{0}^{t} e^{-\frac{\mu}{6}(t-s)}\left[\|\nabla w(s)\|_{2}^{2}+\|w(s)\|_{2(\rho+1)}^{2}\right] d s, \quad t>0 . \tag{3.28}
\end{equation*}
$$

Finally, combining (3.18) and (3.28), we conclude the stabilizability inequality (3.15) by taking $\sigma=\frac{\mu}{6}>0, \widetilde{\varrho}_{B}=\frac{3 K_{B} \widetilde{K}_{B}}{\omega_{1}}>0$ and $\varrho_{B}=\frac{2 K_{B}}{\omega_{1}}>0$. This completes the proof of Proposition 3.4.

### 3.2 The Dynamical System and Attractors

By means of the well-posedness assured by Theorem 2.1 we can define a dynamical system $(\mathcal{H}, S(t))$, where the evolution operator

$$
\begin{equation*}
\mathcal{H} \ni U_{0} \mapsto S(t) U_{0}=U(t) \in \mathcal{H}, \quad t \geq 0, \tag{3.29}
\end{equation*}
$$

is set through the unique mild solution $U(t)=\left(u(t), u_{t}(t)\right)$ of problem (1.4)-(1.6). The following properties are immediately ensured:
I. Dissipative dynamical system. Inequality (3.13) in Corollary 3.3 implies that $(\mathcal{H}, S(t)$ ) has a bounded absorbing set $\mathcal{B} \subset \mathcal{H}$ and, consequently, the existence of bounded positively invariant absorbing sets on $\mathcal{H}$. Hence, $(\mathcal{H}, S(t))$ is a dissipative dynamical system.
II. Gradient dynamical system. Identity (3.3) in Proposition 3.1 implies that $\widetilde{E}(t)$ is a strict Lyapunov functional on $\mathcal{H}$, that is, $(\mathcal{H}, S(t))$ is a gradient dynamical system.

In what follows, we shall prove the existence of attractors, as well as its properties, to the dynamical system $(\mathcal{H}, S(t))$ generated by (3.29). To do so, we shall combine Proposition 3.4 with the abstracts concepts within the theory of infinite-dimensional dynamical systems, see e.g. [1,6-8,11,14,19,27]. More specifically, we use the following notion of quasi-stable dynamical systems, accordingly to [8, Definition 7.9.2] which started with the prior work [6], restricted to our particular dynamical system. Such approach has been recently used in the literature, see for instance Feng et al. [13].

Definition 3.1 The dynamical system $(\mathcal{H}, S(t))$ generated by (3.29) is called to be quasistable on a set $B \subset \mathcal{H}$ if there exist a compact seminorm $n_{X}(\cdot, \cdot)$ on $X:=H_{0}^{2}(\Omega)$ and nonnegative scalar functions $a(t)$ and $c(t)$ locally bounded in $[0, \infty)$, and $b(t) \in L^{1}\left(\mathbb{R}^{+}\right)$ with $\lim _{t \rightarrow \infty} b(t)=0$, such that

$$
\begin{equation*}
\left\|S(t) U_{1}-S(t) U_{2}\right\|_{\mathcal{H}}^{2} \leq a(t)\left\|U_{1}-U_{2}\right\|_{\mathcal{H}}^{2}, \tag{3.30}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|S(t) U_{1}-S(t) U_{2}\right\|_{\mathcal{H}}^{2} \leq b(t)\left\|U_{1}-U_{2}\right\|_{\mathcal{H}}^{2}+c(t) \sup _{s \in[0, t]}\left[n_{X}(u(s), v(s))\right]^{2}, \tag{3.31}
\end{equation*}
$$

for any $U_{1}, U_{2} \in B$, where we denote

$$
S(t) U_{i}=\left(u^{i}(t), u_{t}^{i}(t)\right), i=1,2, \quad \text { and } \quad(u, v)=\left(u^{1}-u^{2}, u_{t}^{1}-u_{t}^{2}\right) .
$$

Proposition 3.4 is, in fact, a key result in order to achieve the quasi-stability property to the dynamical system ( $\mathcal{H}, S(t)$ ) defined in (3.29) and, in particular, its asymptotic smoothness.

Theorem 3.5 Under the assumptions of Proposition 3.4 with $\rho$ satisfying (2.7), the dynamical system ( $\mathcal{H}$, $S(t)$ ) generated by (3.29) is quasi-stable on any bounded positively invariant set $B \subset \mathcal{H}$. In particular, it is also asymptotically smooth.

Proof Let $B \subset \mathcal{H}$ be a bounded positively invariant set of $S(t), U_{1}, U_{2} \in B$, and

$$
S(t) U_{i}=\left(u^{i}(t), u_{t}^{i}(t)\right), i=1,2, \quad u=u^{1}-u^{2} .
$$

Firstly, under the above notations, one sees promptly from (2.9) that (3.30) holds true with $a(t)=e^{C_{B} t}, C_{B}>0$, being locally bounded in $[0, \infty)$. Then, setting

$$
\left[n_{X}(u)\right]^{2}:=\|\nabla u\|_{2}^{2}+\|u\|_{2(\rho+1)}^{2}, \quad X=H_{0}^{2}(\Omega)
$$

and noting that embeddings $H_{0}^{2}(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega), H_{0}^{2}(\Omega) \hookrightarrow H_{0}^{1}(\Omega)$ are compact, it follows that $n_{X}(\cdot)$ is a compact seminorm on $X$. Additionally, from (3.15) one has

$$
\left\|S(t) U_{1}-S(t) U_{2}\right\|_{\mathcal{H}}^{2} \leq b(t)\left\|U_{1}-U_{2}\right\|_{\mathcal{H}}^{2}+c(t) \sup _{s \in[0, t]}\left[n_{X}(u(s))\right]^{2},
$$

where

$$
b(t)=\widetilde{\varrho}_{B} e^{-\sigma t} \quad \text { and } \quad c(t)=\varrho_{B} \int_{0}^{t} e^{-\sigma(t-s)} d s, \quad t \geq 0
$$

Thus, $b \in L^{1}\left(\mathbb{R}^{+}\right)$with $\lim _{t \rightarrow \infty} b(t)=0$, and $c(t)$ is globally bounded

$$
\begin{equation*}
c_{\infty}=\sup _{t \in \mathbb{R}^{+}} c(t)<\infty . \tag{3.32}
\end{equation*}
$$

Hence, condition (3.31) also holds true, which completes the proof that ( $\mathcal{H}, S(t)$ ) is quasi-stable on any bounded positively invariant set in $\mathcal{H}$. In particular, $(\mathcal{H}, S(t))$ is an asymptotically smooth dynamical system from Proposition 7.9.4 in [8].

Theorem 3.6 (Global Attractor). Under the assumptions of Theorem 3.5, we have:
(a) the dynamical system $(\mathcal{H}, S(t))$ generated by (3.29) possesses a unique compact and connected global attractor $\mathfrak{A} \subset \mathcal{H}$;
(b) the global attractor $\mathfrak{A}$ is precisely the unstable manifold

$$
\mathfrak{A}=\mathbb{M}^{u}(\mathcal{N}),
$$

emanating from the set of stationary solutions

$$
\mathcal{N}=\left\{(u, 0) \in \mathcal{H} ; \Delta^{2} u-\left[\beta+\gamma\|\nabla u\|_{2}^{2}\right] \Delta u+f(u)=h\right\} ;
$$

(c) the compact global attractor $\mathfrak{A}$ has finite fractal dimension $\operatorname{dim}_{f}^{\mathcal{H}} \mathfrak{A}$;
(d) every full trajectory $\Gamma=\left\{\left(u(t) ; u_{t}(t)\right) ; t \in \mathbb{R}\right\}$ from the attractor $\mathfrak{A}$ has the following regularity

$$
\left(u_{t}, u_{t t}\right) \in L^{\infty}\left(\mathbb{R} ; H_{0}^{2}(\Omega) \times L^{2}(\Omega)\right) .
$$

Moreover, there exists a constant $\mathfrak{R}=\mathfrak{R}\left(c_{\infty}, n_{X}, \lambda_{1}\right)>0$ such that

$$
\sup _{\Gamma \subset \mathfrak{A}} \sup _{t \in \mathbb{R}}\left(\left\|\Delta u_{t}(t)\right\|_{2}^{2}+\left\|u_{t t}(t)\right\|_{2}^{2}\right) \leq \mathfrak{R}^{2} .
$$

Proof (a) It follows from [8, Theorem 7.2.3], once $(\mathcal{H}, S(t))$ is a dissipative asymptotically smooth dynamical system. See also Corollary 7.9.5 in [8].
(b) It follows from [8, Theorem 7.5.6], since $(\mathcal{H}, S(t))$ is a gradient dynamical system that has a compact global attractor.
(c) It follows from [8, Theorem 7.9.6], because $(\mathcal{H}, S(t))$ possesses a compact global attractor $\mathfrak{A}$ and is quasi-stable on $\mathfrak{A}$ by virtue of Theorem 3.5.
(d) It follows from [8, Theorem 7.9.8], seeing that property (3.31) holds with the function $c(t)$ satisfying (3.32).

Corollary 3.7 (Minimal Attractor). From the items (a), (b) of Theorem 3.6 one has that any trajectory stabilizes to the set of stationary solutions $\mathcal{N}$, that is,

$$
\lim _{t \rightarrow+\infty} \operatorname{dist}_{H}(S(t) z, \mathcal{N})=0, \quad \forall z \in \mathcal{H}
$$

In particular, $\mathcal{N}$ is the global minimal attractor $\mathfrak{A}_{\min }$ to the dynamical system $(\mathcal{H}, S(t))$.
Proof It is a consequence of [8, Theorem 7.5.10].
To finish this work we also explore the concept of fractal exponential attractors for quasistable systems. For the sake of the reader we recall this notion, accordingly to [11], restricted to our particular dynamical system. See also [8, Definition 7.4.4].

Definition 3.2 A compact set $\mathfrak{A}_{\exp } \subset \mathcal{H}$ is said to be a fractal exponential attractor to the dynamical system $(\mathcal{H}, S(t))$ generated by (3.29) if $\mathfrak{A}_{\text {exp }}$ is a positively invariant set of finite fractal dimension in $\mathcal{H}$ and for every bounded set $B \subset \mathcal{H}$ there exist constants $t_{B}, C_{B}, \sigma_{B}>0$ such that

$$
\sup _{U_{0} \in B} \operatorname{dist}_{\mathcal{H}}\left(S(t) U_{0}, \mathfrak{A}_{\exp }\right) \leq C_{B} e^{-\sigma_{B}\left(t-t_{B}\right)}, \quad t \geq t_{B} .
$$

If there exists an exponential attractor only having finite dimension in some extended space $\widetilde{\mathcal{H}} \supseteq \mathcal{H}$, then this exponentially attracting set is called generalized fractal exponential attractor.

Theorem 3.8 (Fractal Exponential Attractor). Under the assumptions of Theorem 3.5, the dynamical system ( $\mathcal{H}, S(t))$ generated by (3.29) possesses a generalized fractal exponential attractor $\mathfrak{A}_{\exp }$ with finite dimension in the extended space

$$
\mathcal{H}_{-1}:=L^{2}(\Omega) \times H^{-2}(\Omega) .
$$

Moreover, there exists a generalized fractal exponential attractor with finite fractal dimension in a smaller extended space $\mathcal{H}_{-r}$, where

$$
\mathcal{H} \subset \mathcal{H}_{-r} \subseteq \mathcal{H}_{-1}, \quad 0<r \leq 1 .
$$

Proof It follows from [8, Theorem 7.9.9], since the dynamical system ( $\mathcal{H}, S(t)$ ) is dissipative and quasi-stable on any bounded positively invariant absorbing set $\mathcal{B}_{0}$, and also the Hölder continuity property of the mapping $t \mapsto S(t) U_{0}$ in $\mathcal{H}_{-r}$, for every $U_{0} \in \mathcal{B}_{0}$, is standard, see e.g. [18, Theorem 2.3-(vi)].

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