Long-Time Behavior for a Class of Semi-linear Viscoelastic Kirchhoff Beams/Plates

B. Feng, M. A. Jorge Silva & A. H. Caixeta

Applied Mathematics & Optimization

ISSN 0095-4616
Volume 82
Number 2

Appl Math Optim (2020) 82:657-686
DOI 10.1007/s00245-018-9544-3

Springer

Applied Mathematics & Optimization (2020) 82:657–686
https://doi.org/10.1007/s00245-018-9544-3

Long-Time Behavior for a Class of Semi-linear Viscoelastic Kirchhoff Beams/Plates

B. Feng[1] · **M. A. Jorge Silva**[2] · **A. H. Caixeta**[2]

Published online: 11 December 2018

Abstract

This is a complementation work of the paper referred in Jorge Silva, Muñoz Rivera and Racke (Appl Math Optim 73:165–194, 2016) where the authors proposed a semi-linear viscoelastic Kirchhoff plate model. While in [28] it is presented a study on well-posedness and energy decay rates in a historyless memory context, here our main goal is to consider the problem in a past history framework and then analyze its long-time behavior through the corresponding autonomous dynamical system. More specifically, our results are concerned with the existence of finite dimensional attractors as well as their intrinsic properties from the dynamical systems viewpoint. In addition, we also present a physical justification of the model under consideration. Hence, our new achievements complement those established in [28] to the case of memory in a history space setting and extend the results in Jorge Silva and Ma (IMA J Appl Math 78:1130–1146, 2013, J Math Phys 54:021505, 2013) to the case of dissipation only given by the memory term.

Keywords Kirchhoff plates · Memory · p-Laplacian · Quasi-stable systems · Attractor

Mathematics Subject Classification 35B40 · 35B41 · 35L56 · 35L76 · 74Dxx

B. Feng has been supported by the National Natural Science Foundation of China, Grant #11701465. M.A. Jorge Silva has been supported by the CNPq, Grant #441414/2014-1. A.H. Caixeta has been supported by the CAPES, Scholarship #1622327.

✉ M. A. Jorge Silva
marcioajs@uel.br

B. Feng
bwfeng@swufe.edu.cn

A. H. Caixeta
arthur-caixeta@hotmail.com

1 Department of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, People's Republic of China

2 Department of Mathematics, State University of Londrina, Londrina, PR 86057-970, Brazil

1 Introduction

In this paper we address the long-time dynamics to the following n-dimensional semi-linear viscoelastic Kirchhoff beam/plate model

$$u_{tt} - \Delta u_{tt} + \chi_g \Delta^2 u - \operatorname{div} F(\nabla u) - \int_{-\infty}^{t} g(t-s)\Delta^2 u(s)\, ds = h \ \text{ in } \ \Omega \times \mathbb{R}^+, \tag{1.1}$$

subject to simply supported boundary condition

$$u = \Delta u = 0 \ \text{ on } \ \partial\Omega \times \mathbb{R}, \tag{1.2}$$

and initial conditions

$$u(x,t) = u_0(x,t), \quad u_t(x,0) = u_1(x), \quad (x,t) \in \Omega \times (-\infty, 0], \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $g : [0,\infty) \to \mathbb{R}^+$ ($\mathbb{R}^+ = (0,\infty)$) corresponds to memory kernel and $F : \mathbb{R}^n \to \mathbb{R}^n$ represents a vector field, whose assumptions for both will be given later (Sect. 3), $\chi_g = 1 + \int_0^\infty g(s)\, ds > 0$ is a constant, and $u_0 : \Omega \times (-\infty, 0] \to \mathbb{R}$ is the prescribed past history of u and $u_1(x) := \partial_t u_0(x,t)|_{t=0}$.

A justification of the particular model (1.1) is presented in Sect. 2 by using classical theory for beams/plates in combination with constitutive laws coming from materials with viscoelastic structures. On the other hand, problem (1.1) was recently introduced in [28] as a viscoelastic Kirchhoff model, see for instance [29, Chap. 6], with lower order perturbation of p-Laplacian type. To be more precise, in [28] the authors deal with null history in the equation (1.1), namely, it is considered $u(\cdot,s) = u_0(\cdot,s) \equiv 0$ for $s < 0$, see e.g. Eq. (1.6) of [28]. Comparisons on the modeling and results will be given later.

Here, our goal is to approach the usual history setting of problem (1.1)–(1.3) and give a treatment on the dynamics of its corresponding autonomous problem. To this end, as in [20,21,23], we introduce the new variable $\eta = \eta^t(x,s)$ corresponding to *relative displacement history*:

$$\eta^t(x,s) := u(x,t) - u(x,t-s), \quad (x,t,s) \in \Omega \times [0,\infty) \times \mathbb{R}^+. \tag{1.4}$$

Thus, formal computations lead to

$$\begin{aligned} \eta_t + \eta_s &= u_t \ \text{ in } \ \Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \\ \eta^0(x,s) &= u_0(x,0) - u_0(x,-s), \quad (x,s) \in \Omega \times \mathbb{R}^+, \\ \eta^t(x,0) &:= \lim_{s\to 0^+} \eta^t(x,s) = 0, \, (x,t) \in \Omega \times [0,\infty), \end{aligned} \tag{1.5}$$

as well as the fourth order memory term can be rewritten as

$$\chi_g \Delta^2 u - \int_{-\infty}^{t} g(t-s)\Delta^2 u(s)\, ds = \Delta^2 u - \int_0^\infty g(s)\Delta^2 \eta(s) ds \ \text{ in } \ \Omega \times \mathbb{R}^+. \tag{1.6}$$

Therefore, through (1.4)–(1.6) we can rewrite (1.1)–(1.3) as in the following equivalent autonomous initial-boundary value problem

$$u_{tt} - \Delta u_{tt} + \Delta^2 u - \operatorname{div} F(\nabla u) + \int_0^\infty g(s)\Delta^2 \eta(s) ds = h \ \text{ in } \ \Omega \times \mathbb{R}^+, \tag{1.7}$$

$$\eta_t + \eta_s = u_t \ \text{ in } \ \Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.8}$$

$$u = \Delta u = 0 \ \text{ on } \ \partial\Omega \times [0, \infty), \tag{1.9}$$

$$\eta = \Delta \eta = 0 \ \text{ on } \ \partial\Omega \times [0, \infty) \times \mathbb{R}^+, \tag{1.10}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.11}$$

$$\eta^0(x,s) = \eta_0(x,s), \quad \eta^t(x,0) = 0, \quad x \in \Omega, \quad s > 0, \quad t \geq 0, \tag{1.12}$$

where we denote

$$u_0(x) = u_0(x,0), \quad u_1(x) = \partial_t u_0(x,t)\big|_{t=0},$$
$$\eta_0(x,s) = u_0(x) - u_0(x,-s), \quad x \in \Omega, \quad s > 0.$$

1.1 A Short Comparison with Existing Literature

Roughly speaking, the results presented in this work complement (and generalize somehow) those provided for viscoelastic problems studied in [1,25,26,28]. The main difference with the existing literature revolves around the nonlinear foundation $\operatorname{div} F(\nabla u)$ whose origin comes from the well-known p-Laplacian term $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and its long-time control by exploiting only the memory dissipation. In what concerns the asymptotic behavior of viscoelastic (and thermo-viscoelastic) plate/beam models encompassing nonlinear source terms like $f(u)$ there is a huge literature on the subject, see for instance [5,11,14–16,22,23,32,33], just to name few. On the other hand, we refer to [7,10,31,39–43] where damped problems addressing plate equations with lower order perturbation of p-Laplacian type are considered, but without memory term. Next, we highlight the main contributions of the present article as well as we provide a brief comparison of the results and methodology with closer works on the subject.

- When compared with papers [1,25,26] we have three huge differences:

  #1. The first one is about the modeling. Indeed, in [1,25,26] the model comes from flows of elastoplastic microstructures in a 1D setting or else from the Kirchhoff-Boussinesq model in a 2D framework, by adding a memory term to get a viscoelastic version of the problem. On the other hand, here our approach on the modeling is totally different being motivated by the theory for viscoelastic beams/plates where the p-Laplacian term appears as lower order perturbation. This is why the strong damping term $-\Delta u_t$ arises in the papers [1,25,26], while here the model appears with rotational inertial term $-\Delta u_{tt}$ instead. Such a model's justification will be clarified in Sect. 2 below.

#2. The second main difference is on the stability of problems. In fact we have that both strong damping $-\Delta u_t$ and rotational inertial $-\Delta u_{tt}$ act as regularizing terms. However, $-\Delta u_t$ also works an additional damping in [1,25,26] and, therefore, it is necessary to define only one perturbed functional in the energy estimates, see e.g. (4.1)–(4.2) on p. 1142 of [25]. Here, the picture is again differently because $-\Delta u_{tt}$ spoils the estimates and a new functional is necessary to control it, see for instance the proof of Proposition 5.5 in Sect. 5. Such approach is already used in the literature as, for example, in [3] where a quasilinear viscoelastic wave equation is considered. Although the methodology employed here does not change the core of the arguments in the viscoelastic context, we observe that nonlinearities of p-Laplacian type $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ can not be addressed in the second order model of [3].

#3. A third issue falls on the attractor obtained in [26] and the main result proved in Sect. 4 (see Theorem 4.1). To be more clear, since in [26] the authors work with the particular p-Laplacian term $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, then to reach a nontrivial attractor it is necessary consider an additional nonlinear source $f(u)$, whereas here we consider a more general assumption on the vector field F so that a nontrivial attractor is ensured only in the presence of the nonlinear foundation $\mathrm{div} F(\nabla u)$.

- When comparing with the previous article [28], we can also argue on the modeling and stability results. Indeed, the physical motivation provided in Sect. 2 to justify problem (1.1) follows different lines from those presented in [28, Sect. 1]. In addition, the problem in [28] is taken within a historyless case, which means the problem is non-autonomous. There, it is addressed issues like well-posedeness and energy decay rates. Here, differently from [28], the model is approached in a history space framework and studied in the context of dynamical systems. Therefore, the attractors and their properties are achieved on an extended phase space by using dissipation provided only by the memory term and a slightly more general assumption on F, which are not considered in [28]. This also extends somehow the results given by [25,26].

The remaining paper is planned as follows. In Sect. 2 we provide a physical justification of the model (1.1)–(1.3). In Sect. 3 we present the notations, initial assumptions and the well-posedness result. In Sect. 4 we set the corresponding dynamical system and state the main result of the paper, namely, Theorem 4.1. Finally, Sect. 5 is dedicated to the proofs by considering some technical results that culminate in the proof of Theorem 4.1. Some examples for F are also provided in a short Appendix at the end.

2 Physical Motivation

In this section, in order to justify the viscoelastic Kirchhoff equation (1.1) as a n-dimensional generalization of concrete 1D-beam and 2D-plate viscoelastic models under the Kirchhoff hypothesis, we provide a physical motivation for the deduction of models that come from the classical theory in Timoshenko beams [37,38] and Mindlin–Timoshenko plates [29,30], in combination with constitutive relations in viscoelasticity where beams/plates are composed of homogeneous linear viscoelastic materials, see for instance [12,35].

2.1 1D Viscoelastic Kirchhoff Beams

We start by considering the classical model in differential equations for vibrations of prismatic beams first introduced by Timoshenko [37,38]:

$$\begin{cases} \rho A\varphi_{tt} - S_x = f_1, \\ \rho I\psi_{tt} - M_x + S = f_2, \end{cases} \tag{2.1}$$

for $(x,t) \in (0,L) \times \mathbb{R}^+$, where ρ means the mass density per area unit, A and I represent, respectively, area and inertial moment of a cross section, $\varphi = \varphi(x,t)$ denotes the vertical displacement and $\psi = \psi(x,t)$ the angle of rotation, M and S designate bending moment and shear force, respectively, and f_1, f_2 may stand for external nonlinear source of distributed loads along the beam of length $L > 0$. The well-known elastic constitutive relations for bending moment and shear force are given, respectively, by

$$M = EI\psi_x, \tag{2.2}$$

$$S = kGA(\phi_x + \psi). \tag{2.3}$$

However, for viscoelastic materials containing hereditary (history) properties, the Boltzmann theory states that the stress is assumed to depend not only on the (instantaneous) strain, but also on the strain history. In such a case, the stress-strain viscoelastic law also depends on a relaxation measure. This is clarified in the next paragraph where our arguments rely on the classical theories by Prüss [35, Chap. 9] and Drozdov and Kolmanovskii [12, Chap. 5].

Let us consider a beam $[0,L] \times \Omega$ of length $L > 0$ and uniform cross section $\Omega \subset \mathbb{R}^2$ made of homogeneous isotropic viscoelastic material with pattern Timoshenko hypotheses:

- $(0,0)$ is the center of Ω, so that $\int_\Omega z dy dz = \int_\Omega y dy dz = 0$,
- the bending takes place only on the (x,z)-plane,
- $\mathrm{diam}\Omega << L$ (thin beams) and normal stresses are negligible in general,
- there are only two relevant stresses σ_{11} and σ_{13} in the stress tensor $\sigma = \{\sigma_{ij}\}$.

In addition, on the basis of Boltzmann principle, the viscoelastic stress-strain relations can be considered as follows

$$\sigma_{11}(x,z,t) = E\left\{\epsilon_{11}(x,z,t) - \int_0^t g_1(t-s)\epsilon_{11}(x,z,s)ds\right\}, \tag{2.4}$$

$$\sigma_{13}(x,z,t) = 2kG\left\{\epsilon_{13}(x,z,t) - \int_0^t g_2(t-s)\epsilon_{13}(x,z,s)ds\right\}, \tag{2.5}$$

where E stands for the Young modulus of elasticity, G is the constant shear modulus, k is a shear correction coefficient and g_1, g_2 are relaxation measures usually called *memory kernels*.

Next, the following notations are also employed:

- $u = u(x,t)$: longitudinal displacement of points lying on the horizontal axis,
- $w_1(x,z,t) = u(x,t) + z\psi(x,t)$: longitudinal displacement,
- $w_2(x,z,t) = \varphi(x,t)$: vertical displacement.

Under these notations, the standard formulas for the components of the infinitesimal strain tensor (see e.g. (2.4) on p. 339 of [12]) can be expressed by

$$\epsilon_{11}(x,z,t) := \frac{\partial w_1}{\partial x} = u_x(x,t) + z\psi_x(x,t), \tag{2.6}$$

$$\epsilon_{13}(x,z,t) := \frac{1}{2}\left(\frac{\partial w_1}{\partial z} + \frac{\partial w_2}{\partial x}\right) = \frac{1}{2}\left(\psi(x,t) + \phi_x(x,t)\right). \tag{2.7}$$

Additionally, the usual formulas to determine bending moment and shear force (see e.g. (9.10)–(9.11) on p. 237 of [35]) are given, respectively, by

$$M(x,t) = \int_\Omega z\sigma_{11}(x,z,t)dydz, \tag{2.8}$$

$$S(x,t) = \int_\Omega \sigma_{13}(x,z,t)dydz. \tag{2.9}$$

As a matter of fact, we have normalized identities (2.8)–(2.9) by the area and inertial moment formulas, namely,

$$A = \int_\Omega dydz \qquad \text{and} \qquad I = \int_\Omega z^2 dydz.$$

Hence, using relations (2.4), (2.6) and (2.8), one can derive the classical (and well-known) viscoelastic law for bending moment

$$\begin{aligned} M = {} & E\overbrace{\left(\int_\Omega zdydz\right)}^{=0}\left(u_x - \int_0^t g_1(t-s)u_x(\cdot,s)ds\right) \\ & + E\underbrace{\left(\int_\Omega z^2dydz\right)}_{=I}\left(\psi_x - \int_0^t g_1(t-s)\psi_x(\cdot,s)ds\right), \end{aligned}$$

that is,

$$M = EI\left(\psi_x - \int_0^t g_1(t-s)\psi_x(\cdot, s)\,ds\right). \tag{2.10}$$

The relations (2.5), (2.7) and (2.9) lead to a viscoelastic law for shear forces that will not be used in the present work. Summarizing, the constitutive relation (2.10) provides bending deformations in the context of Timoshenko beams with viscoelastic materials depending on strain history. Moreover, replacing identities (2.10) and (2.3) in (2.1), we arrive at:

$$\begin{cases} \rho A\varphi_{tt} - kGA(\varphi_x + \psi)_x = f_1, \\ \rho I\psi_{tt} - EI\left(\psi_{xx} - \displaystyle\int_0^t g_1(t-s)\psi_{xx}(\cdot, s)\,ds\right) + kGA(\varphi_x + \psi) = f_2, \end{cases} \tag{2.11}$$

which was first proposed by Ammar-Khodja et al. [2] in the (homogeneous) case $f_1 = f_2 = 0$. Also, for beams with supported end, boundary conditions read as

$$\varphi(0, t) = \varphi(L, t) = \psi_x(0, t) = \psi_x(L, t) = 0, \quad t \geq 0.$$

Now we proceed by taking formally the (distributional) x-derivative in the second equation of (2.11). Before doing so, we regard that standard nonlinear sources usually appear in quantum mechanics with polynomial growth, see for instance [7,10]. In this way, we consider $f_2(\psi) := -|\psi|^{p-2}\psi,\ p \geq 2,\ f_1(x) := h(x)$ for some function h, and $g_1 := g$ for simplicity. Thus, taking the derivative of $(2.11)_2$ and adding to $(2.11)_1$, we deduce

$$\rho A\varphi_{tt} + \rho I\psi_{xtt} - EI\left(\psi_{xxx} - \int_0^t g(t-s)\psi_{xxx}(\cdot, s)\,ds\right) + \left(|\psi|^{p-2}\psi\right)_x = h. \tag{2.12}$$

Additionally, under the limiting case $k \to \infty$ in $(2.11)_2$, one has the following Kirchhoff assumption $\psi = -\varphi_x$ (see, for instance, on p. 238 of [35]), and (2.12) turns into

$$\rho A\varphi_{tt} - \rho I\varphi_{xxtt} + EI\varphi_{xxxx} - \left(|\varphi_x|^{p-2}\varphi_x\right)_x - EI\int_0^t g(t-s)\varphi_{xxxx}(\cdot, s)\,ds = h, \tag{2.13}$$

with boundary condition

$$\varphi(0, t) = \varphi(L, t) = \varphi_{xx}(0, t) = \varphi_{xx}(L, t) = 0, \quad t \geq 0. \tag{2.14}$$

Finally, one sees that the 1D viscoelastic Kirchhoff model (2.13)–(2.14) is a particular case of (1.1)–(1.2) with normalized coefficients, null history $\varphi(\cdot, s) = 0$ for $s \leq 0$, and admissible nonlinearity $F(z) = |z|^{p-2}z,\ z \in \mathbb{R}$.

2.2 2D Viscoelastic Kirchhoff Plates

As a second justification of problem (1.1)–(1.2), we show that it can be obtained from a viscoelastic model for the Mindlin–Timoshenko plate where the p-Laplacian still represents a nonlinear strain of lower order, as done e.g. in Chueshov and Lasiecka [6,7,10]. To do so, we first consider the classical Mindlin–Timoshenko plate model as presented in Lagnese and Lions [29,30] together with constitutive laws in viscoelasticity by Giorgi et al. [17–19].

Let us consider a thin homogeneous plate with uniform thickness d by assuming that it occupies a fixed bounded domain $\mathcal{D} \subset \mathbb{R}^3$. It is also assumed that it has a middle surface that (in its equilibrium) occupies a 2D domain $\Omega = \{(x, y, z) \in \mathcal{D} \,|\, z = 0\} \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$. Thus, following Lagnese [29, Sect. 2.1.2], the governing differential equations for the Mindlin–Timoshenko plate model can be written as

$$\rho d\, w_{tt} - K \mathrm{div}(\nabla w + \mathbf{v}) = h_1, \tag{2.15}$$

$$\frac{\rho d^3}{12}\mathbf{v}_{tt} - D\mathbf{S} + K(\nabla w + \mathbf{v}) = h_2, \tag{2.16}$$

for $(x, y, t) \in \Omega \times \mathbb{R}^+$, where ρ is the material density, $K = \frac{\kappa E d}{2(1+\mu)}$ and $D = \frac{Ed^3}{12(1-\mu^2)}$ stand for shear and flexural rigidity modulus, respectively, E is the Young's modulus, $0 < \mu < 1/2$ represents the Poisson's ratio, κ means the shear coefficient, and h_1, h_2 are forcing terms. In addition, the variables $w = w(x, y, t)$ denotes transverse displacement along z axis and $\mathbf{v} = (\varphi, \psi)$ with $\psi = \psi(x, y, t)$ and $\varphi = \varphi(x, y, t)$ standing for rotations of the transverse normal to the middle surface with respect to x and y axis, respectively. The elastic strain operator corresponding to rotation functions ψ and φ is given by

$$\mathbb{L} = \begin{pmatrix} \psi_x + \mu\varphi_y & \frac{1-\mu}{2}(\varphi_x + \psi_y) \\ \frac{1-\mu}{2}(\varphi_x + \psi_y) & \varphi_y + \mu\psi_x \end{pmatrix},$$

and denoting the matrix of second order differential operators by

$$A = \begin{bmatrix} \partial_{xx} + \frac{1-\mu}{2}\,\partial_{yy} & \frac{1+\mu}{2}\,\partial_{xy} \\ \frac{1+\mu}{2}\,\partial_{xy} & \frac{1-\mu}{2}\,\partial_{xx} + \partial_{yy} \end{bmatrix}, \tag{2.17}$$

then the stress tensor $\mathbf{S}$ is given by the following classical elastic constitutive relation

$$\mathbf{S} = \nabla \cdot \mathbb{L} = A\mathbf{v}. \tag{2.18}$$

On the other hand, in order to generalize the model to a viscoelastic framework, it is proposed by Giorgi and Vegni [18,19] to consider the composition of the plate by an isotropic linear viscoelastic material. Consequently, by following again the Boltzmann principle the viscoelastic stress-strain law comes into the picture for some *relaxation function* μ, namely,

$$\mathbf{S} = \nabla \cdot (\mathbb{L} - \mathbb{L} * \mu) = A\mathbf{v} - (A\mathbf{v} * \mu) = A\mathbf{v} - \int_0^\infty \mu(s) A\mathbf{v}(\cdot, t-s)\, ds, \tag{2.19}$$

where we have (already) assumed that the isotropic tensor $\mathbb{L}$ vanishes for $s < 0$ and involves the independent relaxation measure μ. A more detailed (and stringent) justification of these facts can be found on p. 1006 of [18] under constitutive laws in viscoelasticity. See also identity (2.5) on p. 756 in [19]. Therefore, replacing (2.19) in (2.16) we arrive at the following viscoelastic Mindlin–Timoshenko system:

$$\rho d\, w_{tt} - K \operatorname{div}(\nabla w + \mathbf{v}) = h_1, \tag{2.20}$$

$$\frac{\rho d^3}{12}\mathbf{v}_{tt} - D\left(A\mathbf{v} - \int_0^\infty \mu(s) A\mathbf{v}(\cdot, t-s)\, ds\right) + K(\nabla w + \mathbf{v}) = h_2. \tag{2.21}$$

Also, for simply supported plates the boundary condition reads as

$$w = \operatorname{div} \mathbf{v} = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}.$$

The partially viscoelastic model (2.20)–(2.21) first appeared in [18, Sect. 5] with nonlinear fundations $h_1 = h_1(w)$ and $h_2 = h_2(\mathbf{v})$. For a fully viscoelastic Mindlin–Timoshenko system, we refer to problem **P** in [19, Sect. 3].

Now, by following similar arguments as in Chueshov and Lasiecka [6] in what concerns nonlinear forcing terms, we consider $h_2(\mathbf{v}) := -|\mathbf{v}|^{p-2}\mathbf{v},\ p \geq 2$, and for the sake of simplicity we denote $h_1(x) := h(x)$ and $\mu := g$. Next, applying formally the divergence operator in (2.21) and adding to (2.20) we deduce

$$\rho d\, w_{tt} + \frac{\rho d^3}{12} \operatorname{div} \mathbf{v}_{tt} - D \operatorname{div}\left(A\mathbf{v} - \int_0^\infty g(s) A\mathbf{v}(\cdot, t-s)\, ds\right) + \operatorname{div}\left(|\mathbf{v}|^{p-2}\mathbf{v}\right) = h. \tag{2.22}$$

Dealing with (2.21) in the Kirchhoff limit $\kappa \to \infty$, we then reach once again the Kirchhoff assumption $\mathbf{v} = -\nabla w$. In such a case, regarding operator A set in (2.17) we observe that

$$-\operatorname{div} A\mathbf{v} = \partial_{xx}(\Delta w) + \partial_{yy}(\Delta w) = \Delta(\Delta w) := \Delta^2 w \quad \text{in} \quad \Omega \subset \mathbb{R}^2,$$

and (2.22) turns into

$$\rho d\, w_{tt} - \frac{\rho d^3}{12} \Delta w_{tt} + D\Delta^2 w - \operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right) - D\int_0^\infty g(s)\Delta^2 w(\cdot, t-s)\, ds = h, \tag{2.23}$$

with simply supported boundary condition

$$w = \Delta w = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}. \tag{2.24}$$

Lastly, under a normalization of coefficients, one sees that the 2D viscoelastic Kirchhoff plate model (2.23)–(2.24) is a particular case of problem (1.1)–(1.2) with concrete gradient vector field in the plane $F(z) = |z|^{p-2}z,\ z \in \mathbb{R}^2$.

3 Well-Posedness Result

We start by denoting the functional spaces used throughout this work. For the sake of convenience, we consider the notations and assumptions introduced in [28]. Let

$$V_0 = L^2(\Omega), \quad V_1 = H_0^1(\Omega), \quad V_2 = H^2(\Omega) \cap H_0^1(\Omega),$$

and

$$V_3 = \{u \in H^3(\Omega);\ u = \Delta u = 0 \text{ on } \partial\Omega\},$$

be the Hilbert spaces endowed with their respective norms

$$\|u\|_{V_0} = \|u\|, \quad \|u\|_{V_1} = \|\nabla u\|, \quad \|u\|_{V_2} = \|\Delta u\|, \quad \text{and} \quad \|u\|_{V_3} = \|\nabla\Delta u\|,$$

corresponding to the inner products $(\cdot,\cdot)_{V_i},\ i = 0, 1, 2, 3,$ where $\|\cdot\|$ stands for the usual L^2-norm. Also, $(\cdot,\cdot)_{V_0} = (\cdot,\cdot)$ and $\|\cdot\|_p$ shall denote the L^2-inner product and L^p-norm, respectively. The constants $\lambda_1,\ \lambda_2 > 0$ represent the embedding constants

$$\lambda_1\|u\|^2 \leq \|\Delta u\|^2, \ \ \lambda_2\|\nabla u\|^2 \leq \|\Delta u\|^2, \quad u \in V_2. \tag{3.1}$$

To the relative displacement history, we must consider the L_g^2-weighted Hilbert spaces

$$\mathcal{M}_i = L_g^2(\mathbb{R}^+, V_i) = \left\{\eta : \mathbb{R}^+ \to V_i;\ \int_0^\infty g(s)\|\eta(s)\|_{V_i}^2 ds < \infty\right\},\ i = 0, 1, 2, 3,$$

equipped with inner products and norms

$$(\eta,\zeta)_{\mathcal{M}_i} = \int_0^\infty g(s)(\eta(s),\zeta(s))_{V_i} ds \quad \text{and}$$
$$\|\eta\|_{\mathcal{M}_i}^2 = \int_0^\infty g(s)\|\eta(s)\|_{V_i}^2 ds, \ \ i = 0, 1, 2, 3.$$

The Hilbert phase spaces for solutions along the time are given by

$$\mathcal{H} = V_2 \times V_1 \times \mathcal{M}_2 \quad \text{and} \quad \mathcal{H}_1 = V_3 \times V_2 \times \mathcal{M}_3,$$

equipped with their respective standard norms

$$\|(u, v, \eta)\|_{\mathcal{H}}^2 = \|\Delta u\|^2 + \|\nabla v\|^2 + \|\eta\|_{\mathcal{M}_2}^2$$

and

$$\|(u, v, \eta)\|_{\mathcal{H}_1}^2 = \|\Delta\nabla u\|^2 + \|\Delta v\|^2 + \|\eta\|_{\mathcal{M}_3}^2.$$

Now we give the precise assumptions used in this paper.

(**A1**) $g : [0, \infty) \to \mathbb{R}^+$ is a differentiable function satisfying

$$g(0) > 0, \quad l_0 := \int_0^\infty g(s)ds > 0, \quad l := 1 - l_0 > 0, \tag{3.2}$$

and there exists a constant $k > 0$ such that

$$g'(t) \le -kg(t), \quad t > 0. \tag{3.3}$$

(**A2**) $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-vector field given by $F = (F_1, F_2, \dots, F_n)$ such that

$$F(0) = 0, \quad |\nabla F_j(z)| \le k_j \left(1 + |z|^{\frac{p_j-1}{2}}\right), \quad \forall\, z \in \mathbb{R}^n, \tag{3.4}$$

where, for $j = 1, 2, \dots, n$, we take $k_j > 0$ and p_j satisfying

$$p_j \ge 1 \ \text{ if } \ n = 1, 2 \ \text{ and } \ 1 \le p_j \le \frac{n+2}{n-2} \ \text{ if } \ n \ge 3. \tag{3.5}$$

We additionally suppose that $F = \nabla f$ is a conservative vector field, with $f : \mathbb{R}^n \to \mathbb{R}$ a given function, and there exist constants $\alpha_1 \in [0, \frac{\lambda_2}{2})$ and $\alpha_0 \ge 0$ such that

$$-\alpha_0 - \alpha_1|z|^2 \le f(z) \le F(z)z + \alpha_1|z|^2, \quad \forall\, z \in \mathbb{R}^n. \tag{3.6}$$

Before proceeding with the existence and uniqueness result, let us consider some remarks raised from [28] concerning the assumption (**A2**).

Remark 3.1 1. Condition (3.5) is technical and ensures the following Sobolev embedding $V_2 \hookrightarrow W_0^{1,p_j+1}(\Omega)$, with $\mu_{p_1}, \dots, \mu_{p_n} > 0$ denoting the embedding constants for

$$\|\nabla u\|_{p_j+1} \le \mu_{p_j}\|\Delta u\|, \quad u \in V_2, \quad j = 1, 2, \dots, n. \tag{3.7}$$

2. According to [28, Lemma 4.1], condition (3.4) implies that there exists a positive constant $K = K(k_j, p_j, n)$, $j = 1, 2, \dots, n$, such that

$$|F(z_1) - F(z_2)| \le K \sum_{j=1}^n \left(1 + |z_1|^{\frac{p_j-1}{2}} + |z_2|^{\frac{p_j-1}{2}}\right)|z_1 - z_2|, \quad \forall\, z_1, z_2 \in \mathbb{R}^n. \tag{3.8}$$

3. The nonnegative parameter α_0 in (3.6) will play an important role in the dynamics of the nonlinear evolution operator corresponding to (1.7)–(1.12). Indeed, we observe that the limit situation $\alpha_0 = 0$ was considered in [28, Theorem 2.8] to prove the energy stability. Here, as we are going to clarify later, the condition $\alpha_0 > 0$ will be crucial to guarantee the non-triviality of the attractor whereas the limit case $\alpha_0 = 0$ will lead to a trivial attractor.

Now, for the sake of completeness, we give the definition of weak solution to problem (1.7)–(1.12) for readers not familiar with the subject.

Definition 3.1 Given $T > 0$, $h \in V_0$ and $(u_0, u_1, \eta_0) \in \mathcal{H}$, we say a function in the class $U = (u, u_t, \eta) \in C([0, T], \mathcal{H})$ is a weak solution of the problem (1.7)–(1.12) on $[0, T]$ if $U(0) = (u_0, u_1, \eta_0)$ and

$$\frac{d}{dt}\Big[(u_t(t), \omega) + (\nabla u_t(t), \nabla\omega)\Big] + (\Delta u(t), \Delta\omega) + (F(\nabla u(t)), \nabla\omega) + (\eta^t, \omega)_{\mathcal{M}_2}$$
$$= (h, \omega) \text{ a.e. in } [0, T], \ \forall\, \omega \in V_2, \tag{3.9}$$
$$(\partial_t\eta^t + \partial_s\eta^t, \xi)_{\mathcal{M}_2} = (u_t(t), \xi)_{\mathcal{M}_2} \text{ a.e. in } [0, T], \ \forall\, \xi \in \mathcal{M}_2. \tag{3.10}$$

With this notion of solution, we can state the Hadamard well-posedness of problem (1.7)–(1.12) as follows.

Theorem 3.1 *Let Assumptions* (**A1**)-(**A2**) *be in force and take* $h \in V_0$.

(i) *If* $(u_0, u_1, \eta_0) \in \mathcal{H}$*, then problem* (1.7)–(1.12) *has a weak solution*

$$(u, u_t, \eta) \in C(0, T; \mathcal{H}), \ \textit{with} \ (I - \Delta)u_{tt} \in L^\infty(0, T; V_2'), \ \ \forall\, T > 0, \tag{3.11}$$

where V_2' *stands for the topological dual of* V_2.

(ii) *If* $(u_0, u_1, \eta_0) \in \mathcal{H}_1$*, then problem* (1.7)–(1.12) *has a more regular weak solution in the class*

$$(u, u_t, \eta) \in C(0, T; \mathcal{H}) \cap L^\infty(0, T; \mathcal{H}_1),$$
$$\textit{with} \ (I - \Delta)u_{tt} \in L^\infty(0, T; V_1'), \ \ \forall\, T > 0,$$

where V_1' *stands for the topological dual of* V_1.

(iii) *In both cases, one has continuous dependence on initial data in* $\mathcal{H}$*. More precisely, given any two weak solutions* $U_j(t) = (u^j(t), u_t^j(t), \eta^{j,t})$, $j = 1, 2$, *of problem* (1.7)–(1.12)*, then*

$$\|U_1(t) - U_2(t)\|_{\mathcal{H}}^2 \le e^{c_0 t}\|U_1(0) - U_2(0)\|_{\mathcal{H}}^2, \quad \forall\, t \in [0, T], \ T > 0, \tag{3.12}$$

for some constant $c_0 > 0$ *depending on* $\mathcal{H}$*-initial data. In particular, problem* (1.7)–(1.12) *has a unique solution.*

Proof The proof can be done by using the Faedo-Galerkin method and combining the arguments from [14,20,21,25,28]. Therefore, we omit it here. □

4 Main Result

Let us define the one-parameter family of operators $S(t) : \mathcal{H} \to \mathcal{H}$ by

$$S(t)(u_0, u_1, \eta_0) = (u(t), u_t(t), \eta^t), \quad t \geq 0, \tag{4.1}$$

where (u, u_t, η) is the unique solution of problem (1.7)–(1.12) ensured by Theorem 3.1. Thus, the pair $(\mathcal{H}, S(t))$ constitutes a dynamical system that will describe the long-time behavior of problem (1.7)–(1.12).

Before introducing the main result of this article, let us first consider some preliminary concepts coming from the general theory that can be applied to our particular dynamical system $(\mathcal{H}, S(t))$ generated by (4.1). Here, just for a reason of adaptation, we follow the book by Chueshov and Lasiecka [8,9]. However, for readers interested in other approaches within the general theory in dynamical systems, we also refer to [4,13,24,36], among others.

- The dynamical system $(\mathcal{H}, S(t))$ given in (4.1) is called *quasi-stable* on a set $B \subset \mathcal{H}$ (in accordance with [9, Definition 7.9.2]) if there exist a compact seminorm $n_X(\cdot, \cdot)$ on $X \supset V_2$ and nonnegative scalar functions $a(t)$ and $c(t)$ locally bounded in $[0, \infty)$, and $b(t) \in L^1(\mathbb{R}^+)$ with $\lim_{t\to\infty} b(t) = 0$, such that

$$\|S(t)U_1 - S(t)U_2\|^2_{\mathcal{H}} \leq a(t)\|U_1 - U_2\|^2_{\mathcal{H}}, \tag{4.2}$$

and

$$\|S(t)U_1 - S(t)U_2\|^2_{\mathcal{H}} \leq b(t)\|U_1 - U_2\|^2_{\mathcal{H}} + c(t) \sup_{s\in[0,t]} \Big[n_X(u^1(s) - u^2(s))\Big]^2, \tag{4.3}$$

for any $U_1, U_2 \in B$, where we have denoted

$$S(t)U_j = U_j(t) = (u^j(t), u^j_t(t), \eta^{j,t}), \quad j = 1, 2.$$

- A *global attractor* for $(\mathcal{H}, S(t))$ is a bounded closed set $\mathcal{A} \subset \mathcal{H}$ which is fully invariant and uniformly attracting, namely, $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and

$$\lim_{t\to+\infty} \text{dist}_{\mathcal{H}}(S(t)B, \mathcal{A}) = 0, \quad \text{for every bounded subset } B \subset \mathcal{H}.$$

- A *global minimal attractor* for $(\mathcal{H}, S(t))$ is a bounded closed set $\mathcal{A}_{\min} \subset \mathcal{H}$ which is positively invariant ($S(t)\mathcal{A}_{\min} \subseteq \mathcal{A}_{\min}$) and attracts uniformly every point, that is,

$$\lim_{t\to+\infty} \text{dist}_{\mathcal{H}}(S(t)U_0, \mathcal{A}_{\min}) = 0, \quad \text{for any } U_0 \in \mathcal{H},$$

and $\mathcal{A}_{\min}$ has no proper subsets possessing these two properties.

- The *unstable manifold* emanating from a set $\mathcal{N}$, denoted by $\mathbb{M}_+(\mathcal{N})$, is a set of $\mathcal{H}$ such that for each $U_0 \in \mathbb{M}_+(\mathcal{N})$ there exists a full trajectory $\Gamma = \{\mathbf{U}(t) \mid t \in \mathbb{R}\}$ satisfying

$$\mathbf{U}(0) = U_0 \quad \text{and} \quad \lim_{t\to-\infty} \mathrm{dist}_{\mathcal{H}}(\mathbf{U}(t), \mathcal{N}) = 0.$$

- The *fractal dimension* of a compact set $\mathcal{A} \subset \mathcal{H}$ is defined by

$$\dim^f_{\mathcal{H}} \mathcal{A} = \limsup_{\varepsilon\to 0} \frac{\ln n(\mathcal{A}, \varepsilon)}{\ln(1/\varepsilon)},$$

where $n(\mathcal{A}, \varepsilon)$ is the minimal number of closed balls in $\mathcal{H}$ of radius ε which covers $\mathcal{A}$. It is very well-known that the Hausdorff dimension does not exceed the fractal one ([24, Chap. 2]) so that it is enough to achieve finiteness of the fractal dimension.
- A compact set $\mathcal{A}_{\exp} \subset \mathcal{H}$ is said to be a *fractal exponential attractor* of the dynamical system $(\mathcal{H}, S(t))$ if $\mathcal{A}_{\exp}$ is a positively invariant set of finite fractal dimension in $\mathcal{H}$ and for every bounded set $B \subset \mathcal{H}$ there exist positive constants t_B, C_B and σ_B such that

$$\sup_{U_0\in B} \mathrm{dist}_{\mathcal{H}}(S(t)U_0, \mathcal{A}_{\exp}) \leq C_B\, e^{-\sigma_B(t-t_B)}, \quad t \geq t_B.$$

If there exists an exponential attractor only having finite dimension in some extended space $\widetilde{\mathcal{H}} \supseteq \mathcal{H}$, then this exponentially attracting set is called *generalized fractal exponential attractor*.

On the light of the above concepts, we are now in position to state the main result on long-time dyanmics as follows.

Theorem 4.1 *Under the assumptions of Theorem 3.1, with the additionally hypotheses* $p_j < \dfrac{n+2}{n-2}$ *if* $n \geq 3$ *in* (3.5) *and* $\alpha_0 > 0$ *in* (3.6)*, we have:*

I. *The dynamical system $(\mathcal{H}, S(t))$ given in (4.1) is quasi-stable on any bounded positively invariant set $B \subset \mathcal{H}$.*

II. *The dynamical system $(\mathcal{H}, S(t))$ possesses a unique compact global attractor $\mathcal{A} \subset \mathcal{H}$, which is characterized by the unstable manifold $\mathcal{A} = \mathbb{M}_+(\mathcal{N})$, emanating from the set $\mathcal{N} = \big\{(u, 0, 0) \in \mathcal{H};\ \Delta^2 u - \mathrm{div} F(\nabla u) = h\big\}$ of stationary solutions.*

III. *Every trajectory stabilizes to the set $\mathcal{N}$, namely, for any $U \in \mathcal{H}$ one has*

$$\lim_{t\to+\infty} \mathit{dist}_{\mathcal{H}}(S(t)U, \mathcal{N}) = 0.$$

In particular, there exists a global minimal attractor $\mathcal{A}_{\min}$ given by $\mathcal{A}_{\min} = \mathcal{N}$.

IV. *The above global minimal attractor $\mathcal{A}_{\min} = \mathcal{N}$ is nontrivial. In other words, even if $h \equiv 0$, the set $\mathcal{N}$ has at least two stationary solutions.*

V. *The attractor $\mathcal{A}$ has finite fractal and Hausdorff dimension $\dim^f_{\mathcal{H}} \mathcal{A}$.*

VI. *Every trajectory* $\Gamma = \{(u(t), u_t(t), \eta^t)\ ;\ t \in \mathbb{R}\}$ *from the attractor* $\mathcal{A}$ *has the smoothness property*

$$(u_t, u_{tt}, \eta_t) \in L^\infty(\mathbb{R}; \mathcal{H}). \tag{4.4}$$

Moreover, there exists a constant $R > 0$ *such that*

$$\sup_{\Gamma \subset \mathcal{A}} \sup_{t \in \mathbb{R}} \left(\|\nabla u_{tt}(t)\|_2^2 + \|\Delta u_t(t)\|_2^2 + \|\eta_t^t\|_{\mathcal{M}_2}^2 \right) \le R^2. \tag{4.5}$$

VII. *The dynamical system* $(\mathcal{H}, S(t))$ *possesses a generalized fractal exponential attractor* $\mathcal{A}_{\exp}$ *with finite dimension in the extended space*

$$\widetilde{\mathcal{H}} := V_1 \times V_0 \times \mathcal{M}_1.$$

In addition, the fractal exponential attractor $\mathcal{A}_{\exp}$ *has finite fractal dimension in a smaller extended space* $\mathcal{H}_\delta$, *where*

$$\mathcal{H} := \mathcal{H}^0 \subsetneq \mathcal{H}^\delta \subseteq \mathcal{H}^1 := \widetilde{\mathcal{H}}, \quad 0 < \delta \le 1.$$

VIII. *In the limit case* $\alpha_0 = 0$ *and* $h \equiv 0$, *the attractor* $\mathcal{A}$ *is trivial. More precisely,* $\mathcal{A} = \{(0, 0, 0)\}$ *with exponential attraction*

$$\|S(t)U_0\|_{\mathcal{H}} \le C\, e^{-ct} \tag{4.6}$$

for any initial data U_0 *lying in bounded sets* $B \subset \mathcal{H}$ *and some constants* $C = C(B) > 0$, $c = c(B) > 0$ *depending on* B.

The proof of Theorem 4.1 shall be done later, in Sect. 5.2, as a consequence of several technical results along with abstract theory coming from dynamical systems.

5 Proofs

5.1 Technical Results

We start by defining the energy functional corresponding to the weak solution $(u, u_t, \eta) \in \mathcal{H}$ of problem (1.7)–(1.12) as

$$E(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{2}\|\eta^t\|_{\mathcal{M}_2}^2 + \int_\Omega f(\nabla u(t))dx - \int_\Omega hu(t)dx. \tag{5.1}$$

Lemma 5.1 *Under the above notations, there exist positive constants* $\beta_0 > 0$ *and* $\overline{K} = \overline{K}(|\Omega|, \|h\|) > 0$ *such that the energy* $E(t)$ *satisfies*

$$E(t) \ge \beta_0(\|u_t(t)\|^2 + \|\nabla u_t(t)\|^2 + \|\Delta u(t)\|^2 + \|\eta^t\|_{\mathcal{M}_2}^2) - \overline{K}, \quad t \ge 0. \tag{5.2}$$

Proof Let us denote

$$\tilde{E}(t) = E(t) + \overline{K},$$

where

$$\overline{K} = \alpha_0|\Omega| + \frac{1}{2\lambda_1\left(\frac{1}{2} - \frac{\alpha_1}{\lambda_2}\right)}\|h\|^2. \tag{5.3}$$

From (3.6), (3.1) and Young's inequality with $\varrho > 0$ it follows that

$$\int_\Omega f(\nabla u(t))dx \geq \int_\Omega \left(-\alpha_0 - \alpha_1|\nabla u(t)|^2\right)dx \geq -\alpha_0|\Omega| - \frac{\alpha_1}{\lambda_2}\|\Delta u(t)\|^2,$$

and

$$-\int_\Omega hu(t)dx \geq -\frac{\varrho}{4}\|\Delta u(t)\|^2 - \frac{1}{\lambda_1\varrho}\|h\|^2,$$

Now, since $\alpha_1 \in [0, \frac{\lambda_2}{2})$, then choosing $\varrho = 2\left(\frac{1}{2} - \frac{\alpha_1}{\lambda_2}\right) > 0$ we have

$$\begin{aligned}\tilde{E}(t) &\geq \frac{1}{2}\left(\frac{1}{2} - \frac{\alpha_1}{\lambda_2}\right)\|\Delta u(t)\|^2 + \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{1}{2}\|\eta^t\|^2_{\mathcal{M}_2}\\ &\geq \frac{1}{2}\left(\frac{1}{2} - \frac{\alpha_1}{\lambda_2}\right)\left[\|\Delta u(t)\|^2 + \|u_t(t)\|^2 + \|\nabla u_t(t)\|^2 + \|\eta^t\|^2_{\mathcal{M}_2}\right].\end{aligned}$$

Therefore, (5.2) follows by taking $\beta_0 = \frac{1}{2}\left(\frac{1}{2} - \frac{\alpha_1}{\lambda_2}\right) > 0$, which concludes the proof. □

Lemma 5.2 *Under the above notations, we have:*

- *there exists a strict Lyapunov functional φ for the dynamical system $(\mathcal{H}, S(t))$ given in (4.1). In other words, $(\mathcal{H}, S(t))$ is gradient;*
- *the Lyapunov functional Φ is bounded from above on any bounded subset of $\mathcal{H}$;*
- *the set $\Phi_R = \{U \in \mathcal{H}\,;\ \Phi(U) \leq R\}$ is bounded in $\mathcal{H}$ for every $R > 0$.*

Proof Firstly, taking the multiplier u_t with equation (1.7), integrating by parts, using equation (1.8) and boundary conditions, and also condition (3.3), then the energy $E(t) = E(u(t), u_t(t), \eta^t)$ set in (5.1) satisfies

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta^t(s)\|^2 ds \leq -\frac{k}{2}\int_0^\infty g(s)\|\Delta\eta^t(s)\|^2 ds \leq 0, \quad t > 0, \tag{5.4}$$

which implies that the mapping $t \mapsto E(t)$ is non-increasing. In this way, let us take the functional $\Phi := E$ and from notation introduced in (4.1) one reads that

$t \mapsto \Phi(S(t)U_0)$ is non-increasing for every $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}$. Moreover, from (5.4) one gets

$$\Phi(S(t)U_0) + \frac{k}{2}\int_0^t \int_0^\infty g(s)\|\Delta\eta^t(s)\|^2 ds dt \leq \Phi(U_0), \quad U_0 \in \mathcal{H}. \tag{5.5}$$

Thus, supposing that $\Phi(S(t)U_0) = \Phi(U_0)$ for any $t > 0$, yields

$$\|\Delta\eta^t(s)\| = 0 \quad \text{a.e. } t, s > 0.$$

from where we deduce that $\eta^t(s) = 0$ for $t, s > 0$. Also, from Eq. (1.8) we conclude $\partial_t u(t) = 0$ for $t > 0$, and so $u(t) = u_0$ for all $t \geq 0$. Therefore, $S(t)U_0 = (u_0, 0, 0)$ is a stationary solution, that is, $S(t)U_0 = U_0$ for all $t > 0$. This proves that Φ is a strict Lyapunov functional for the dynamical system $(\mathcal{H}, S(t))$.

Additionally, from (5.5) we have $\Phi(S(t)U_0) \leq \Phi(U_0)$ and, therefore, it is trivial to conclude that Φ is bounded from above on bounded subsets of $\mathcal{H}$.

Finally, given any weak solution $(u(t), u_t(t), \eta^t) = S(t)U_0 \in \mathcal{H}$ of problem (1.7)–(1.12) such that $\Phi(S(t)U_0) \leq R$, then we infer from (5.2) that

$$\|S(t)U_0\|^2_{\mathcal{H}} \leq \frac{1}{\beta_0}(R + \overline{K}), \quad t \geq 0,$$

from where one concludes that Φ_R is a bounded set of $\mathcal{H}$ for every $R > 0$. □

Lemma 5.3 *Under the above notations, we have:*

- *The set $\mathcal{N} = \left\{(u, 0, 0) \in \mathcal{H};\ \Delta^2 u - div F(\nabla u) = h\right\}$ of stationary solutions to problem (1.7)–(1.12) is bounded in $\mathcal{H}$;*
- *In particular, if $\alpha_0 = 0$ in (3.6) and $h \equiv 0$, then $\mathcal{N} = \{(0, 0, 0)\} \subset \mathcal{H}$ is trivial.*

Proof Taking the multiplier u in the equation $\Delta^2 u - \text{div} F(\nabla u) = h$, and integrating by parts, we have

$$\|\Delta u\|^2 = -\int_\Omega F(\nabla u)\nabla u\, dx + \int_\Omega hu\, dx. \tag{5.6}$$

Using condition (3.6) and also (3.1), we infer

$$-\int_\Omega F(\nabla u)\nabla u dx \leq \alpha_0|\Omega| + \frac{2\alpha_1}{\lambda_2}\|\Delta u\|^2.$$

From Young's inequality with $\varrho > 0$ and again from (3.1), we deduce

$$\int_\Omega hu dx \leq \frac{\varrho}{4}\|\Delta u\|^2 + \frac{1}{\lambda_1\varrho}\|h\|^2.$$

Going back to (5.6), we obtain

$$\left(1 - \frac{2\alpha_1}{\lambda_2} - \frac{\varrho}{4}\right)\|\Delta u\|^2 \leq \alpha_0|\Omega| + \frac{1}{\lambda_1\varrho}\|h\|^2. \tag{5.7}$$

Taking $\varrho > 0$ small enough, one concludes that $\mathcal{N}$ is bounded.

In particular, for $\alpha_0 = 0$ and $h \equiv 0$, then (5.7) implies that $u = 0$ and $\mathcal{N}$ is trivial. □

Lemma 5.4 *Under the above notations, if* $\alpha_0 > 0$ *in* (3.6) *and* $h \equiv 0$*, then the set*

$$\mathcal{N}_0 = \left\{(u, 0, 0) \in \mathcal{H};\ \Delta^2 u - divF(\nabla u) = 0\right\}$$

has a nontrivial weak solution $u \neq 0$*. In conclusion, the set* $\mathcal{N}_0$ *(and therefore* $\mathcal{N}$*) has at least two stationary solutions.*

Proof Since $F(0) = 0$, then obviously $u = 0$ is the trivial stationary solution of problem $\Delta^2 u - \operatorname{div} F(\nabla u) = 0$. Let us deal with the case of nontrivial weak solution for $\mathcal{N}_0$.

To fix the idea, let us take the concrete case

$$F(z) = |z|^q z - \lambda |z|^r z, \quad q > r > 0, \quad \lambda > 0, \tag{5.8}$$

with proper $q, r, \lambda > 0$. Therefore, $F = \nabla f$, where $f(z) = \frac{1}{q+2}|z|^{q+2} - \frac{\lambda}{r+2}|z|^{r+2}$, satisfy assumption (**A2**) with $\alpha_0 > 0$ in (3.6). This will be clarified in Example A.3 later.

Now, let us consider the elliptic problem

$$\begin{cases} \Delta^2 u - \operatorname{div}\left(|\nabla u|^q \nabla u - \lambda |\nabla u|^r \nabla u\right) = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.9}$$

where $\lambda > 0$ and $0 < r < q < \frac{4}{n-2}$ if $n \geq 3$. In what follows, we are going to prove that there exists a nontrivial weak solution $u \in V_2 = H^2(\Omega) \cap H_0^1(\Omega)$ for the particular problem (5.9) and, therefore, the same happens with the more general problem as well.

We define the functional I_M whose Euler-Lagrange equation corresponds to (5.9). In this case, $I_M : V_2 \to \mathbb{R}$ is given by

$$I_M(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx + \frac{1}{q+2}\int_\Omega |\nabla u|^{q+2} dx - \frac{\lambda}{r+2}\int_\Omega |\nabla u|^{r+2} dx.$$

We first claim that for all $\lambda > 0$, I_M is coercive and bounded from below. Indeed, since $0 < r < q$, then from the embedding $L^q \hookrightarrow L^r$ (with constant $C > 0$) we get

$$\begin{aligned} I_M(u) &\geq \frac{1}{2}\|\Delta u\|^2 + \frac{1}{q+2}\|\nabla u\|_{q+2}^{q+2} - \frac{\lambda C}{r+2}\|\nabla u\|_{q+2}^{r+2} \\ &\geq \frac{1}{2}\|\Delta u\|^2 + C_0, \end{aligned} \tag{5.10}$$

where

$$C_0 = \inf_{\tau \geq 0}\left\{\frac{\tau^{q+2}}{q+2} - \lambda C \frac{\tau^{r+2}}{r+2}\right\}.$$

Thus, from (5.10) one sees that I_M is clearly coercive and bounded from below. On the other hand, by fixing $0 \neq u \in V_2$, we note that there exists $\lambda_0 > 0$ such that $I_M(u) < 0$. Then, for such λ_0, by taking a minimizing sequence, i.e., a sequence $(u_n) \subset V_2$ such that

$$\lim_{n\to+\infty} I_M(u_n) = \alpha := \inf_{V_2} I_M,$$

from the coerciveness it follows that (u_n) is bounded and then, up to a subsequence, that $u_n \rightharpoonup u$. From the compactness of the embeddings of V_2 in $W_0^{1,q+2}(\Omega)$ and $W_0^{1,r+2}(\Omega)$, it follows that

$$\alpha \le I_M(u) \le \liminf_{n\to+\infty} I_M(u_n) = \alpha,$$

which this implies that u is a global minimizer and then a nontrivial critical point of I_M. Since critical points of I_M correspond to weak solutions of (5.9), the result follows. $\square$

The next result will be crucial to reach the existence of a global attractor for the dynamical system $(\mathcal{H}, S(t))$ and its properties as well. It provides an inequality usually called *stabilizability inequality* that will be a key point for all achievements stated in Theorem 4.1.

Proposition 5.5 (Stabilizability inequality) *Under the above notations and assumptions of Theorem (4.1), let* $S(t)U^i = (u^i(t), u_t^i(t), \eta^{i,t})$, $t \ge 0$, *for each* $i = 1, 2$, *be the weak solution of problem* (1.7)–(1.12) *with initial data* U^i *lying in a bounded set* $B \subset \mathcal{H}$. *Then, there exist constants* $b_0 > 0$ *and* γ_B, $C_B > 0$ *such that*

$$\begin{aligned}&\|S(t)U^1 - S(t)U^2\|_{\mathcal{H}}^2\\ &\quad \le b_0 e^{-\gamma_B t}\|U^1 - U^2\|_{\mathcal{H}}^2 + C_B \int_0^t e^{-\gamma_B(t-s)}\|\nabla(u_1 - u_2)(s)\|_{p_j+1}^2 ds,\end{aligned} \tag{5.11}$$

for every $t \ge 0$.

Proof Let us first denote $u = u^1 - u^2$ and $\eta = \eta^1 - \eta^2$. Thus, the triplet $(u(t), u_t(t), \eta^t) = S(t)U^1 - S(t)U^2$, $t \ge 0$, is a solution for

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + \int_0^\infty g(s)\Delta^2\eta(s)ds = \operatorname{div} F(\nabla u^1) - \operatorname{div} F(\nabla u^2), \tag{5.12}$$

$$\eta_t + \eta_s = u_t, \tag{5.13}$$

with initial conditions

$$(u(0), u_t(0), \eta^0) = U^1 - U^2. \tag{5.14}$$

The energy functional corresponding to system (5.12)–(5.13) is defined as

$$G(t)=\frac{1}{2}\|u_t(t)\|^2+\frac{1}{2}\|\nabla u_t(t)\|^2+\frac{1}{2}\|\Delta u(t)\|^2+\frac{1}{2}\|\eta(t)\|^2_{\mathcal{M}_2}. \tag{5.15}$$

Since the proof of (5.11) is not so short and technical, then we will work in some steps as follows. Besides, in some large formulas we will omit the parameter t for convenience.
Step 1. Given $\epsilon>0$, we claim that there exists a constant $C_{B,\epsilon}>0$, depending on B and $\epsilon>0$, such that

$$\begin{aligned} G'(t)\le \epsilon\|\nabla u_t(t)\|^2+\frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta(s)\|^2ds \\ +C_\epsilon\|\nabla u(t)\|^2_{p_j+1},\quad j=1,2,\ldots,n. \end{aligned} \tag{5.16}$$

Indeed, taking the multiplier u_t in (5.12), using (5.13) and integration by parts, we get

$$G'(t)=\frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta^t(s)\|^2ds-\int_\Omega \nabla u_t(t)(F(\nabla u^1(t))-F(\nabla u^2(t)))dx. \tag{5.17}$$

Applying (3.8) and using Hölder and Young's inequalities, we conclude for every $j=1,2,\ldots,n$, and for any $\epsilon>0$, that

$$\begin{aligned} &-\int_\Omega \nabla u_t(t)\ (F(\nabla u^1(t))-F(\nabla u^2(t)))dx \\ &\le C_1\sum_{j=1}^n\int_\Omega\left(1+|\nabla u^1|^{\frac{p_j-1}{2}}+|\nabla u^2|^{\frac{p_j-1}{2}}\right)|\nabla u||\nabla u_t|dx \\ &\le C_1\sum_{j=1}^n\left(|\Omega|^{\frac{p_j-1}{2(p_j+1)}}+\|\nabla u^1\|^{\frac{p_j-1}{2}}_{p_j+1}+\|\nabla u^2\|^{\frac{p_j-1}{2}}_{p_j+1}\right)\|\nabla u\|_{p_j+1}\|\nabla u_t\| \\ &\le C_B\|\nabla u\|_{p_j+1}\|\nabla u_t\| \\ &\le \epsilon\|\nabla u_t\|^2+C_{B,\epsilon}\|\nabla u\|^2_{p_j+1}. \end{aligned} \tag{5.18}$$

Thus (5.16) follows from (5.17) and (5.18).
Step 2. Let us define the functional

$$\phi(t)=\int_\Omega\Big(u_t(t)-\Delta u_t(t)\Big)u(t)dx.$$

Then, there exists a constant $C_1>0$ such that for every $j=1,2,\ldots,n$ and for any $t>0$,

$$\phi'(t) \leq -G(t) - \frac{1}{4}\|\Delta u(t)\|^2 + \frac{3}{2}\|u_t(t)\|^2 + \frac{3}{2}\|\nabla u_t(t)\|^2$$
$$+ C_1 \int_0^\infty g(s)\|\Delta\eta(s)\|^2 ds + C_B\|\nabla u(t)\|^2_{p_j+1}. \tag{5.19}$$

In fact, taking the derivative of ϕ and using (5.12) we obtain

$$\phi'(t) = \|u_t\|^2 + \|\nabla u_t\|^2 - \|\Delta u\|^2 - \int_\Omega \Delta u(t) \int_0^\infty g(s)\Delta\eta(s) ds dx$$
$$- \int_\Omega (F(\nabla u^1(t)) - F(\nabla u^2(t)))\nabla u dx. \tag{5.20}$$

Hölder and Young's inequality with $\delta > 0$, gives

$$- \int_\Omega \Delta u(t) \int_0^\infty g(s)\Delta\eta(s) ds dx \leq \delta\|\Delta u\|^2 + \frac{1-l}{4\delta} \int_0^\infty g(s)\|\Delta\eta(s)\|^2 ds. \tag{5.21}$$

Using the same estimate as in (5.18), we infer that for any $\delta > 0$,

$$- \int_\Omega (F(\nabla u^1(t)) - F(\nabla u^2(t)))\nabla u dx \leq C_B\|\nabla u\|_{p_j+1}\|\nabla u\|^2$$
$$\leq \delta\|\Delta u\|^2 + \frac{C_B}{4\delta\lambda_2}\|\nabla u\|^2_{p_j+1}. \tag{5.22}$$

Replacing (5.21)–(5.22) in (5.20) and regarding (5.15), we have that for any $\delta > 0$,

$$\phi'(t) \leq -G(t) - \left(\frac{1}{2} - 2\delta\right)\|\Delta u\|^2 + \frac{3}{2}\|u_t\|^2 + \frac{3}{2}\|\nabla u_t\|^2$$
$$+ \left(\frac{1}{2} + \frac{1-l}{4\delta}\right) \int_0^\infty g(s)\|\Delta\eta(s)\|^2 ds + C_B\|\nabla u\|^2_{p_j+1}. \tag{5.23}$$

Taking $\delta > 0$ small enough so that $\frac{1}{2} - 2\delta > \frac{1}{4}$, then we obtain (5.19) with $C_1 = \frac{1}{2} + \frac{1-l}{4\delta}$.
Step 3. Let us define the functional

$$\psi(t) = - \int_\Omega \left(u_t(t) - \Delta u_t(t)\right) \left(\int_0^\infty g(s)\eta^t(s) ds\right) dx.$$

Then, there exists a positive constant C_2 such that for every $j = 1, 2, \ldots, n$ and for any $\delta_1 > 0$,

$$\psi'(t) \leq -\frac{3}{4} l_0\|u_t(t)\|^2 - \frac{3}{4} l_0\|\nabla u_t(t)\|^2 + \delta_1\|\Delta u(t)\|^2$$
$$-C_2 \int_0^\infty g'(s)\|\Delta\eta^t(s)\|^2 ds + C_B\|\nabla u(t)\|^2_{p_j+1}. \tag{5.24}$$

In fact, taking the derivative of $\psi(t)$ and using (5.12), we have

$$
\begin{aligned}
\psi'(t) = {} & \underbrace{\int_\Omega \Delta u(t) \int_0^\infty g(s)\Delta\eta^t(s)dsdx}_{:=I_1} + \underbrace{\int_\Omega \left(\int_0^\infty g(s)\Delta\eta^t(s)ds\right)^2}_{:=I_2} \\
& + \underbrace{\int_\Omega (F(\nabla u^1) - F(\nabla u^2)) \int_0^\infty g(s)\nabla\eta^t(s)dsdx}_{:=I_3} \\
& \underbrace{- \int_\Omega (u_t - \Delta u_t) \int_0^\infty g(s)\eta_t^t(s)ds}_{:=I_4}.
\end{aligned}
\tag{5.25}
$$

By using again Hölder's inequality and Young's inequality with $\delta_1 > 0$, we infer

$$
I_1 \le \delta_1 \|\Delta u\|^2 + \frac{1-l}{4\delta_1} \int_0^\infty g(s)\|\Delta\eta(s)\|^2 ds, \tag{5.26}
$$

$$
I_2 \le (1-l) \int_0^\infty g(s)\|\Delta\eta(s)\|^2 ds, \tag{5.27}
$$

and

$$
\begin{aligned}
I_3 & \le C_B \|\nabla u\|_{p_j+1} \int_0^\infty g(s)\,\|\nabla\eta(s)\|\,ds \\
& \le \frac{C_B}{2}\|\nabla u\|_{p_j+1}^2 + \frac{C_B(1-l)}{2\lambda_2} \int_0^\infty g(s)\|\Delta\eta(s)\|^2 ds.
\end{aligned}
\tag{5.28}
$$

Noting that

$$
\int_0^\infty g(s)\eta_t^t(s)ds = \int_0^\infty g'(s)\eta^t(s)ds + l_0 u_t,
$$

we can easily get for any $t > 0$,

$$
\begin{aligned}
I_4 & = -l_0\|u_t\|^2 - l_0\|\nabla u_t\|^2 - \int_\Omega u_t \int_0^\infty g'(s)\eta(s)ds - \int_\Omega \nabla u_t \int_0^\infty g'(s)\nabla\eta(s)ds \\
& \le -\frac{3}{4} l_0\|u_t\|^2 - \frac{3}{4} l_0\|\nabla u_t\|^2 \\
& \quad + \frac{1}{l_0}\int_\Omega \left(\int_0^\infty -g'(s)ds\right)\left(\int_0^\infty -g'(s)|\eta(s)|^2 ds\right)dx \\
& \quad + \frac{1}{l_0}\int_\Omega \left(\int_0^\infty -g'(s)ds\right)\left(\int_0^\infty -g'(s)|\nabla\eta(s)|^2 ds\right)dx \\
& \le -\frac{3}{4} l_0\|u_t\|^2 - \frac{3}{4} l_0\|\nabla u_t\|^2 - \left(\frac{g(0)}{l_0\lambda_1} + \frac{g(0)}{l_0\lambda_2}\right)\int_0^\infty g'(s)\|\Delta\eta(s)\|^2 ds.
\end{aligned}
\tag{5.29}
$$

Combining (5.26)–(5.29) with (5.25), we have the next estimate for any $\delta_1 > 0$

$$\begin{aligned}\psi'(t) \le & -\frac{3}{4}l_0\|u_t\|^2 - \frac{3}{4}l_0\|\nabla u_t\|^2 + \delta_1\|\Delta u\|^2 + C_B\|\nabla u\|^2_{p_j+1} \\ & + \left(\frac{1-l}{4\delta_1} + (1-l) + \frac{C_B(1-l)}{2\lambda_2}\right)\int_0^\infty g(s)\|\Delta\eta(s)\|^2 ds \\ & - \left(\frac{g(0)}{l_0\lambda_1} + \frac{g(0)}{l_0\lambda_2}\right)\int_0^\infty g'(s)\|\Delta\eta(s)\|^2 ds.\end{aligned}$$

Now, taking into account (3.3), we finally obtain (5.24) with

$$C_2 = \frac{1-l}{4\delta_1 k} + \frac{1-l}{k} + \frac{C_B(1-l)}{2\lambda_2 k} + \frac{g(0)}{l_0\lambda_1} + \frac{g(0)}{l_0\lambda_2}.$$

Step 4. Defining the Lyapunov perturbed functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) := G(t) + \varepsilon_1\phi(t) + \varepsilon_2\psi(t), \tag{5.30}$$

with $\varepsilon_1, \varepsilon_2 > 0$ to be determined later, then it holds that

$$\frac{1}{2}G(t) \le \mathcal{L}(t) \le \frac{3}{2}G(t), \quad t \ge 0, \tag{5.31}$$

for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough.

Indeed, by using Hölder's inequality and Young's inequality with $\varrho > 0$, one gets

$$\begin{aligned}|\mathcal{L}(t) - G(t)| \le & \ \varepsilon_1\left(\varrho\|u_t\|^2 + \frac{1}{4\varrho}\|u\|^2\right) + \varepsilon_1\left(\varrho\|\nabla u_t\|^2 + \frac{1}{4\varrho}\|\nabla u\|^2\right) \\ & + \varepsilon_2\left(\varrho\|u_t\|^2 + \frac{l_0}{4\varrho}\int_0^\infty g(s)\|\eta(s)\|^2 ds\right) \\ & + \varepsilon_2\left(\varrho\|\nabla u_t\|^2 + \frac{l_0}{4\varrho}\int_0^\infty g(s)\|\nabla\eta(s)\|^2 ds\right) \\ \le & \ \varrho(\varepsilon_1+\varepsilon_2)\|u_t\|^2 + \varrho(\varepsilon_1+\varepsilon_2)\|\nabla u_t\|^2 + \left(\frac{\varepsilon_1}{4\varrho\lambda_1} + \frac{\varepsilon_1}{4\varrho\lambda_2}\right)\|\Delta u\|^2 \\ & + \left(\frac{\varepsilon_2 l_0}{4\varrho\lambda_1} + \frac{\varepsilon_2 l_0}{4\varrho\lambda_2}\right)\|\eta^t\|_{\mathcal{M}_2}.\end{aligned}$$

Then there exists a constant $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2) > 0$ such that

$$|\mathcal{L}(t) - G(t)| \le \varepsilon G(t),$$

which implies (5.31) by taking $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ sufficiently small.

Step 5. Conclusion of the proof. Combining (5.16), (5.20) and (5.24), we have the following estimate for every $j = 1, 2, \dots, n$ and for any $t > 0$, $\epsilon > 0$,

$$\begin{aligned} \mathcal{L}'(t) &= G'(t) + \varepsilon_1 \phi'(t) + \varepsilon_2 \psi'(t) \\ &\leq -\varepsilon_1 G(t) - \left(\frac{3l_0}{4}\varepsilon_2 - \frac{3}{2}\varepsilon_1\right)\|u_t(t)\|^2 - \left(\frac{3l_0}{4}\varepsilon_2 - \frac{3}{2}\varepsilon_1 - \epsilon\right)\|\nabla u_t(t)\|^2 \\ &\quad - \left(\frac{1}{4}\varepsilon_1 - \delta_1\varepsilon_2\right)\|\Delta u(t)\|^2 + \left(\frac{1}{2} - \frac{C_1\varepsilon_1}{k} - C_2\varepsilon_2\right)\int_0^\infty g'(s)\|\Delta\eta(s)\|^2 ds \\ &\quad + C_B\|\nabla u(t)\|^2_{p_j+1}. \end{aligned} \tag{5.32}$$

At this point, we choose $\delta_1 > 0$ small enough such that $\delta_1 < \frac{l_0}{8}$, which implies

$$\frac{\varepsilon_1}{4\delta_1} > \frac{2\varepsilon_1}{l_0}.$$

Then, for $\delta_1 > 0$ fixed, we pick up $\varepsilon_1 > 0$ small enough such that (5.31) holds, and further

$$\varepsilon_1 < \min\left\{\frac{k}{4C_1}, \frac{l_0}{16C_2}\right\} \Rightarrow \frac{1}{2} - \frac{C_1\varepsilon_1}{k} > \frac{1}{4}, \quad \frac{1}{8C_2} > \frac{2\varepsilon_1}{l_0}.$$

In addition, fixed numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$, we take $\varepsilon_2 > 0$ small enough so that (5.31) holds, and also

$$\frac{2\varepsilon_1}{l_0} < \varepsilon_2 < \min\left\{\frac{\varepsilon_1}{4\delta_1}, \frac{1}{8C_2}\right\},$$

which implies

$$\frac{3l_0}{4}\varepsilon_2 - \frac{3}{2}\varepsilon_1 > 0, \quad \frac{1}{4}\varepsilon_1 - \delta_1\varepsilon_2 > 0, \quad \frac{1}{4} - C_2\varepsilon_2 > \frac{1}{8}.$$

At last, we take $\epsilon > 0$ small enough so that

$$\frac{3l_0}{4}\varepsilon_2 - \frac{3}{2}\varepsilon_1 - \epsilon > 0.$$

In light of above estimates, we deduce from (5.32) and then (5.31) that there exists a constant $\gamma_0 > 0$ such that

$$\mathcal{L}'(t) \leq -2\gamma_0\mathcal{L}(t) + C_B\|\nabla u(t)\|^2_{p_j+1},$$

which gives us

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-2\gamma_0 t} + C_B\int_0^t e^{-2\gamma_0(t-s)}\|\nabla u(s)\|^2_{p_j+1} ds.$$

Using again (5.31), we get

$$G(t) \leq 3G(0)e^{-2\gamma_0 t} + C_B \int_0^t e^{-2\gamma_0(t-s)} \|\nabla u(s)\|_{p_j+1}^2 ds, \tag{5.33}$$

and recalling that

$$2G(t) = \|(u(t), u_t(t), \eta^t)\|_{\mathcal{H}}^2 = \|S(t)U^1 - S(t)U^2\|_{\mathcal{H}}^2, \quad t \geq 0,$$

then (5.11) follows by renaming the constants in (5.33). Hence, the proof of Proposition 5.5 is complete. □

By virtue of the above technical results we have gathered the tools to prove our main result as follows.

5.2 Proof of Theorem 4.1: Completion

Proof of Theorem 4.1 – *item* **I**. We must show that the dynamical system $(\mathcal{H}, S(t))$ defined in (4.1) satisfy the properties (4.2) and (4.3). To do so, we first consider a bounded positively invariant set $B \subset \mathcal{H}$ with respect to $S(t)$, denote $S(t)U^i = (u^i(t), u_t^i(t), \eta^{i,t})$ for $U^i \in B$, $\quad i = 1, 2$, and set $u = u^1 - u^2$, as before.

Firstly, from (3.12) in Theorem 3.1-(iii) we see that (4.2) follows promptly with $a(t) = e^{c_B t} > 0$, for some constant $c_B > 0$ depending on B, which is locally bounded in $[0, \infty)$.

Secondly, to conclude (4.3), we consider the seminorm

$$n_X(u) = \|\nabla u\|_{p_j+1}, \quad X := W^{1,p_j}(\Omega), \; j = 1, \ldots, n.$$

From assumptions of Theorem 4.1, the embedding $V_2 = H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow X$ is compact, and so $n_X(\cdot)$ is a compact seminorm on X. In addition, from (5.11) in Proposition 5.5 it follows that

$$\|S(t)U^1 - S(t)U^2\|_{\mathcal{H}}^2 \leq b(t)\|U^1 - U^2\|_{\mathcal{H}}^2 + c(t) \sup_{0<s<t} [n_X(u(s))]^2,$$

with

$$b(t) = b_0 e^{-\gamma_B t} \quad \text{and} \quad c(t) = C_B \int_0^t e^{-\gamma_B(t-s)} ds, \quad t > 0.$$

It is easy to verify that: $b \in L^1(\mathbb{R}^+)$, $\lim_{t\to\infty} b(t) = 0$ and $c_\infty = \sup_{t\in\mathbb{R}^+} c(t) \leq \dfrac{C_B}{\gamma_B} < \infty$. Therefore, condition (4.3) also holds true as desired, which concludes the proof that $(\mathcal{H}, S(t))$ is quasi-stable on any bounded positively invariant set in $\mathcal{H}$. □

Proof of Theorem 4.1 – *item* **II**. Applying Theorem 4.1-**I** and Proposition 7.9.4 in [9], then $(\mathcal{H}, S(t))$ is asymptotically smooth. Thus, combining Lemmas 5.2 and 5.3 with

Corollary 7.5.7 in [9], we conclude that $(\mathcal{H}, S(t))$ has a compact global attractor given by $\mathcal{A} = \mathbb{M}_+(\mathcal{N})$. □

***Proof of Theorem 4.1** – items* **III** *and* **IV**. It is an immediate consequence of Theorem 4.1-**II** and Theorem 7.5.10 in [9], and also Lemma 5.4. □

***Proof of Theorem 4.1** – items* **V** *and* **VI**. From the above, $(\mathcal{H}, S(t))$ is quasi-stable on the attractor $\mathcal{A}$. Thus, as a consequence of Theorem 7.9.6 in [9], it follows that $\mathcal{A}$ has finite fractal dimension $\dim_{\mathcal{H}}^f \mathcal{A}$. Besides, since condition (4.3) holds with $c_\infty = \sup_{t\in\mathbb{R}^+} c(t) < \infty$, then the smoothness (4.4)–(4.5) can be achieved by using Theorem 7.9.8 in [9]. □

***Proof of Theorem 4.1** – item* **VII**. Let us take $\mathcal{B} = \{U \in \mathcal{H};\ \ \Phi(U) \leq R\}$ for any given $R > 0$, where $\Phi = E$ is the strict Lyapunov functional for the dynamical system $(\mathcal{H}, S(t))$. For R sufficiently large it is possible to conclude that $\mathcal{B}$ is a positively invariant bounded absorbing set and then $(\mathcal{H}, S(t))$ is a dissipative dynamical system. Moreover, from Theorem 4.1-**I** $(\mathcal{H}, S(t))$ is quasi-stable on $\mathcal{B}$. Given $U_0 = (u_0, u_1, \eta_0) \in \mathcal{B}$, from (3.11) in Theorem 3.1, and also from equations (1.7)–(1.8), along with (1.12), we infer

$$(u_t, u_{tt}, \eta_t) \in L^\infty_{loc}(\mathbb{R}^+, \widetilde{\mathcal{H}}), \quad \widetilde{\mathcal{H}} := V_1 \times V_0 \times \mathcal{M}_1.$$

From this and following analogous arguments as presented in [5,34], one can prove that the mapping

$$t \mapsto S(t)U_0 \equiv U(t), \quad \text{for any} \quad U_0 \in \mathcal{B},$$

is Hölder continuous in $\widetilde{\mathcal{H}}$ with exponent $\delta = 1$. Hence, from Theorem 7.9.9 in [9] the dynamical system $(\mathcal{H}, S(t))$ has a generalized fractal exponential attractor $\mathcal{A}_{\exp}$ with finite dimension in the extended space $\widetilde{\mathcal{H}}$. The remaining conclusion follows after applying interpolation theorem, which can be done again as in [5,34]. See also [27, Theorem 2.3]. □

***Proof of Theorem 4.1** – item* **VIII**. For $\alpha_0 = 0$ and $h \equiv 0$, it follows from (5.2)–(5.3) in Lemma 5.1 that

$$\|S(t)U_0\|^2_{\mathcal{H}} \leq \frac{1}{\beta_0} E(t), \quad t \geq 0, \tag{5.34}$$

for any solution $S(t)U_0 = (u(t), u_t(t), \eta^t)$ corresponding to initial data $U_0 = (u_0, u_1, \eta_0)$. In addition, by following the same arguments as in the proof of Proposition 5.5 (or else the proof, as a especial case with exponential kernels, of Theorem 2.8 in [28]), one can prove that the energy $E(t)$ set in (5.1)—neglecting h—satisfies the following estimate

$$E(t) \leq C_0 e^{-c_0 t}, \quad t \geq 0, \tag{5.35}$$

for some constants $C_0, c_0 > 0$ depending on initial data U_0. Therefore, for $U_0 \in B$ with $B \subset \mathcal{H}$ being a bounded set, we conclude from (5.34)–(5.35) that (4.6) holds true by rearranging the constants. Finally, from (4.6), Lemma 5.3 and Theorem 4.1-**II**, we conclude that $\mathcal{A}$ is trivial in the limit situation $\alpha_0 = 0$ along with null function $h = 0$.

Hence, the proof of Theorem 4.1 is complete. □

Appendix: Examples for *F*

We finish this work by giving some examples of C^1-vector fields $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfying Assumption (**A2**), or else, more generally that:

(a) There exist positive constants $k_1, \ldots, k_n$ and $q_1, \ldots, q_n$ such that

$$|\nabla F_j(z)| \leq k_j(1 + |z|^{q_j}), \quad \forall \quad z \in \mathbb{R}^n, \quad \forall \quad j = 1, \ldots, n. \tag{A.1}$$

(b) $F = \nabla f$ with $f : \mathbb{R}^n \to \mathbb{R}$ so that

$$-a_0 - a_1|z|^2 \leq f(z) \leq F(z)z + a_2|z|^2, \quad \forall\, z \in \mathbb{R}^n, \tag{A.2}$$

for some nonnegative constants $a_0, a_1, a_3 \geq 0$.

We remark that it is important to consider at least one example such that (A.2) holds true with $a_0 > 0$, by differing of the examples presented in [28, Sect. 4.2]. For the sake of completeness, we also provide an example where $a_0 = 0$. Also, since condition (A.1) is only technical, we shall omit comments on it in the next examples.

Example A.1 (Example 4.11 in [28]) Let us first consider

$$F(z) = |z|^q z, \quad F = \nabla f \;\text{ with }\; f(z) = \frac{1}{q+2}|z|^{q+2}, \quad q \geq 0.$$

Then, condition (A.2) is readily verified for any $a_1, a_2 \geq 0$ and $a_0 = 0$. In this case, the vector field generates the p-Laplacian operator

$$\operatorname{div} F(\nabla u) = \operatorname{div}\left(|\nabla u|^q \nabla u\right),$$

with power $p = 2q + 1$ that must satisfy condition (3.5).

Example A.2 (Example 4.12 in [28]) Let $F = \nabla f$ be a conservative vector field, where

$$f(z) = \frac{\kappa}{q+2}|z|^{q+2} + \tau z,$$

with $q \geq 0$, $\kappa > 0$, and $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n$. Thus, condition (A.2) is fulfilled with $a_0 = \frac{|\tau|^2}{2}$, $a_1 = \frac{1}{2}$, and any $a_2 \geq 0$.

Example A.3 Let us take $F(z) = |z|^q z - \lambda|z|^r z$, $q > r > 0, \lambda > 0$. Then, $F = \nabla f$, where

$$f(z) = \frac{1}{q+2}|z|^{q+2} - \frac{\lambda}{r+2}|z|^{r+2}.$$

Let us verify (A.2). We first check that there exist $a_0, a_1 \geq 0$ such that

$$-a_0 - a_1|z|^2 \leq f(z), \quad \forall z \in \mathbb{R}^n.$$

In fact, for a fixed $a_1 \geq 0$, it is enough to choose $a_0 \geq 0$ such that

$$a_0 \geq -\min_{t \geq 0} \left\{ \frac{t^{q+2}}{q+2} - \frac{\lambda t^{r+2}}{r+2} + a_1 t^2 \right\}.$$

To the second inequality in (A.2), it is enough to choose $a_2 \geq 0$ such that

$$a_2 \geq \max_{t \geq 0} \left\{ \left(\frac{1}{q+2} - 1 \right) t^q - \lambda \left(\frac{1}{r+2} - 1 \right) t^r \right\}.$$

Note that since $(\frac{1}{q+2} - 1) < 0$ and $r < q$, such maximum there exists. In this case, the vector field generates the q, r-Laplacian operator provided in the elliptic problem (5.9).

Acknowledgements The authors would like to thank Professors M. T. O. Pimenta for remarkable suggestions on elliptic problems and T.F. Ma for fruitful comments on the modeling of viscoelastic Kirchhoff problems.

References

1. Andrade, D., Jorge Silva, M.A., Ma, T.F.: Exponential stability for a plate equation with p-Laplacian and memory terms. Math. Methods Appl. Sci. **35**, 417–426 (2012)
2. Ammar-Khodja, F., Benabdallah, A., Muñoz Rivera, J .E., Racke, R.: Energy decay for Timoshenko systems of memory type. J. Differ. Equ. **194**(1), 82–115 (2003)
3. Araújo, R.O., Ma, T.F., Qin, Y.: Long-time behavior of a quasilinear viscoelastic equation with past history. J. Differ. Equ. **254**, 4066–4087 (2013)
4. Babin, A.V., Vishik, M.I.: Attractors of Evolution Equations. Studies in Mathematics and Its Application, vol. 25. North-Holland, Amsterdam (1992)
5. Barbosa, A.R.A., Ma, T.F.: Long-time dynamics of an extensible plate equation with thermal memory. J. Math. Anal. Appl. **416**, 143–165 (2014)
6. Chueshov, I., Lasiecka, I.: Global attractors for Mindlin-Timoshenko plates and for their Kirchhoff limits. Milan J. Math. **74**, 117–138 (2006)
7. Chueshov, I., Lasiecka, I.: Existence, uniqueness of weak solutions and global attractors for a class of nonlinear 2D Kirchhoff-Boussinesq models. Discret. Contin. Dyn. Syst. **15**, 777–809 (2006)
8. Chueshov, I., Lasiecka, I.: Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping, vol. 195, p. 912. Memoirs of the American Mathematical Society, Providence (2008)
9. Chueshov, I., Lasiecka, I.: Von Karman Evolution Equations: Well-Posedness and Long-Time Dynamics. Springer Monographs in Mathematics. Springer, New York (2010)
10. Chueshov, I., Lasiecka, I.: On global attractors for 2D Kirchhoff-Boussinesq model with supercritical nonlinearity. Commun. Partial Differ. Equ. **36**, 67–99 (2011)

11. Conti, M., Geredeli, P.: Existence of smooth global attractors for nonlinear viscoelastic equation with memory. J. Evol. Equ. **15**, 533–538 (2015)
12. Drozdov, A.D., Kolmanovskii, V.B.: Stability in Viscoelasticity. North-Holland, Amsterdam (1994)
13. Eden, A., Foias, C., Nicolaenko, B., Temam, R.: Exponential Attractors for Dissipative Evolution Equations. RAM: Research in Applied Mathematics, vol. 37. Masson, Paris (1994)
14. Fatori, H., Jorge Silva, M.A., Ma, T.F., Yang, Z.: Long-time behavior of a class of thermoelastic plates with nonlinear strain. J. Differ. Equ. **259**, 4831–4862 (2015)
15. Feng, B.: Well-posedness and exponential stability for a plate equation with time-varying delay and past history. Z. Angew. Math. Phys. **68**, 24 (2017)
16. Feng, B.: Long-time dynamics of a plate equation with memory and time delay. Bull. Braz. Math. Soc. (2017). https://doi.org/10.1007/s00574-017-0060-x
17. Giorgi, C., Naso, M.G.: Mathematical models of Reissner-Mindlin thermoviscoelastic plates. J. Therm. Stress. **29**, 699–716 (2006)
18. Giorgi, C., Vegni, F.: Uniform energy estimates for a semilinear evolution equation of the Mindlin-Timoshenko beam with memory. Math. Comput. Model. **39**, 1005–1021 (2004)
19. Giorgi, C., Vegni, F.: The longtime behavior of a nonlinear Reissner-Mindlin plate with exponentially decreasing memory kernels. J. Math. Anal. Appl. **326**, 754–771 (2007)
20. Giorgi, C., Marzocchi, A., Pata, V.: Asymptotic behavior of a semilinear problem in heat conduction with memory. Nonlinear Differ. Equ. Appl. NoDEA **5**, 333–354 (1998)
21. Giorgi, C., Grasseli, M., Pata, V.: Well-posedness and longtime behavior of the phase-field model with memory in a history space setting. Q. Appl. Math. **59**, 701–736 (2001)
22. Giorgi, C., Muñoz Rivera, J.E., Pata, V.: Global attractors for a semilinear hyperbolic equation in viscoelasticity. J. Math. Anal. Appl. **260**, 83–99 (2001)
23. Grasselli, M., Pata, V.: Uniform attractors of nonautonomous dynamical systems with memory. Prog. Nonlinear Differ. Equ. Appl. **50**, 155–178 (2002)
24. Hale, J.K.: Asymptotic Behavior of Dissipative Systems. Mathematical Surveys and Monographs, vol. 25. American Mathematical Society, Providence (1988)
25. Jorge Silva, M.A., Ma, T.F.: On a viscoelastic plate equation with history setting and perturbation of p-Laplacian type. IMA J. Appl. Math. **78**, 1130–1146 (2013)
26. Jorge Silva, M.A., Ma, T.F.: Long-time dynamics for a class of Kirchhoff models with memory. J. Math. Phys. **54**, 021505 (2013)
27. Jorge Silva, M.A., Narciso, V.: Attractors and their properties for a class of nonlocal extensible beams. Discret. Contin. Dyn. Syst. **35**, 985–1008 (2015)
28. Jorge Silva, M.A., Muñoz Rivera, J.E., Racke, R.: On a classes of nonlinear viscoelastic Kirchhoff plates: well-posedness and generay decay rates. Appl. Math. Optim. **73**, 165–194 (2016)
29. Lagnese, J.E.: Boundary Stabilization of Thin Plates. SIAM Studies in Applied Mathematics, vol. 10. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (1989)
30. Lagnese, J., Lions, J.-L.: Modelling, Analysis and Control of Thin Plates. Recherches en Mathématiques Appliquées, vol. 6. Masson, Paris (1988)
31. Ma, T.F., Pelicer, M.L.: Attractors for weakly damped beam equations with p-Laplacian. Discret. Contin. Dyn. Syst. Suppl. **15**, 525–534 (2013)
32. Marzocchi, A., Vuk, E.: Global attractor for damped semilinear elastic beam equations with memory. Z. Angew. Math. Phys. **54**, 224–234 (2003)
33. Narciso, V.: Long-time behavior of a nonlinear viscoelastic beam equation with past history. Math. Methods Appl. Sci. **38**, 775–784 (2014)
34. Potomkin, M.: Asymptotic behavior of thermoviscoelastic Berger plate. Commun. Pure Appl. Anal. **9**, 161–192 (2010)
35. Prüss, J.: Evolutionary Integral Equations and Applications. Monographs in Mathematics, vol. 87. Birkhäuser, Basel (1993)
36. Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Applied Mathematical Sciences, vol. 68. Springer, New York (1988)
37. Timoshenko, S.P.: On the correction for shear of the differential equation for transverse vibrations of prismatic bars. Philos. Mag. Ser. 6 **41**(245), 744–746 (1921)
38. Timoshenko, S.P.: Vibration Problems in Engineering. Van Nostrand, New York (1955)
39. Yang, Z.: Longtime behavior for a nonlinear wave equation arising in elasto-plastic flow. Math. Methods Appl. Sci. **32**, 1082–1104 (2009)

40. Yang, Z.: Global attractor and their Hausdorff dimensions for a class of Kirchhoff models. J. Math. Phys. **51**, 032701 (2010)
41. Yang, Z.: Finite-dimensional attractors for the Kirchhoff models. J. Math. Phys. **51**, 092703 (2010)
42. Yang, Z.: Finite-dimensional attractors for the Kirchhoff models with critical exponents. J. Math. Phys. **53**, 032702 (2012)
43. Yang, Z., Jin, B.: Global attractor for a class of Kirchhoff models. J. Math. Phys. **50**, 032701 (2009)