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A general stability result for the semilinear viscoelastic wave model under localized effects

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ABSTRACT

Our main goal in the present work is to address an integro-differential model under localized viscoelastic and frictional effects arising in the Boltzmann theory of viscoelasticity. More precisely, we consider a general version in the history context of the pioneer localized viscoelastic problem approached by Cavalcanti and Oquendo (2003) in the null history scenario, and more recently by Cavalcanti et al. (2018) in the history framework. By means of a new observability inequality, we prove a general stability result to the model under a weaker assumption on the localized frictional damping and a slower condition on the decreasing memory kernel (of polynomial type) than the previously mentioned works. To achieve such stability results, we still work in a general setting by removing the assumption on *complementary damping mechanisms* and show, in some reasonable situations concerning the density coefficient, that the localized viscoelastic effect is enough to ensure the general stability (of polynomial type) to the problem.

1. Introduction

1.1. Localized viscoelastic model

In the present work, motivated by the semilinear wave model with localized memory and linear frictional terms proposed by Cavalcanti et al. [1,2], we are going to study the following autonomous n-dimensional

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initial–boundary value problem

$$\begin{cases} \rho(x)u_{tt} - \operatorname{div}[\kappa(x)\nabla u] - \displaystyle\int_0^\infty g(s)\operatorname{div}[a(x)\nabla\eta(s)]\,ds + b(x)h(u_t) + f(u) = 0 \text{ in } \Omega\times(0,\infty), \\ \eta_t = -\eta_s + u_t \quad \text{in} \quad \Omega\times(0,\infty)\times(0,\infty), \\ u = 0 \quad \text{on} \quad \partial\Omega\times(0,\infty), \quad \eta = 0 \quad \text{on} \quad \partial\Omega\times(0,\infty)\times(0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ \eta^0(x,s) = \eta_0(x,s), \quad x\in\Omega, \ s\in(0,\infty), \\ \eta^t(x,0) = 0, \quad x\in\Omega, \ t\in[0,\infty). \end{cases} \tag{1.1}$$

where

- Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$;
- $\rho > 0$ is a non-constant positive function related to material density;
- $a \geq 0$ is a smooth bounded function that can vanish in a proper subset $A \subset \Omega$;
- $g > 0$ is the well-known memory kernel with total mass $\int_0^\infty g(s)\,ds := g_0 \in (0, \|a\|^{-1}_{L^\infty(\Omega)})$;
- $\kappa(x) = 1 - g_0 a(x), \quad x \in \Omega$;
- $b \geq 0$ is a bounded function that will act in A when ρ is general;
- f and h are real functions satisfying standard properties.

All assumptions on the functions $a,\ b,\ \rho,\ g,\ f$ and h will be precisely stated in Sections 2 and 3. In the above problem (1.1), the function $u = u(x,t)$ represents the *displacement* and the variable $\eta = \eta^t(x,s)$ stands for the *relative displacement history*. By following Grasselli and Pata [3, Sects. 3 and 4], one can show

$$\eta^t(\cdot,s) = \begin{cases} u(\cdot,t) - u_0(\cdot,t-s), & s \geq t, \\ u(\cdot,t) - u(\cdot,t-s), & s < t, \end{cases} \tag{1.2}$$

where $u_0 : \Omega\times(-\infty,0] \to \mathbb{R}$ is the prescribed *past history* of the displacement u. Thus, one knows that system (1.1) can be deduced from the following viscoelastic wave problem with history

$$\begin{cases} \rho(x)u_{tt} - \Delta u + \displaystyle\int_{-\infty}^t g(t-s)\operatorname{div}[a(x)\nabla u(s)]\,ds + b(x)h(u_t) + f(u) = 0 \text{ in } \Omega\times(0,\infty), \\ u = 0 \quad \text{on} \quad \partial\Omega\times\mathbb{R}, \\ u(x,s) = u_0(x,s), \ u_t(x,0) = \partial_t u_0(x,s)|_{s=0}, \quad (x,s)\in\Omega\times(-\infty,0], \end{cases} \tag{1.3}$$

and conversely. We refer to [1, Sects. 1.1 and 2.2] to a more accurate description of the model as well as to the one-to-one correspondence between problems (1.3) and (1.1) with proper initial data such as $\eta_0(\cdot,s) = u_0(\cdot) - u_0(\cdot,-s)$, $s > 0$. At this point, it is worth mentioning that the localized memory does not interfere in the core of the equivalence between these related problems. Indeed, all the arguments presented in [1] are somehow similar and rely on the statements previously introduced e.g. in [3–5]. Here, our main goal is to analyze the general stability of problem (1.1). In what follows, we are going to bring the attention to problems with localized memory and then highlight our main contributions on the subject.

1.2. Viscoelastic problems under localized effects

We initially notice that there is a vast literature dealing with asymptotic and long-time behavior of problems with linear memory without localizing coefficient, say $a \equiv 1$ in (1.3). Indeed, we refer to [3–16] for problems involving the history case and [11,16–27] for models where the null history case is considered, just to quote a few of them. Nonetheless, according to our best knowledge, there are only a few papers addressing the stabilization of (1.3) (with or without history) under the localized viscoelastic effect, namely, with coefficient $a \geq 0$ possibly vanishing in a suitable subset $A \subset \Omega$. Below, we are going to quote such works (and their results) in order to compare with the present one.

In Cavalcanti and Oquendo [2], the authors consider the following equation with null history

$$u_{tt} - \kappa_0 \Delta u + \int_0^t g(t-s)\mathrm{div}[a(x)\nabla u(s)]\,ds + b(x)h(u_t) + f(u) = 0 \;\text{ in }\; \Omega \times (0,\infty), \tag{1.4}$$

with initial–boundary conditions and appropriate assumptions on the nonlinearities h and f. Both viscoelastic and frictional localized coefficients a and b act only in a portion of $\Omega \subset \mathbb{R}^n$, but the authors assumed an additional hypothesis like *complementary damping mechanisms,* namely,

$$a(x) + b(x) \geq \delta > 0, \quad \forall\, x \in \Omega, \tag{1.5}$$

which means that both viscoelastic and frictional terms cooperate with each other by constituting an effectively (full) damping in the whole domain. Therefore, under the assumption (1.5), the authors prove that the corresponding solution of (1.4) decays exponentially or polynomially to zero, provided the memory kernel g decays exponentially or polynomially, respectively.

Since then, some works have appeared in the literature with the same cooperating assumption (1.5) on the coefficients. In fact, in Cavalcanti et al. [28], the authors approach, in a more general setting, the following model

$$u_{tt} - \Delta u + \int_0^t g(t-s)\mathrm{div}[a(x)\nabla u(s)]\,ds + b(x)h(u_t) = 0 \;\text{ in }\; M \times (0,+\infty), \tag{1.6}$$

where M is a compact Riemannian manifold with boundary. Still under the hypothesis (1.5), they prove a general stability to the energy associated with (1.6) by considering milder assumptions on g and h than [2]. In general, the decay rate is determined by the “poorer” decay considered on the viscoleastic and frictional damping effects.

Now, regarding problems in the history framework, we refer to Cavalcanti, Fatori and Ma [29] where the following problem is studied

$$u_{tt} - \Delta u + \int_{-\infty}^t g(t-s)\mathrm{div}[a(x)\nabla u(s)]\,ds + b(x)u_t + f(u) = h \;\text{ in }\; \Omega \times (0,+\infty), \tag{1.7}$$

with initial–boundary conditions like in (1.3) and $\Omega \subset \mathbb{R}^3$. Under the complementary damping condition (1.5) and exponential kernel g, the authors prove the existence of a finite dimensional compact global attractor to the dynamical system corresponding to problem (1.7). This shows that the assumption (1.5) as well as the frictional damping $b(x)u_t$ play a crucial role in their results, which characterizes a full damping in the overall computations and follows similar arguments as in the previous null history cases addressed by [2,28]. Therefore, only the multiplier technique is required to reach the long-time behavior of solutions. More recently, Shomberg [30] has complemented the results proposed in [29] by providing further regularity for the above achieved global attractor under extra assumptions on the nonlinear term $f(u)$, still considering exponential memory kernel g, and again exploiting the assumption (1.5).

We remark that in all the above mentioned works [2,28–30], the assumption (1.5) is essential, since it allows to consider the frictional and viscoelastic terms as a complementary damping acting in the whole domain, namely, a full damping in the end. In this way, the semigroup property is no longer necessary to the stability of solutions, for instance, and multipliers are usually enough to conclude all proofs on stabilization. Therefore, one question that arises is whether it would be possible stabilize problem (1.3) without assuming (1.5) or even neglecting the frictional coefficient b, that is, by taking $b \equiv 0$ in Ω.

To the best of our knowledge, the first work that gives a positive answer to this issue is due to Liu and Liu [31] in the one dimensional case ($n = 1$). In this scenario, inspired by their own puzzling results proved previously in [32], the authors consider, in addition to the one-dimensional linear wave equation with the Kelvin–Voigt damping and smooth coefficients, the Boltzmann problem under discontinuity of material properties at the interfaces, see for instance the linear models (1.3) and (1.4) therein. Thus, by assuming exponential memory kernel g, they prove exponential energy decay in both cases. Their arguments rely on the strength of the one-dimensional case combined with linear semigroup theory, once both problems are linear.

Other previous and recent attempts in the stabilization of systems by means of localized viscoleastic effects can be found in Muñoz Rivera et al. [33,34] and Cavalcanti et al. [35,36]. However, in the latter papers the authors deal with null history, namely, memory term defined on the range $(0,t)$, so that the regarded semigroup property seems to be not appropriate. Therefore, it seems hard to find a proper comparison with such results afterwards and, for this reason, we omit them.

More recently, Cavalcanti et al. [1] study a n-dimensional ($n \geq 2$) history problem with localized memory. By assuming exponential kernel g, linear frictional damping in (1.1) (i.e. $h(s) = s$), standard hypotheses on the nonlinear source $f(u)$, and suitable conditions on the density $\rho(x)$ and on the viscoelastic and frictional damping coefficients $a(x)$ and $b(x)$ (see e.g. [1, Rem. 3.3]), the authors prove that the energy related to the autonomous problem (1.1) decays exponentially whenever the initial energy is taken in bounded sets of the phase space, see for instance the main results in [1, Thms. 3.1 and 4.1]. In this occasion, the assumption (1.5) is not regarded and even in the absence of frictional damping effects ($b \equiv 0$), the localized viscoelastic damping is enough to ensure the (locally) exponential stabilization of the system. As far as we know, this consists a very weaker dissipation than those given by [2,28–31] in the context of localized memory with past and null history. Thus, to the proof of their main results, the authors show two observability inequalities in terms of g' (for the linear and semilinear problems) by means of contraction arguments along with powerful tools such as Unique Continuation Property (UCP), Geometric Control Condition (GCC) and Microlocal Analysis (MA).

Motivated by the aforementioned papers [1,2,28–31], in especial by [1,2], and in order to go further, the present article aims to promote a generalization of the stability results provided in [1] and to give a general version in the history scenario of the previous viscoelastic problem in [2]. To this purpose, we work in a more general condition with respect to frictional and viscoelastic damping. Our main contributions are highlighted below.

1.3. Contributions and article structure

Under the previous statements, we are now in position to stress the main novelties of this paper.

(1) In Section 2, we introduce our preliminary assumptions and notations and the well-posedness result as well. We first remark that, as in [1], we do not employ the strength of the complementary damping assumption (1.5). Thus, we work in a more general (and harder) scenario than [2,28–31]. This fact will be clarified in Remark 2.1 (see also Remark 3.2).

(2) We also observe that, under Assumptions 2.1 and 2.2 given in Section 2, one sees that our present hypotheses are milder than those ones considered in [1] in three aspects:

  (a) we assume a more general hypothesis on the localized frictional damping $b(x)h(u_t)$;
  (b) we consider a slower decay (of polynomial type) to the memory kernel g;
  (c) we reach the critical Sobolev exponent in what concerns the growth of the nonlinear source $f(u)$.

  Therefore, our damping is weaker and slower than that one considered in [1].

(3) In Section 3, we state our main stability results, namely, Theorem 3.3 and Proposition 3.5. While in Theorem 3.3 we address the general stability concerning the energy related to problem (1.1), in Proposition 3.5 we feature a new observability inequality involving the kernel g. The latter constitutes the principal difference when compared with the observability inequality provided by [1] in terms of g' (see Eqs. (3.12) and (3.13) therein). Indeed, their result does not apply in our case since our assumption on the memory kernel (see Assumption 2.2) does not reflect the same exponential property to both g

and g', as done e.g. in [1]. Hence, a new proof is required, which still combines contradiction arguments with the effective UCP, GCC and MA tools.

(4) We also note that for some suitable choices of the density coefficient ρ, see e.g. Assumption 3.1–II, the energy stability is addressed under the sole dissipation given by the (polynomial) memory term (i.e. $b \equiv 0$), see e.g. Theorem 3.3–II. In such a case, we believe it is a very weak damping that has been considered in the literature in what concerns viscoelastic wave models with localized memory in the history framework.

(5) In Section 4, we provide all details of the proofs of the main results. Finally, in Appendix we recall some important (and useful) existing results in the literature to help with proofs and make this work as clear as possible.

2. Preliminary concepts

2.1. Assumptions

In order to state our main results on the asymptotic behavior of problem (1.1), let us first consider the assumptions and notations to be used throughout this paper as well as the well-posedness result concerning (1.1).

Assumption 2.1. With respect to the functions a, b, ρ, g, f and h, we initially assume:

$(\mathbf{A_1})$ $a \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ is a non-negative function such that there exists a closed connected set $A \subset \Omega$ verifying

$$a(x) = 0 \iff x \in A.$$

$(\mathbf{A_2})$ $b \in L^\infty(\Omega)$ is a non-negative function $b \geq 0$ on Ω.

$(\mathbf{A_3})$ $\rho \in C^\infty(\Omega)$ is bounded function such that

$$0 < a_1 \leq \rho(x) \leq a_2, \quad \forall\, x \in \Omega, \tag{2.1}$$

for some positive constants a_1, a_2.

$(\mathbf{A_4})$ $g \in L^1([0,\infty)) \cap C^1([0,\infty))$ is a positive non-increasing function satisfying

$$l := 1 - g_0\|a\|_{L^\infty(\Omega)} > 0, \quad \text{where} \quad g_0 = \int_0^\infty g(s)\, ds. \tag{2.2}$$

$(\mathbf{A_5})$ $f \in C^2(\mathbb{R})$ is function such that $f(0) = 0$ and

(a) the primitive $F(s) = \int_0^s f(\tau)d\tau$ satisfies

$$-\frac{\beta}{2}|s|^2 \leq F(s) \leq f(s)s + \frac{\beta}{2}|s|^2, \quad \forall\, s \in \mathbb{R}, \tag{2.3}$$

for $\beta \in [0, \lambda_1)$, where $\lambda_1 > 0$ is the first eigenvalue corresponding to problem

$$\begin{cases} -\mathrm{div}[(1 - g_0 a(x))\nabla u] = \lambda u \quad \text{in} \quad \Omega, \\ u = 0 \text{ on } \partial\Omega; \end{cases}$$

(b) there exists $c > 0$ such that

$$|f^{(j)}(s)| \leq c(1 + |s|)^{p-j}, \quad \forall\, s \in \mathbb{R},\ j = 1, 2, \tag{2.4}$$

where

$$p \geq 1 \ \text{ if } \ n = 2 \quad \text{ and } \quad 1 \leq p \leq \frac{n}{n-2} \ \text{ if } \ n \geq 3. \tag{2.5}$$

($\mathbf{A_6}$) h is continuous and monotone increasing function such that

(i) $h(s)s > 0$, for all $s \neq 0$;
(ii) $M_1 s^2 \leq h(s)s \leq M_2 s^2$, for all $|s| > 1$, where M_1, M_2 are positive constants.

Assumption 2.2. Concerning the memory kernel g, we additionally assume:

($\mathbf{G_1}$) there exist constants $p_2 \geq p_1 > 1$ and C_1, $C_2 > 0$ such that

$$- C_1[g(s)]^{p_1} \leq g'(s) \leq -C_2[g(s)]^{p_2}, \quad \forall\, s > 0; \tag{2.6}$$

($\mathbf{G_2}$) there exists $\alpha_0 \in (0, 1)$ such that

$$\int_0^\infty [g(s)]^{1-\alpha_0} ds < \infty. \tag{2.7}$$

Before proceeding, let us give some comments on the above assumptions as follows.

Remark 2.1. In relation to the conditions ($\mathbf{A_1}$)–($\mathbf{A_6}$) and ($\mathbf{G_1}$)–($\mathbf{G_2}$) regarded above, we would like to observe the following issues.

1. We first notice that Assumptions ($\mathbf{A_1}$)–($\mathbf{A_2}$) do not require a condition like complementary damping coefficients (1.5). Such an assumption will not be requested even for stabilization purposes.
2. Hypothesis ($\mathbf{A_5}$) is quite standard in the literature. A peculiar example is given by $f(s) = -\xi \arctan(s)$ with proper coefficient $\xi > 0$. We highlight that in [1] the growth p like in (2.4) is only considered in the sub-critical case. Here, the critical Sobolev exponent is achieved as one sees in (2.5). Additionally, we observe that the growth condition on f also implies that

$$|f(s)| \leq c(p)|s| + c(p)|s|^p. \tag{2.8}$$

   We still note that (2.3) implies $f'(0) + \beta \geq 0$ as well.
3. Condition ($\mathbf{A_6}$) has its origins in the work [37]. It is crucial to construct a convex strictly increasing function $H : [0, \infty) \to [0, \infty)$ vanishing at $x = 0$ and so that

$$s^2 + [h(s)]^2 \leq H^{-1}(sh(s)) \quad \text{for} \quad |s| \leq 1.$$

   Several examples of h-functions as well as decay rates were presented in [38] in a more general framework. Such a property will be very important in the (general) energy stabilization when one considers the case of non-vanishing complementary damping coefficient $b > 0$.
4. Assumptions ($\mathbf{G_1}$)–($\mathbf{G_2}$) will be only required for stabilization purposes. It is worth mentioning that (2.6) leads to decreasing memory kernels of polynomial type, which correspond to slower decay rates than the (optimal) exponential one addressed in [1, Assump. 1.1].

2.2. Notations

Now we consider the well-known Hilbert space $H_0^1(\Omega)$ endowed with the topology given by

$$\|u\|_1^2 = \int_\Omega (1 - g_0 a(x))|\nabla u|^2 \, dx,$$

which is equivalent to the usual norm of $H_0^1(\Omega)$ due to (2.2). From the Poincaré inequality we get

$$\|u\|_{L^2(\Omega)}^2 \leq \lambda_1^{-1} \|\nabla u\|_{L^2(\Omega)}^2, \quad \forall\, u \in H_0^1(\Omega),$$

where $\lambda_1 > 0$ is the first eigenvalue of the Laplace operator with Dirichlet boundary condition and $\|\cdot\|_{L^2(\Omega)}$ stands for the usual norm in $L^2(\Omega)$.

Given a satisfying $(\mathbf{A_1})$, we also define the Hilbert space

$$H_a^1 = \left\{u \in L^2(\Omega);\ \int_\Omega a(x)|\nabla u|^2\,dx < \infty,\ u|_{\partial\Omega} = 0\right\}$$

with respective inner-product and norm

$$(u,v)_{H_a^1} = \int_\Omega a(x)\nabla u\cdot\nabla v\,dx + \int_\Omega u\,v\,dx, \quad \|u\|_{H_a^1}^2 = \int_\Omega a(x)|\nabla u|^2\,dx + \int_\Omega |u|^2\,dx.$$

The regularity imposed on the function a is the main ingredient to prove that H_a^1 is a Hilbert space and, consequently, it makes sense to consider the trace of order zero of any function u belonging to this space. In addition, we define the g-weighted spaces with values on H_a^1 as follows

$$L_g^2(\mathbb{R}^+, H_a^1) = \left\{\eta : \mathbb{R}^+ \to H_a^1;\ \int_0^\infty g(s)\|\eta(s)\|_{H_a^1}^2\,ds < \infty\right\}$$

endowed with the inner-product and norm

$$(\eta,\zeta)_{L_g^2(\mathbb{R}^+,H_a^1)} = \int_0^\infty g(s)(\eta(s),\zeta(s))_{H_a^1}\,ds, \quad \|\eta\|_{L_g^2(\mathbb{R}^+,H_a^1)}^2 = \int_0^\infty g(s)\|\eta(s)\|_{H_a^1}^2\,ds.$$

Furthermore, under the assumption $(\mathbf{A_3})$ on ρ, we set the space

$$L_\rho^2(\Omega) = \left\{u : \Omega \to \mathbb{R};\ \int_\Omega \rho(x)|u(x)|^2 dx < \infty\right\}$$

endowed with the norm

$$\|u\|_\rho = \left(\int_\Omega \rho(x)|u(x)|^2 dx\right)^{1/2}.$$

Thus, due to (2.1), it is clear that

$$u \in L_\rho^2(\Omega) \iff u \in L^2(\Omega).$$

Under the above notations, we finally define the following Hilbert phase space

$$\mathcal{H} = H_0^1(\Omega) \times L_\rho^2(\Omega) \times L_g^2(\mathbb{R}^+, H_a^1)$$

endowed with the norm

$$\|(u,v,\eta)\|_{\mathcal{H}}^2 = \|u\|_1^2 + \|v\|_\rho^2 + \|\eta\|_{L_g^2(\mathbb{R}^+,H_a^1)}^2, \quad \forall\,(u,v,\eta) \in \mathcal{H}.$$

2.3. Well-posedness

Denoting by U the vector-valued function $U = (u,v,\eta)$, where $v = u_t$, then problem (1.1) is equivalent to the next Cauchy problem

$$\begin{cases} \dfrac{d}{dt}U(t) + \mathcal{A}U(t) + \mathcal{F}U(t) = 0, & t > 0, \\ U(0) = (u_0,u_1,\eta_0) := U_0, \end{cases} \tag{2.9}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator

$$\mathcal{A}U = \begin{pmatrix} -v \\ -\dfrac{1}{\rho(x)}\left\{\operatorname{div}[\kappa(x)\nabla u] + \displaystyle\int_0^\infty g(s)\operatorname{div}[a\nabla\eta(s)]\,ds - b(x)h(v)\right\} \\ \eta_s - v \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \Big\{(u,v,\eta) \in \mathcal{H};\ v \in H_0^1(\Omega), \eta_s \in L_g^2(\mathbb{R}^+, H_a^1), \eta(0) = 0,$$
$$\operatorname{div}[\kappa(x)\nabla u] + \int_0^\infty g(s)\operatorname{div}[a(x)\nabla \eta(s)]\,ds \in L^2(\Omega)\Big\}.$$

which is well-defined due to the growth of h, and $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is set by

$$\mathcal{F}(U) = \left(0, \frac{1}{\rho(x)} f(u), 0\right)^T,$$

being also well-defined by virtue of the growth condition on f and standard Sobolev embedding.

The Hadamard well-posedness of problem (2.9) and, consequently, of the original system (1.1), reads as follows.

Theorem 2.2 (*Global Well-posedness*). *Under the Assumption 2.1 we have:*

(i) *If $U_0 = (u_0, u_1, \eta_0) \in D(\mathcal{A})$, then there exists a unique regular solution $U = (u, u_t, \eta)$ of (2.9) such that*

$$u \in W^{2,\infty}(0,T; L_\rho^2(\Omega)) \cap W^{1,\infty}(0,T; H_0^1(\Omega)),\ \eta \in W^{1,\infty}(0,T; \mathcal{M}),$$

with $U(t) = (u(t), u_t(t), \eta^t) \in D(\mathcal{A})$, for all $t \in [0,T]$, for a given $T > 0$.

(ii) *If $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}$, then there exists a unique mild solution $U = (u, u_t, \eta)$ of (2.9) such that*

$$u \in C^1([0,T]; L_\rho^2(\Omega)) \cap C([0,T]; H_0^1(\Omega)),\ \eta \in C([0,T], \mathcal{M}),$$

for all $T > 0$ given.

(iii) *Moreover, these solutions are continuously dependent of the initial data, in the norm of $C([0,T], \mathcal{H})$, for all $T > 0$.*

The proof of Theorem 2.2 relies on very similar arguments as those presented in [1,29], with minor adjustments on the nonlinear frictional damping $b(x)h(u_t)$ that can be handled analogously to [37,38]. Therefore, we shall omit the proof of Theorem 2.2.

3. Main stability results

Let us consider $U(t) = (u, u_t, \eta)$ the unique global solution of problem (2.9) (resp. (1.1)). The associated energy functional is given by

$$E_{u,\eta}(t) = \frac{1}{2}\int_\Omega \rho(x)|u_t(t)|^2\,dx + \frac{1}{2}\int_\Omega \kappa(x)|\nabla u(t)|^2\,dx + \frac{1}{2}\int_0^\infty g(s)\|\sqrt{a}\nabla\eta^t(s)\|^2 ds + \int_\Omega F(u(t))dx. \quad (3.1)$$

In order to prove locally uniform decay rates for $E_{u,\eta}(t)$, we deal with two possibilities for the frictional damping coefficient $b \geq 0$ below, depending on the density coefficient ρ. Firstly, we observe that a straight forward computation leads to

$$\frac{d}{dt}E_{u,\eta}(t) = \frac{1}{2}\int_0^\infty g'(s)\|\sqrt{a}\nabla\eta^t(s)\|^2 ds - \int_\Omega b(x)h(u_t(t))u_t(t)\,dx, \quad (3.2)$$

which reduces to

$$\frac{d}{dt}E_{u,\eta}(t) = \frac{1}{2}\int_0^\infty g'(s)\|\sqrt{a}\nabla\eta^t(s)\|^2 ds, \quad (3.3)$$

for the case of vanishing coefficient $b \equiv 0$.

Remark 3.1. At this point, let us give some comments on the identities (3.2)–(3.3). In both cases, due to the Assumption 2.1 (see ($\mathbf{A_2}$), ($\mathbf{A_4}$) and ($\mathbf{A_6}$)), one sees that the energy is non-increasing with $E_{u,\eta}(t) \leq E_{u,\eta}(0)$ for all $t > 0$. Additionally, in case of (3.3) one has that the only dissipativity comes from the memory term and due to technical reasons we must take some specific features for the density coefficient $\rho(x)$. In this part we follow the same assumptions as in [1, Rem. 3.3]. On the other hand, in a more general scenario with $\rho(x)$ satisfying only ($\mathbf{A_3}$), the presence of the frictional damping is necessary with proper condition on $b(x)$. In both situations, the Assumption 2.2 will play an important role in the stability result.

Under the above remark, we are going to consider two cases as follows. For this purpose, we follow the same assumptions as regarded in [1], see Remarks 3.2 and 3.3 therein.

Assumption 3.1. Concerning the density and frictional coefficients, we assume either:

I. General Density $\rho(x)$. If $\rho(x)$ is a general function, then we assume that $b(x)$ is effective at the whole set A, that is, there exists $b_0 > 0$ such that

$$b(x) \geq b_0 > 0 \quad \text{a.e.} \quad x \in A. \tag{3.4}$$

II. Specific Density $\rho(x)$. Let $K = (K_{i,j})$ be a matrix given by $K_{i,j}(x) = \kappa(x)\delta_{ij}$, where δ_{ij} is the Kronecker delta function, and set $\omega' = \Omega \setminus A$. If $\rho(x)$ satisfies one of the following statements, then we can consider $b \equiv 0$ in (1.1).

- **Constant Case:** $\rho(x) = \rho_0 > 0$ for $n \geq 2$. In this scenario, we observe that the geodesics of the metric $G = \left(\frac{K}{\rho(x)}\right)^{-1}$ are straight lines in A and, therefore, every geodesic of the G metric which enters A does not remain inside A. Thus, ω' satisfies the Geometric Control Condition (GCC) according to [1, Appendix 5].
- **Non-constant Case:** $\rho(x) = (1 - g_0 a(x))^{\frac{n}{n-2}}$ for $n \geq 3$. In this case, we note that $\rho(x) = \kappa(x) = 1$, for all $x \in A$. Then, if we consider the Riemannian metric $G = \left(\frac{K}{\rho(x)}\right)^{-1}$, we still observe that the geodesics are straight lines in A. Therefore, every geodesic which crosses A does not remain in A. Consequently, the set ω' also satisfies the GCC. We refer again to [1, Appendix 5] to the proof that every geodesic in Riemannian metric G meets $\partial\Omega$ as well as to find other examples of non-constant density $\rho(x)$ in the particular dimension $n = 2$.

Remark 3.2. It is worth pointing out that, even under the Assumption 2.1–($\mathbf{A_1}$)–($\mathbf{A_2}$) and Assumption 3.1–I, the condition (1.5) is not regarded. Moreover, in view of Assumption 3.1–II we are going to provide a general stability (of polynomial type) to the energy defined in (3.1) under the sole localized viscoelastic effect. To illustrate the Assumptions 3.1 and 2.1 we consider Fig. 1.

Our main stability result reads as follows.

Theorem 3.3 (*Main Result*)**.** *Let us take on the Assumptions 2.1 and 2.2, and let $R > 0$ be given such that $E_{u,\eta}(0) \leq R$ and $\sup_{\tau<0} \|\sqrt{a}\nabla u_0(\tau)\| < R$. We have:*

I. *Under the additional Assumption 3.1–I, then there exists a time $T_0 > 0$ such that*

$$E_{u,\eta}(t) \leq S\left(\frac{t}{T_0} - 1\right), \quad \forall\, t > T_0, \quad \textit{with} \quad \lim_{t\to\infty} S(t) = 0, \tag{3.5}$$

where $S(t)$ is the solution of the ODE

$$\frac{d}{dt}S(t) + q_1(S(t)) = 0, \quad S(0) = E_{u,\eta}(0), \tag{3.6}$$

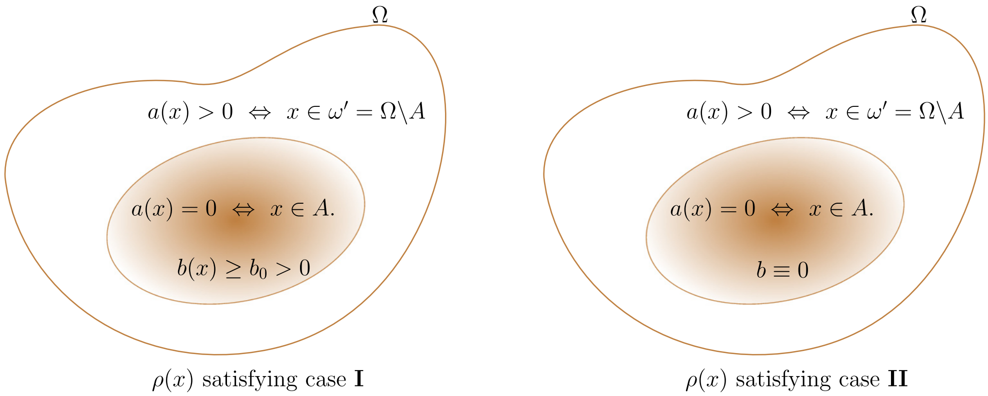

Fig. 1. The viscoelastic effect acts in $\omega' = \Omega\backslash A$ while the frictional one is only located in the closed connected set $A \subset \Omega$ in case **I** and may vanish in case **II**.

with $q_1(s) = s - (Id + p_1)^{-1}(s)$, and for some constant $C = C(R,T) > 0$ we have

$$\begin{aligned} p_1(s) &= \left[C\left(J + \widetilde{H}^{-1} + \left(\frac{1}{M_1} + M_2\right) Id\right)\right]^{-1}(s),\\ \widetilde{H}(s) &= T\|b\|_{L^1(\Omega)} H\left(\frac{1}{T\|b\|_{L^1(\Omega)}} s\right), \ T > T_0,\\ J(s) &= \frac{g(0)^{(1-\alpha_0)\frac{\alpha_0}{p_2}}}{C_2^{\frac{\alpha_0}{p_2}}} \left(\sup_{t\geq 0}\int_0^\infty g(r)^{1-\alpha_0}\|\sqrt{a}\nabla\eta^t(r)\|^2 dr + 1\right)^{1-\frac{\alpha_0}{p_2}} T\left(\frac{s}{T}\right)^{\frac{\alpha_0}{p_2}}, \ T > T_0. \end{aligned}$$

II. *Under the additional Assumption 3.1–II, then there exists a time $T_0 > 0$ such that*

$$E_{u,\eta}(t) \leq S\left(\frac{t}{T_0} - 1\right), \quad \forall t > T_0, \quad \text{with} \quad \lim_{t\to\infty} S(t) = 0, \tag{3.7}$$

where $S(t)$ is now the solution of the ODE

$$\frac{d}{dt}S(t) + q_2(S(t)) = 0, \quad S(0) = E_{u,\eta}(0), \tag{3.8}$$

and for some constant $\tilde{C} = \tilde{C}(R,T) > 0$ we have

$$\begin{aligned} q_2(s) &= s - (Id + p_2)^{-1}(s),\\ p_2(s) &= \tilde{C}\cdot J^{-1}(s), \quad J \text{ set as above.} \end{aligned}$$

Remark 3.4. Before proceeding with the main result that leads to the proof of the decay rates (3.5) in case **I** (under the Assumption 3.1–I) and (3.7) in case **II** (under the Assumption 3.1–II), let us first give some comments on examples of stability.

I. Let us split this case into four sub-cases:

(1) For exponential kernel g (which would mean $p_2 = p_1 = 1$ in (2.6)) and linear frictional damping (i.e. taking $h(u_t) = u_t$ in (1.1)), then (3.5) falls on the exponential stability. Indeed, this is exactly the case approached in [1, Sects. 3 and 4], see Theorem 3.1 (for $f = 0$) and Theorem 4.1 (for $f \neq 0$) therein.

(2) Still considering exponential memory kernel g, but taking h as a non-linear frictional damping satisfying ($\mathbf{A_6}$), then (3.5) is carried out by the frictional effect through the function H constructed in Remark 2.1-3, and several examples of decay rates can be found e.g. in [28,38].

(3) For non-exponential kernel g satisfying (2.6) and linear frictional damping ($h(u_t) = u_t$), then the stability (3.5) is driven by the viscoelastic effect coming from the memory kernel g, and by virtue of the assumptions $(\mathbf{G_1})$–$(\mathbf{G_2})$, the concrete polynomial decay rates can be achieved similar to that presented in [21, Sect. IV].

(4) In a general situation where g satisfies $(\mathbf{G_1})$–$(\mathbf{G_2})$ and h satisfies $(\mathbf{A_6})$, then the decay (3.5) is conducted by the worst scenario, namely, the worst decay rate estimate provided by g and h, as considered in [28, Sect. 1], see Theorem 1.4 and Remarks 4–6 therein.

II. In this case, the stability coming from (3.7) depends only on the (polynomial) behavior of the memory kernel g, and the decay rate is already clarified in [21, Sect. IV], as above.

The main tool in the conclusion of the proof of Theorem 3.3 is given by the next result. Indeed, it will provide new and key observability inequalities that play a crucial role in the proofs of the estimates (3.5) in case **I** and (3.7) in case **II**. More precisely, we have:

Proposition 3.5 (*Observability Inequality*)**.** *Let us take on the Assumptions 2.1 and 2.2, and let $R > 0$ be given such that $E_{u,\eta}(0) \leq R$. We have:*

I. *Under the further Assumption 3.1–I, then for all $T > T_0 > 0$ there exists a constant $C = C(T, R) > 0$ such that*

$$E_{u,\eta}(0) \leq C \left(\int_0^T \int_0^\infty g(s) \|\sqrt{a} \nabla \eta^t(s)\|^2 ds dt + \int_0^T \int_\Omega b(x)(|u_t(t)|^2 + |h(u_t(t))|^2) dx dt \right). \tag{3.9}$$

II. *Under the further Assumption 3.1–II, then for all $T > T_0 > 0$ there exists a constant $\tilde{C} = \tilde{C}(T, R) > 0$ such that*

$$E_{u,\eta}(0) \leq \tilde{C} \int_0^T \int_0^\infty g(s) \|\sqrt{a} \nabla \eta^t(s)\|^2 ds dt. \tag{3.10}$$

Both Proposition 3.5 and Theorem 3.3 will be proved in the next section. To the proof of Proposition 3.5 we rely on contradiction arguments in combination with microlocal analysis. For the latter we use the theory developed by Gérard in [39], whose results are briefly recalled in Appendix A.2 to make this paper more self-contained as possible. To this end, we quote and use again [1, Appendix 5] where a more complete range of results in microlocal analysis are built and adjusted to the present problem.

4. Proofs

4.1. Proof of the observability inequality: Part I

Let us start by proving inequality (3.9) in Proposition 3.5. Indeed, if it does not hold, then there exists a time $T > T_0 > 0$ and a sequence of solutions (u_k, η_k) for (1.1) verifying

$$E_{u_k,\eta_k}(0) \leq R, \tag{4.1}$$

and

$$\lim_{k\to\infty} \frac{E_{u_k,\eta_k}(0)}{\displaystyle\int_0^T \int_0^\infty g(s) \|\sqrt{a} \nabla \eta_k^t(s)\|^2 ds dt + \int_0^T \int_\Omega b(x)(|u_k'(t)|^2 + |h(u_k'(t))|^2) dx dt} = +\infty, \tag{4.2}$$

where $E_{u_k,\eta_k}(t)$ is the energy defined in (3.1) associated with the solution (u_k, η_k) of (1.1). Also, from now on, the notation u_k' stands for the time derivative $\partial_t u_k$. Combining (4.1) and (4.2) it follows that

$$\lim_{k\to\infty} \left[\int_0^T \int_0^\infty g(s) \|\sqrt{a} \nabla \eta_k^t(s)\|^2 ds dt + \int_0^T \int_\Omega b(x)(|u_k'(t)|^2 + |h(u_k'(t))|^2) dx dt \right] = 0. \tag{4.3}$$

Now we claim that

$$\lim_{k\to\infty}\int_0^T\int_0^\infty -g'(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2dsdt=0. \tag{4.4}$$

In fact, since $p_1>1$, we can write $p_1=\boldsymbol{\zeta}+\xi$ where $\boldsymbol{\zeta}\geq 1$ and $0<\xi<1$, and then we have

$$\begin{aligned}
&\int_0^T\int_0^\infty(-g'(s))\|\sqrt{a}\nabla\eta_k^t(s)\|^2dsdt\\
&\quad=\int_0^T\int_0^\infty\frac{-g'(s)}{g(s)^\xi}g(s)^\xi\|\sqrt{a}\nabla\eta_k^t(s)\|^2dsdt\\
&\quad\leq\int_0^T\left[\int_0^\infty\frac{-g'(s)}{g(s)^\xi}\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\right]^{1-\xi}\left[\int_0^\infty\frac{-g'(s)}{g(s)^\xi}g(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\right]^\xi dt\\
&\quad=\int_0^T\left[\int_0^\infty\frac{-g'(s)}{g(s)^\xi}\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\right]^{1-\xi}\left[\int_0^\infty -g'(s)g(s)^{1-\xi}\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\right]^\xi dt.
\end{aligned}$$

From the hypothesis $(\mathbf{G_1})$, we also see

$$\frac{-g'(s)}{g(s)^\xi}\leq C_1\frac{g(s)^{p_1}}{g(s)^\xi}=C_1g(s)^{p_1-\xi}=C_1g(s)^{\boldsymbol{\zeta}}\text{ with }\boldsymbol{\zeta}\geq 1.$$

If $g(0)\leq 1$, then $g(s)^{\boldsymbol{\zeta}}\leq g(s)$ for all $s\geq 0$, once g is decreasing. Combining the latter with the preceding inequality, we get

$$\int_0^\infty\frac{-g'(s)}{g(s)^\xi}\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\leq C_1\int_0^\infty g(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds.$$

On the other hand, if $g(0)>1$, there exists $s_0>0$ such that $g(s_0)=1$ and $g(s)\leq 1$ for all $s\geq s_0$. Thus,

$$\begin{aligned}
&\int_0^\infty\frac{-g'(s)}{g(s)^\xi}\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\\
&\quad\leq C_1\int_0^\infty g(s)^{\boldsymbol{\zeta}}\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\\
&\quad=C_1\int_0^{s_0}g(s)^{\boldsymbol{\zeta}}\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds+C_1\int_{s_0}^\infty g(s)^{\boldsymbol{\zeta}}\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\\
&\quad\leq C_1\frac{g(0)^{\boldsymbol{\zeta}}}{g(s_0)}\int_0^{s_0}g(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds+C_1\int_{s_0}^\infty g(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\\
&\quad=C_1\left(1+g(0)^{\boldsymbol{\zeta}}\right)\int_0^\infty g(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds.
\end{aligned}$$

In both situations, we obtain

$$\int_0^\infty\frac{-g'(s)}{g(s)^\xi}\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\leq C_1\left(1+g(0)^{\boldsymbol{\zeta}}\right)\int_0^\infty g(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds. \tag{4.5}$$

In addition, we still have

$$-g'(s)g(s)^{1-\xi}\leq C_1g(s)^{p_1}g(s)^{1-\xi}=C_1g(s)^{1-\xi+p_1}=C_1g(s)^{1+\boldsymbol{\zeta}},$$

and following similar arguments as in (4.5), we infer

$$\int_0^\infty -g'(s)g(s)^{1-\xi}\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\leq\left(C_1+\frac{g(0)^{\boldsymbol{\zeta}}}{g(s_0)}\right)\int_0^\infty g(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds.$$

Therefore,

$$\begin{aligned}
&\int_0^T\int_0^\infty(-g'(s))\|\sqrt{a}\nabla\eta_k^t(s)\|^2dsdt\\
&\leq\int_0^T\left[\int_0^\infty\frac{-g'(s)}{g(s)^\xi}\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\right]^{1-\xi}\left[\int_0^\infty-g'(s)g(s)^{1-\xi}\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\right]^\xi dt\\
&\leq C_1\left(1+\frac{g(0)^\zeta}{g(s_0)}\right)\int_0^T\left[\int_0^\infty g(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\right]^{1-\xi}\left[\int_0^\infty g(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2ds\right]^\xi dt\\
&=C_1\left(1+g(0)^\zeta\right)\int_0^T\int_0^\infty g(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2dsdt,
\end{aligned}$$

from where (4.4) follows by means of the limit (4.3). Keeping these both convergences (4.3)–(4.4) in mind, we proceed as follows.

Since $E_{u_k,\eta_k}(t)\leq E_{u_k,\eta_k}(0)\leq R$, for all $t\geq0$, it follows that there exists a subsequence $\{(u_k,\eta_k)\}$, still denoted by $\{(u_k,\eta_k)\}$, such that

$$u_k\overset{*}{\rightharpoonup}u \text{ in } L^\infty(0,T;H_0^1(\Omega)),\tag{4.6}$$
$$u_k'\overset{*}{\rightharpoonup}u' \text{ in } L^\infty(0,T;L^2(\Omega)),\tag{4.7}$$

and, from Aubin–Lions Theorem it follows that

$$u_k\to u \text{ in } L^2(0,T;L^2(\Omega)).\tag{4.8}$$

To achieve the desired contradiction, let us consider two cases.

Case 1: $u\neq0$. For each $k\in\mathbb{N}$, (u_k,η_k) is a solution of the following problem

$$\begin{cases}\rho(x)u_k''-\operatorname{div}[\kappa(x)\nabla u_k]-\displaystyle\int_0^\infty g(s)\operatorname{div}[a(x)\nabla\eta_k^t(s)]\,ds+b(x)h(u_k')+f(u_k)=0\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{in } \Omega\times(0,T),\\ \eta_k'=-\partial_s\eta_k+u_k' \qquad\qquad\qquad\qquad\qquad\text{in } \Omega\times(0,T)\times(0,\infty),\end{cases}\tag{4.9}$$

with initial and boundary conditions

$$\begin{aligned}&u_k=0 \text{ on } \partial\Omega\times(0,T),\quad \eta_k=0 \text{ on } \partial\Omega\times(0,T)\times(0,\infty),\\ &u_k(0)=u_{0k},\ u_k'(0)=u_{1k},\ \eta_k^0(s)=\eta_{0k}(s) \text{ in } \Omega,\ s\in(0,\infty),\\ &\eta_k^t(0)=0 \text{ in } \Omega,\ t\in[0,T).\end{aligned}\tag{4.10}$$

From (2.8) and $H_0^1(\Omega)\hookrightarrow L^{2p}(\Omega)$ we obtain

$$\|f(u_k)\|_{L^\infty(0,T;L^2(\Omega))}\leq c\|u_k\|^2_{L^\infty(0,T;H_0^1(\Omega))}+c\|u_k\|^{2p}_{L^\infty(0,T;H_0^1(\Omega))}\leq c,\quad\forall k\in\mathbb{N},\tag{4.11}$$

and, in particular, this implies that $\{f(u_k)\}$ is bounded in $L^2(0,T;L^2(\Omega))$. Besides, $u_k\to u$ almost everywhere in $\Omega\times(0,T)$, and from the continuity of f it follows that $f(u_k)\to f(u)$ almost everywhere in $\Omega\times(0,T)$. Then, from Lions' Lemma

$$f(u_k)\rightharpoonup f(u) \text{ in } L^2(0,T;L^2(\Omega)).\tag{4.12}$$

From (4.3), (4.6) and (4.12) results

$$\rho(x)u''-\operatorname{div}[\kappa(x)\nabla u]+f(u)=0 \text{ in } L^2(0,T;H^{-1}(\Omega)).\tag{4.13}$$

Now, motivated by [1] (see on page 6555 therein), we define the auxiliary function

$$z_k \;=\; \kappa(x)u_k + a(x)\int_0^\infty g(s)\eta_k^t(s)\,ds, \quad k\in\mathbb{N}. \tag{4.14}$$

Using the second equation of (4.9), integration by parts and $(4.10)_3$ we have, for almost every $t>0$,

$$\begin{aligned} z_k'(t) &= \kappa(x)u_k'(t) + a(x)\int_0^\infty g(s)[\eta_k^t(s)]'\,ds \\ &= \kappa(x)u_k'(t) + a(x)\int_0^\infty g(s)[-\partial_s\eta_k^t(s) + u_k'(t)]\,ds \\ &= u_k'(t) + a(x)\int_0^\infty g'(s)\eta_k^t(s)\,ds. \end{aligned}$$

Now, let ω^* be a closed subset of Ω such that $A \subset\subset \omega^*$. Then there exists $c>0$ such that

$$\int_{\Omega\setminus\omega^*} |\eta_k^t(x,s)|^2 dx \le \frac{1}{a_0\lambda_1}\int_{\Omega\setminus\omega^*} a(x)|\nabla\eta_k^t(x,s)|^2 dx, \quad \forall s>0, \tag{4.15}$$

where $a(x)\ge a_0>0$ for all $x\in\overline{\Omega\setminus\omega^*}$. Then, from (4.4), (4.7) and (4.15), we infer

$$z_k' \rightharpoonup u' \;\text{ in }\; L^2(0,T;L^2(\Omega\setminus\omega^*)). \tag{4.16}$$

Moreover, an analogous calculation implies, in view of (4.3), (4.8) and (4.15), that

$$z_k \to \kappa(\cdot)u \;\text{ in }\; L^2(0,T;L^2(\Omega\setminus\omega^*)) \hookrightarrow \mathcal{D}'(0,T;L^2(\Omega\setminus\omega^*)),$$

and consequently

$$z_k' \to \kappa(\cdot)u' \;\text{ in }\; \mathcal{D}'(0,T;L^2(\Omega\setminus\omega^*)). \tag{4.17}$$

From (4.16)–(4.17) we obtain $\kappa(\cdot)u' = u'$ in $L^2(0,T;L^2(\Omega\setminus\omega^*))$. As $\kappa(x)-1\neq 0$ in ω^*,

$$u' \equiv 0 \quad \text{a.e. in} \quad \omega^*\times(0,T), \;\text{for all}\; \omega^*\supset\supset A,$$

which implies

$$u'\equiv 0 \text{ a.e. in } (\Omega\setminus A)\times(0,T). \tag{4.18}$$

On the other hand, from (4.3) and using that $b(x)\ge b_0>0$ in A, we obtain

$$u'\equiv 0 \text{ a.e. in } A\times(0,T). \tag{4.19}$$

Consequently, from (4.18) and (4.19) we deduce

$$u'\equiv 0 \text{ a.e. in } \Omega\times(0,T),$$

which implies $u''\equiv 0$ in $\Omega\times(0,T)$, and also

$$\operatorname{div}[\kappa(x)\nabla u(t)] = f(u(t)),$$

which in turn implies, along with condition $(\mathbf{A_5})$-(a), that

$$\|u(t)\|_1^2 - \frac{\beta}{2}\|u(t)\|^2 = -\int_\Omega f(u(t))u\,dx - \frac{\beta}{2}\|u(t)\|^2 \le \frac{\beta}{2}\|u(t)\|^2,$$

and

$$\lambda_1\|u(t)\|^2 \le \beta\|u(t)\|^2.$$

Since $\beta < \lambda_1$, one concludes

$$u \equiv 0 \text{ in } \Omega \times (0,T),$$

which is a contradiction.

Case 2: $u = 0$. In this case, for each $k \in \mathbb{N}$, we initially define

$$\alpha_k = \left[E_{u_k,\eta_k}(0)\right]^{1/2}, \qquad v_k = \frac{1}{\alpha_k}\, u_k, \qquad \zeta_k = \frac{1}{\alpha_k}\, \eta_k. \tag{4.20}$$

Then, (v_k, ζ_k) is a solution of

$$\begin{cases} \rho(x)v_k'' - \operatorname{div}[\kappa(x)\nabla v_k] - \displaystyle\int_0^\infty g(s)\operatorname{div}[a(x)\nabla\zeta_k^t(s)]\, ds + b(x)\frac{h(\alpha_k v_k')}{\alpha_k} + \frac{f(\alpha_k v_k)}{\alpha_k} = 0 & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega \times (0,T), & \\ \{\zeta_k^t\}' = -\partial_s \zeta_k^t + v_k' \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{in } \Omega \times (0,T) \times (0,\infty), & \end{cases} \tag{4.21}$$

with initial and boundary conditions

$$\begin{aligned} & v_k = 0 \ \text{ on } \ \partial\Omega \times (0,T), \ \ \zeta_k = 0 \ \text{ on } \ \partial\Omega \times (0,T) \times (0,\infty), \\ & v_k(0) = v_{0k}, \ v_k'(0) = v_{1k}, \ \zeta_k^0(s) = \zeta_{0k}(s) \ \text{ in } \ \Omega, \ s \in (0,\infty), \\ & \zeta_k^t(0) = 0 \ \text{ in } \ \Omega, \ t \in [0,T). \end{aligned} \tag{4.22}$$

The energy functional associated with (4.21)–(4.22) is given by

$$\begin{aligned} E_{v_k,\zeta_k}(t) \ = \ & \frac{1}{2}\int_\Omega \rho(x)|v_k'(t)|^2\, dx + \frac{1}{2}\int_\Omega \kappa(x)|\nabla v_k(t)|^2\, dx + \frac{1}{2}\int_0^\infty g(s)\|\sqrt{a}\nabla\zeta_k^t\|^2 ds \\ & + \int_\Omega \frac{F(\alpha_k v_k(t))}{\alpha_k^2} dx. \end{aligned} \tag{4.23}$$

We first notice that hypothesis $(\mathbf{A_5})$-(a) yields

$$\begin{aligned} \frac{1}{2}\|v_k(t)\|_1^2 + \int_\Omega \frac{F(\alpha_k v_k(t))}{\alpha_k^2} dx \ \ & \geq \ \ \frac{1}{2}\|v_k(t)\|_1^2 - \frac{1}{\alpha_k^2}\frac{\beta}{2}\|\alpha_k v_k(t)\|^2 \\ & = \ \ \frac{1}{2}\|v_k(t)\|_1^2 - \frac{\beta}{2}\|v_k(t)\|^2 \\ & = \ \ \frac{1}{2}\|v_k(t)\|_1^2 - \frac{\beta}{2\lambda_1}\|v_k(t)\|_1^2 \geq 0, \end{aligned}$$

and then $E_{v_k,\zeta_k}(t) \geq 0$, from all $k \in \mathbb{N}$ and all $t \in [0,T]$. We also observe that the energy functional $E_{v_k,\zeta_k}(t)$ satisfies

$$\frac{d}{dt}E_{v_k,\zeta_k}(t) = \frac{1}{2}\int_0^\infty g'(s)\|\sqrt{a}\nabla\zeta_k^t\|^2 ds - \int_\Omega b(x)v_k'(t)\frac{h(\alpha_k v_k'(t))}{\alpha_k}\, dx \leq 0, \quad t \in [0,T].$$

Then, $E_{v_k,\eta_k}(t)$ is a non-increasing function such that

$$E_{v_k,\zeta_k}(0) = E_{v_k,\zeta_k}(T) + \frac{1}{2}\int_0^T \int_0^\infty (-g'(s))\|\sqrt{a}\nabla\zeta_k^t\|^2 ds + \int_\Omega b(x)v_k'\frac{h(\alpha_k v_k'(t))}{\alpha_k}\, dxdt. \tag{4.24}$$

Besides, from (4.20) we have

$$E_{v_k,\zeta_k}(t) = \frac{1}{\alpha_k^2}E_{u_k,\eta_k}(t) = \frac{1}{E_{u_k,\eta_k}(0)}E_{u_k,\eta_k}(t), \quad t \in [0,T],$$

and then

$$E_{v_k,\zeta_k}(0) \ = \ 1, \quad \forall\, k \in \mathbb{N}. \tag{4.25}$$

Accordingly to (4.3)–(4.4),

$$\lim_{k\to+\infty}\left[\int_0^T\int_0^\infty g(s)\|\sqrt{a}\nabla\zeta_k^t\|^2 dsdt+\int_0^T\int_\Omega b(x)|v_k'(x,t)|^2+b(x)\frac{|h(\alpha_k v_k')|^2}{\alpha_k^2}\,dxdt\right]=0, \tag{4.26}$$

and

$$\lim_{k\to+\infty}\int_0^T\int_0^\infty -g'(s)\|\sqrt{a}\nabla\zeta_k^t\|^2 dsdt=0. \tag{4.27}$$

Thus, passing (4.24) to the limit, and observing (4.25)–(4.27), we conclude

$$1=\lim_{k\to+\infty}E_{v_k,\zeta_k}(0)=\lim_{k\to+\infty}E_{v_k,\zeta_k}(T). \tag{4.28}$$

On the other hand, if we show that $E_{v_k,\zeta_k}(T)$ goes to zero, that is,

$$\lim_{k\to+\infty}E_{v_k,\zeta_k}(T)=0, \tag{4.29}$$

then desired contradiction is achieved in this case and, therefore, the proof of (3.9) in the part **I** is complete.

In what follows, we aim to obtain (4.29). Indeed, since $E_{v_k,\zeta_k}(t)\le E_{v_k,\zeta_k}(0)=1$ for all $t\in[0,T]$, there exists a subsequence of $\{(v_k,\zeta_k)\}$, still denoted by $\{(v_k,\zeta_k)\}$, such that

$$v_k \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,T;H_0^1(\Omega)), \tag{4.30}$$

$$v_k' \overset{*}{\rightharpoonup} v' \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{4.31}$$

$$v_k \to v \text{ in } L^2(0,T;L^2(\Omega)). \tag{4.32}$$

Firstly, let us prove that $v\equiv 0$ in $\Omega\times(0,T)$. Since $\alpha_k=[E_{u_k,\eta_k}(0)]^{1/2}$, from (4.1) results that there exists $\alpha\ge 0$ such that

$$\alpha_k\to\alpha.$$

We consider again two possibilities, namely, $\alpha>0$ or $\alpha=0$.

P1. $\alpha>0$. Here, from (4.8) and (4.20) result that $\alpha_k v_k\to 0$ in $L^2(0,T;L^2(\Omega))$, and following similar arguments as in the proof of (4.12), we obtain

$$\frac{f(\alpha_k v_k)}{\alpha_k}\rightharpoonup 0 \text{ in } L^2(0,T;L^2(\Omega)). \tag{4.33}$$

From (4.26), (4.30), (4.31), (4.33), and passing $(4.21)_1$ to the limit, we obtain

$$\rho(x)v''-\operatorname{div}[\kappa(x)\nabla v]=0 \text{ in } L^2(0,T;H^{-1}(\Omega)). \tag{4.34}$$

Using analogous computations as in (4.14)–(4.18), from (4.26) and hypothesis (3.4) we conclude that $v'\equiv 0$ in $\Omega\times(0,T)$. Therefore, standard multipliers with (4.34) lead to $v\equiv 0$ in $\Omega\times(0,T)$.

P2. $\alpha=0$. By using Taylor's formula along with hypotheses $f(0)=0$ and ($\mathbf{A_5}$)-(b), we get

$$f(s)=f'(0)s+R(s), \tag{4.35}$$

where

$$|R(s)|\le c|s|^2+c|s|^p,\quad \forall s\in\mathbb{R}. \tag{4.36}$$

From (4.36), the fact that $\alpha=0$ and $\{v_n\}$ is bounded in $L^\infty(0,T;H_0^1(\Omega))$, we deduce

$$\frac{R(\alpha_k v_k(t))}{\alpha_k}\to 0 \text{ in } L^\infty(0,T;L^1(\Omega)). \tag{4.37}$$

Passing (4.21)$_1$ to the limit, and using (4.26), (4.35), (4.37), we obtain

$$\rho(x)v'' - \operatorname{div}[\kappa(x)\nabla v] + f'(0)v = 0 \text{ in } L^2(0,T;H^{-1}(\Omega)). \tag{4.38}$$

Now, we define

$$\tilde{z}_k \;=\; \kappa(x)v_k + a(x)\int_0^\infty g(s)\zeta_k^t(s)\,ds, \quad k \in \mathbb{N}.$$

Again proceeding as in (4.14)–(4.18), we have $v' \equiv 0$ in $(\Omega \setminus A)\times(0,T)$. Taking the derivative of (4.38) in t and writing $v' = w$, we arrive at

$$\begin{cases} \rho(x)w'' - \operatorname{div}[\kappa(x)\nabla w] + f'(0)w = 0 \;\text{ in }\; \Omega\times(0,T), \\ w \equiv 0 \text{ in } (\Omega\setminus A)\times(0,T). \end{cases} \tag{4.39}$$

At this moment we use the strength of the unique continuation property provided by Theorem A.1. In fact, from the latter, it results that $v' \equiv 0$ in $\Omega\times(0,T)$. Then, multiplying (4.38) by v, integrating by parts in Ω and integrating in $(0,T)$, we infer

$$\lambda_1\|v\|^2_{L^2(0,T;L^2(\Omega))} + f'(0)\|v\|^2_{L^2(0,T;L^2(\Omega))} = 0. \tag{4.40}$$

Regarding that $0 \le \beta < \lambda_1$ and $\beta + f'(0) \ge 0$, then (4.40) yields

$$0 \le \beta\|v\|^2_{L^2(0,T;L^2(\Omega))} + f'(0)\|v\|^2_{L^2(0,T;L^2(\Omega))} < \lambda_1\|v\|^2_{L^2(0,T;L^2(\Omega))} + f'(0)\|v\|^2_{L^2(0,T;L^2(\Omega))} = 0, \tag{4.41}$$

which implies $v \equiv 0$ in $\Omega\times(0,T)$. Consequently, we have proved that $v \equiv 0$ in $\Omega\times(0,T)$ as desired, and from (4.30)–(4.32), we infer

$$v_k \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty(0,T;H_0^1(\Omega)), \tag{4.42}$$

$$v_k' \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{4.43}$$

$$v_k \to 0 \text{ in } L^2(0,T;L^2(\Omega)). \tag{4.44}$$

To the next estimates, let us take $\theta \in C_0^\infty(0,T)$ and $\varphi \in C_0^\infty(\Omega)$ such that supp$\varphi \subset (\Omega\setminus A)$. From this,

$$\int_\Omega (\varphi(x) + |\nabla\varphi(x)|)|\zeta_k^t|^2\,dx \le c_\varphi \int_\Omega a(x)|\nabla\zeta_k^t|^2\,dx. \tag{4.45}$$

In addition, setting $\phi_k(x,t) = \int_0^\infty g(s)\zeta_k^t(x,s)\,ds$, multiplying the first equation of (4.21) by $\theta\varphi(x)\phi_k$, and integrating by parts on $\Omega\times(0,T)$, we get

$$\begin{aligned} 0 \;=\; & -\int_0^T\int_\Omega \rho(x)v_k'\phi_k'\varphi(x)\theta(t)\,dxdt - \int_0^T\int_\Omega \rho(x)v_k'\phi_k\varphi(x)\theta'(t)\,dxdt \\ & + \int_0^T\int_\Omega \kappa(x)\nabla v_k\cdot\nabla\phi_k\,\varphi(x)\theta(t)\,dxdt + \int_0^T\int_\Omega \kappa(x)\nabla v_k\cdot\nabla\varphi(x)\phi_k\theta(t)\,dxdt \\ & + \int_0^T\int_0^\infty g(s)\int_\Omega a(x)\nabla\zeta_k^t(s)\cdot\nabla\varphi(x)\phi_k\theta(t)\,dxdsdt \\ & + \int_0^T\int_0^\infty g(s)\int_\Omega a(x)\nabla\zeta_k^t\cdot\nabla\phi_k\,\varphi(x)\theta(t)\,dxdsdt \\ & + \int_0^T\int_\Omega b(x)\frac{h(\alpha_k v_k')}{\alpha_k}\phi_k\varphi(x)\theta(t)\,dxdt \\ & + \int_0^T\int_\Omega \frac{f(\alpha_k v_k)}{\alpha_k}\phi_k\varphi(x)\theta(t)dxdt := J_{1k} + J_{2k} + \cdots + J_{6k} + J_{7k} + J_{8k}. \end{aligned} \tag{4.46}$$

From the convergences (4.26), (4.42)–(4.44) and (4.45) we directly see that

$$\lim_{k\to\infty} J_{2k} = \cdots = \lim_{k\to\infty} J_{6k} = 0. \tag{4.47}$$

In order to obtain a limit for J_{7k}, we take (4.26) into account and the following inequalities

$$\begin{aligned}
&\left|\int_0^T \int_0^\infty g(s) \int_\Omega b(x)\frac{h(\alpha_k v_k')}{\alpha_k}\zeta_k^t(x,s)\varphi(x)\theta(t)\,dxdtds\right| \\
&\le c(b,\varphi,\theta)\int_0^T \int_0^\infty g(s)\left[\int_\Omega b(x)\left|\frac{h(\alpha_k v_k')}{\alpha_k}\right|^2 dx\right]^{\frac{1}{2}}\left[\int_\Omega |\varphi(x)||\zeta_k^t(x,s)|^2 dx\right]^{\frac{1}{2}} dsdt \\
&\le c(b,\varphi,\theta,g_0)\left[\int_0^T \int_\Omega b(x)\left|\frac{h(\alpha_k v_k')}{\alpha_k}\right|^2 dxdt + \int_0^T \int_0^\infty g(s)\int_\Omega a(x)|\nabla\zeta_k^t(x,s)|^2 dxdsdt\right],
\end{aligned} \tag{4.48}$$

to conclude that $\lim\limits_{k\to\infty} J_{7k} = 0$.

To compute a limit for J_{8k} we use (2.8) to have

$$\left|\frac{f(\alpha_k v_k(t))}{\alpha_k}\right| \le c(p)|v_k(t)| + c(p)|\alpha_k|^{p-1}|v_k(t)|^p, \quad \forall k\in\mathbb{N}.$$

Consequently, there exists a constant $c>0$ such that

$$\int_0^T \int_\Omega \left|\frac{f(\alpha_k v_k(t))}{\alpha_k}\right|^2 dxdt < c, \quad \forall k\in\mathbb{N} \quad \forall t\in[0,T].$$

Then, from (4.26) and

$$\int_0^T \int_\Omega \left|\int_0^\infty g(s)\zeta_k^t(x,s)\varphi(x)\theta(t)ds\right|^2 dxdt \le c(\theta,g_0,a)\int_0^T \int_0^\infty g(s)\|\sqrt{a}\nabla\zeta_k^t(s)\|^2 dsdt,$$

we get $\lim\limits_{k\to\infty} J_{8k} = 0$. Going back to (4.46), one sees that $\lim J_{1k} = 0$. Now, remembering that $\{\zeta_k^t\}' = -\partial_s\zeta_k^t + v_k'$ and writing

$$\begin{aligned}
J_{1k} &= -g_0\int_0^T \theta(t)\int_\Omega \varphi(x)\rho(x)|v_k'|^2 dxdt - \int_0^T \theta(t)\int_0^\infty g'(s)\int_\Omega \varphi(x)\rho(x)\zeta_k(s)v_k'(t)dxdsdt \\
&:= L_{1k} + L_{2k},
\end{aligned}$$

and since $\lim\limits_{k\to\infty} L_{2k} = 0$, we conclude

$$\lim_{k\to\infty}\int_0^T \int_\Omega \rho(x)\theta(t)\,\varphi(x)|v_k'|^2\,dxdt = 0. \tag{4.49}$$

Returning to (4.21), multiplying its first equation by $\theta(t)\varphi(x)v_k$ and integrating by parts on $\Omega\times(0,T)$, yields

$$\begin{aligned}
0 \;=\; & -\int_0^T \theta(t)\int_\Omega \varphi(x)\rho(x)|v_k'(t)|^2\,dxdt - \int_0^T \theta'(t)\int_\Omega \varphi(x)\rho(x)v_k'(t)v_k(t)\,dxdt \\
& + \int_0^T \theta(t)\int_\Omega \varphi(x)\kappa(x)|\nabla v_k(t)|^2\,dxdt + \int_0^T \int_\Omega \kappa(x)\nabla v_k\cdot\nabla\varphi(x)v_k\theta(t)dxdt \\
& + \int_0^T \int_0^\infty g(s)\int_\Omega a(x)\nabla\zeta_k^t(s)\cdot\nabla\varphi(x)v_k\theta(t)dxdsdt \\
& + \int_0^T \theta(t)\int_\Omega \left(\int_0^\infty g(s)a(x)\nabla\zeta_k^t(s)\,ds\right)\cdot\nabla v_k(t)\,dxdt \\
& + \int_0^T \int_\Omega b(x)\frac{h(\alpha_k v_k'(t))}{\alpha_k}v_k(t)\varphi(x)\theta(t)\,dxdt \\
& + \int_0^T \int_\Omega \frac{f(\alpha_k v_k(t))}{\alpha_k}v_k(t)\varphi(x)\theta(t)dxdt.
\end{aligned} \tag{4.50}$$

Combining (4.26), (4.42)–(4.44), (4.49) and (4.50), we infer

$$\lim_{k\to\infty}\int_0^T \theta(t)\int_\Omega \varphi(x)\,\kappa(x)|\nabla v_k(t)|^2 dx\,dt = 0. \tag{4.51}$$

From (4.49) and (4.51), defining $\psi(x,t) = \theta(t)\varphi(x)$ with $\theta(t) \in C_0^\infty(0,T)$ and $\varphi(x) \in C_0^\infty(\Omega)$, where $\operatorname{supp}\varphi \subset (\Omega\backslash A)$, we derive

$$\lim_{k\to\infty}\int_0^T\int_\Omega \psi(x,t)\left[\rho(x)|v_k'|^2 + \rho(x)|\nabla v_k|^2\right] dx\,dt = 0. \tag{4.52}$$

Now, considering $\varepsilon > 0$ small enough and such that $0 \le \theta \le 1$, $\theta = 1$ in $(\varepsilon, T-\varepsilon)$ and $\operatorname{supp}\theta \subset (0,T)$, then (4.52) implies

$$\lim_{k\to\infty}\partial_t v_k = 0 \text{ in } L^2((0,T)\times(\Omega\backslash A)) \text{ and } \lim_{k\to\infty}\nabla v_k = 0 \ \text{ in } \ [L^2((0,T)\times(\Omega\backslash A))]^n. \tag{4.53}$$

In what follows, we shall use precisely the results in microlocal analysis recalled in the appendix. Indeed, let us first consider the microlocal defect measure μ associated with $\{v_k\}$ in $H^1((0,T)\times(\Omega\setminus A))$. Thus, (4.53) in combination with Remark A.5 imply that $\mu = 0$ in $(0,T)\times(\Omega\backslash A)$, that is, $\operatorname{supp}\mu \subset A$.

On the other hand, from identity (4.21) we have

$$\rho(x)v_k'' - \operatorname{div}[\kappa(x)\nabla v_k] = \int_0^\infty g(s)\operatorname{div}[\sqrt{a}\nabla\zeta_k^t(s)]ds - b(x)\frac{h(\alpha_k v_k')}{\alpha_k} - \frac{f(\alpha_k v_k)}{\alpha_k} \text{ in } \Omega\times(0,T). \tag{4.54}$$

Moreover, the convergence (4.26) leads to

$$\lim_{k\to\infty}\int_0^\infty g(s)\sqrt{a}\frac{\partial}{\partial x_i}\zeta_k^t(s)ds = \lim_{k\to\infty} b(x)\frac{h(\alpha_k v_k')}{\alpha_k} = 0 \text{ in } L^2(\Omega\times(0,T)), \tag{4.55}$$

for each $i = 1,\ldots,n$, from where it follows

$$\lim_{k\to\infty}\frac{\partial}{\partial t}\int_0^\infty g(s)\operatorname{div}[\sqrt{a}\nabla\zeta_k^t(s)]ds = \lim_{k\to\infty}\frac{\partial}{\partial t}\left[b(x)\frac{h(\alpha_k v_k')}{\alpha_k}\right] = 0 \text{ in } H^{-1}(\Omega\times(0,T)). \tag{4.56}$$

From this, we also note

$$\lim_{k\to\infty}\frac{\partial}{\partial t}\frac{f(\alpha_k v_k)}{\alpha_k} = 0 \text{ in } H^{-1}_{loc}(\Omega\times(0,T)). \tag{4.57}$$

Indeed, since $H_0^1(\Omega) \hookrightarrow L^{2p}(\Omega)$, it follows that

$$\left\|\frac{f(\alpha_k v_k)}{\alpha_k}\right\|_{L^\infty(0,T;L^2(\Omega))} \le c\|v_k\|_{L^\infty(0,T;L^2(\Omega))} + c\alpha_k^{p-1}\|v_k\|_{L^\infty(0,T;H_0^1(\Omega))},$$

and then, with

$$\left\|\frac{\partial}{\partial t}\frac{f(\alpha_k v_k)}{\alpha_k}\right\|_{H^{-1}(\Omega\times(0,T))} \le c\left\|\frac{f(\alpha_k v_k)}{\alpha_k}\right\|_{L^2(\Omega\times(0,T))},$$

we obtain (4.57). Therefore, from (4.54), (4.56) and (4.57), yields

$$\Box v_k' = \frac{\partial}{\partial t}\left(\rho(x)v_k'' - \operatorname{div}[\kappa(x)\nabla v_k]\right) \to 0 \text{ in } H^{-1}_{loc}(\Omega\times(0,T)). \tag{4.58}$$

According to Theorem A.6, the convergence (4.58) implies

$$\operatorname{supp}\mu \subset \left\{(t,x,\tau,\xi) : \tau^2 = \frac{\kappa(x)}{\rho(x)}|\xi|^2\right\},$$

and from Theorem A.9 one obtains that $\operatorname{supp}\mu$ is the union of curves which are the bicharacteristics of the principal symbol $p(t,x,\tau,\xi) = \tau^2 - \dfrac{\kappa(x)}{\rho(x)}|\xi|^2$. We refer to Appendix A.3 for related definitions and characterization of the principal symbol of the wave operator and its bicharacteristics. Since $T > T_0$, every bicharacteristic ray enters the region $\omega' = \Omega\backslash A$ before the time T, and then $\mu = 0$ in Ω. Consequently, we deduce

$$\lim_{k\to\infty} \partial_t v_k = 0 \text{ in } L^2((0,T)\times\Omega). \tag{4.59}$$

Now, let us show that

$$\lim_{k\to\infty} \nabla v_k = 0 \text{ in } [L^2(\Omega\times(0,T))]^n. \tag{4.60}$$

Indeed, multiplying (4.21)$_1$ by θv_k, where $\theta \in C_0^\infty(0,T)$ is such that $0\le\theta\le 1$ and $\theta = 1$ in $(\varepsilon, T-\varepsilon)$ for $\varepsilon\in(0,T)$ fixed and arbitrary, and integrating in $\Omega\times(0,T)$, we obtain

$$\begin{aligned} 0 \;=\; & -\int_0^T \theta(t)\int_\Omega \rho(x)|v_k'(t)|^2\,dxdt - \int_0^T \theta'(t)\int_\Omega \rho(x)v_k'(t)v_k(t)\,dxdt \\ & + \int_0^T \theta(t)\int_\Omega \kappa(x)|\nabla v_k(t)|^2\,dxdt \\ & + \int_0^T\int_0^\infty g(s)\int_\Omega a(x)\nabla\zeta_k^t(s)\cdot\nabla v_k\theta(t)dxdsdt \\ & + \int_0^T\int_\Omega b(x)\frac{h(\alpha_k v_k')}{\alpha_k}v_k\theta(t)\,dxdt \\ & + \int_0^T\int_\Omega \frac{f(\alpha_k v_k)}{\alpha_k}v_k\theta(t)dxdt. \end{aligned} \tag{4.61}$$

From (4.26), (4.42)–(4.44), (4.59) and (4.61) we deduce

$$\lim_{k\to\infty}\int_0^T\int_\Omega \theta(t)\kappa(x)|\nabla v_k|^2dxdt = 0,$$

and since

$$\int_\varepsilon^{T-\varepsilon}\int_\Omega \kappa(x)|\nabla v_k|^2dxdt \le \int_0^T\int_\Omega \theta(t)\kappa(x)|\nabla v_k|^2dxdt, \tag{4.62}$$

with arbitrary $\varepsilon > 0$, we get (4.60) as desired.

Now, from (2.8) we have

$$|F(s)| \le c|s|^2 + c|s|^{p+1}, \quad \forall s\in\mathbb{R},$$

which implies

$$\int_\Omega \left|\frac{F(\alpha_k v_k(t))}{\alpha_k}\right| dx \le c\alpha_k\|v_k\|^2_{L^\infty(0,T;L^2(\Omega))} + c\alpha_k^p\|v_k\|^{p+1}_{L^\infty(0,T;L^{p+1}(\Omega))}, \quad \forall t\in[0,T],$$

and then

$$\lim_{k\to\infty}\int_0^T\int_\Omega \frac{F(\alpha_k v_k(t))}{\alpha_k}dxdt = 0. \tag{4.63}$$

From (4.26), (4.59), (4.60) and (4.63) we obtain

$$\lim_{k\to\infty}\int_0^T E_{v_k,\zeta_k}(t)\,dt = 0.$$

Since the energy is non-increasing we have

$$\int_0^T E_{v_k,\zeta_k}(t)\,dt \ge T E_{v_k,\zeta_k}(T),$$

which finally proves the desired contradiction limit (4.29).

This completes the proof of Proposition 3.5-**I**. □

4.2. Proof of the observability inequality: Part II

Now we prove inequality (3.10) in Proposition 3.5. We follow the same strategy as above but regarding now that $b \equiv 0$. In fact, let us suppose that (3.10) does not hold. Thus, there exists a time $T > T_0 > 0$ and a sequence of solutions (u_k, η_k) for (1.1) such that

$$E_{u_k,\eta_k}(0) \leq R \tag{4.64}$$

and

$$\lim_{k\to\infty} \frac{E_{u_k,\eta_k}(0)}{\displaystyle\int_0^T \int_0^\infty g(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2 ds dt} = +\infty, \tag{4.65}$$

where $E_{u_k,\eta_k}(t)$ is the energy defined in (3.1) associated with the solution (u_k, η_k) of (1.1) with $b \equiv 0$. Below, u_k' still denotes $\partial_t u_k$. Combining (4.64) and (4.65) we obtain

$$\lim_{k\to\infty} \int_0^T \int_0^\infty g(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2 dsd = 0, \tag{4.66}$$

and following step by step the proof of (4.4), we also obtain

$$\lim_{k\to\infty} \int_0^T \int_0^\infty -g'(s)\|\sqrt{a}\nabla\eta_k^t(s)\|^2 dsdt = 0. \tag{4.67}$$

On the other hand, since $E_{u_k,\eta_k}(t) \leq E_{u_k,\eta_k}(0) \leq R$, for all $t \geq 0$, there exists a subsequence of $\{(u_k, \eta_k)\}$, still denoted by$\{(u_k, \eta_k)\}$, such that

$$u_k \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0, T; H_0^1(\Omega)), \tag{4.68}$$

$$u_k' \overset{*}{\rightharpoonup} u' \text{ in } L^\infty(0, T; L^2(\Omega)), \tag{4.69}$$

and, by Aubin–Lions' Theorem,

$$u_k \to u \text{ in } L^2(0, T; L^2(\Omega)). \tag{4.70}$$

As before, to reach the expected contradiction we assume two cases as follows.

Case 1: $u \neq 0$. For each $k \in \mathbb{N}$, (u_k, η_k) is the solution of

$$\begin{cases} \rho(x)u_k'' - \operatorname{div}[\kappa(x)\nabla u_k] - \displaystyle\int_0^\infty g(s)\operatorname{div}[a(x)\nabla\eta_k^t(s)]\, ds + f(u_k) = 0 \text{ in } \Omega \times (0, T), \\ \eta_k' = -\partial_s\eta_k + u_k' \text{ in } \Omega \times (0, T) \times (0, \infty), \end{cases} \tag{4.71}$$

with initial–boundary conditions

$$\begin{aligned} &u_k = 0 \text{ on } \partial\Omega \times (0, T), \quad \eta_k = 0 \text{ on } \partial\Omega \times (0, T) \times (0, \infty), \\ &u_k(0) = u_{0k},\ u_k'(0) = u_{1k},\ \eta_k^0(s) = \eta_{0k}(s) \text{ in } \Omega,\ s \in (0, \infty), \\ &\eta_k^t(0) = 0 \text{ in } \Omega,\ t \in [0, T). \end{aligned} \tag{4.72}$$

As in (4.12), we obtain

$$f(u_k) \rightharpoonup f(u) \text{ in } L^2(0, T; L^2(\Omega)), \tag{4.73}$$

and combining (4.66), (4.68) e (4.73), yields

$$\rho(x)u'' - \operatorname{div}[\kappa(x)\nabla u] + f(u) = 0 \text{ in } L^2(0, T; H^{-1}(\Omega)). \tag{4.74}$$

Defining again the auxiliary function

$$z_k = \kappa(x)u_k + a(x)\int_0^\infty g(s)\eta_k^t(s)\, ds, \quad k \in \mathbb{N}, \tag{4.75}$$

and using similar arguments as in (4.18), we also conclude

$$u' \equiv 0 \text{ a.e. in } (\Omega \setminus A) \times (0,T). \tag{4.76}$$

Since u satisfies

$$\begin{cases} \rho(x)u'' - \operatorname{div}[\kappa(x)\nabla u] + f(u) = 0 \quad \text{in} \quad \Omega \times (0,T), \\ u' \equiv 0 \text{ in } (\Omega \setminus A) \times (0,T), \end{cases} \tag{4.77}$$

then taking its time derivative and denoting $u' = v$, we get

$$\begin{cases} \rho(x)v'' - \operatorname{div}[\kappa(x)\nabla v] + f'(u)v = 0 \quad \text{in} \quad \Omega \times (0,T), \\ v \equiv 0 \text{ in } (\Omega \setminus A) \times (0,T). \end{cases} \tag{4.78}$$

According to hypothesis $(\mathbf{A_5})$-(b) one sees that $f'(u) \in L^\infty(0,T;L^n(\Omega))$, and using again Theorem A.1, we have $v \equiv 0$ in $\Omega \times (0,T)$, which implies $u'' \equiv 0$ in $\Omega \times (0,T)$. Consequently,

$$\operatorname{div}[\kappa(x)\nabla u(t)] = -f(u(t)) \;\text{ a.e. in }\; [0,T]. \tag{4.79}$$

Multiplying (4.79) by u, integrating by parts on Ω and using the hypothesis $(\mathbf{A_5})$-(a) we obtain

$$\lambda_1 \|u(t)\|^2 \le \beta \|u(t)\|^2,$$

from where it follows that

$$u \equiv 0 \text{ in } \Omega \times (0,T).$$

This reaches the desired contradiction in Case 1.

Case 2: $u = 0$. As previously, for each $k \in \mathbb{N}$ we define

$$\alpha_k = \left[E_{u_k,\eta_k}(0)\right]^{1/2}, \qquad v_k = \frac{1}{\alpha_k}\,u_k, \qquad \zeta_k = \frac{1}{\alpha_k}\,\eta_k. \tag{4.80}$$

Thus, (v_k,ζ_k) is the solution of problem

$$\begin{cases} \rho(x)v_k'' - \operatorname{div}[\kappa(x)\nabla v_k] - \displaystyle\int_0^\infty g(s)\operatorname{div}[a(x)\nabla \zeta_k^t(s)]\,ds + \frac{f(\alpha_k v_k)}{\alpha_k} = 0 \;\text{ in }\; \Omega \times (0,T), \\ \{\zeta_k^t\}' = -\partial_s \zeta_k^t \;\text{ in }\; \Omega \times (0,T) \times (0,\infty), \end{cases} \tag{4.81}$$

with initial–boundary conditions

$$\begin{aligned} & v_k = 0 \;\text{ on }\; \partial\Omega \times (0,T), \quad \zeta_k = 0 \;\text{ on }\; \partial\Omega \times (0,T) \times (0,\infty), \\ & v_k(0) = v_{0k},\; v_k'(0) = v_{1k},\; \zeta_k^0(s) = \zeta_{0k}(s) \;\text{ in }\; \Omega,\; s \in (0,\infty), \\ & \zeta_k^t(0) = 0 \;\text{ in }\; \Omega,\; t \in [0,T). \end{aligned} \tag{4.82}$$

The energy functional associated with (4.81)–(4.82) is given by

$$\begin{aligned} E_{v_k,\zeta_k}(t) \;=\; & \frac{1}{2}\int_\Omega \rho(x)|v_k'(t)|^2\,dx + \frac{1}{2}\int_\Omega \kappa(x)|\nabla v_k(t)|^2\,dx + \frac{1}{2}\int_0^\infty g(s)\|\sqrt{a}\nabla\zeta_k^t\|^2 ds \\ & + \int_\Omega \frac{F(\alpha_k v_k(t))}{\alpha_k^2}dx, \end{aligned} \tag{4.83}$$

with $E_{v_k,\zeta_k}(t) \ge 0$, for all $k \in \mathbb{N}$ and for every $t \in [0,T]$.

From (4.80) we get

$$E_{v_k,\zeta_k}(t) = \frac{1}{\alpha_k^2}E_{u_k,\eta_k}(t) = \frac{1}{E_{u_k,\eta_k}(0)}E_{u_k,\eta_k}(t), \quad t \in [0,T],$$

and then

$$E_{v_k,\zeta_k}(0) \;=\; 1, \quad \forall\, k \in \mathbb{N}. \tag{4.84}$$

Besides, (4.66) and (4.80) yield

$$\lim_{k\to\infty}\int_0^T\int_0^\infty g(s)\|\sqrt{a}\nabla\zeta_k^t\|^2 ds dt = 0, \tag{4.85}$$

and from the definition of ζ_k in (4.80) along with the limit (4.67), we obtain

$$\lim_{k\to\infty}\int_0^T\int_0^\infty -g'(s)\|\sqrt{a}\nabla\zeta_k^t\|^2 ds dt = 0. \tag{4.86}$$

A straightforward computation shows that the energy functional $E_{v_k,\zeta_k}(t)$ satisfies

$$\frac{d}{dt}E_{v_k,\zeta_k}(t) = \frac{1}{2}\int_0^\infty g'(s)\|\sqrt{a}\nabla\zeta_k\|^2 ds \le 0, \quad t\in[0,T],$$

from where $E_{v_k,\eta_k}(t)$ is a non-increasing function verifying

$$E_{v_k,\zeta_k}(0) = E_{v_k,\zeta_k}(T) + \frac{1}{2}\int_0^T\int_0^\infty (-g'(s))\|\sqrt{a}\nabla\zeta_k^t\|^2 ds dt. \tag{4.87}$$

Passing (4.87) to the limit, when $k\to\infty$, and using (4.84), (4.86) e (4.87), it follows that

$$1 = \lim_{k\to+\infty} E_{v_k,\zeta_k}(0) = \lim_{k\to+\infty} E_{v_k,\zeta_k}(T). \tag{4.88}$$

Our wish is again to prove

$$\lim_{k\to+\infty} E_{v_k,\zeta_k}(T) = 0, \tag{4.89}$$

and the contradiction is also obtained in this case, which completes the proof of (3.10).

In what follows, the effort will be to prove (4.89). Indeed, since $E_{v_k,\zeta_k}(t)\le E_{v_k,\zeta_k}(0)=1$ for all $t\in[0,T]$, there exists a subsequence $\{(v_k,\zeta_k)\}$, still denoted by $\{(v_k,\zeta_k)\}$, such that

$$v_k \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,T;H_0^1(\Omega)), \tag{4.90}$$

$$v_k' \overset{*}{\rightharpoonup} v' \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{4.91}$$

$$v_k \to v \text{ in } L^2(0,T;L^2(\Omega)). \tag{4.92}$$

Firstly, we show that $v\equiv 0$ in $\Omega\times(0,T)$. Since $\alpha_k=[E_{u,\eta}(0)]^{1/2}$, from (4.64) follows that there exists $\alpha\ge 0$ such that $\alpha_k\to\alpha$. We consider two situations as follows.

S1. $\alpha>0$. Here, combining (4.70) and (4.80) we obtain $\alpha_k v_k\to 0$ in $L^2(0,T;L^2(\Omega))$, and using similar arguments as used in (4.73) we also see

$$\frac{f(\alpha_k v_k)}{\alpha_k} \rightharpoonup 0 \text{ in } L^2(0,T;L^2(\Omega)).$$

Using this convergence, (4.85) and passing $(4.81)_1$ to the limit we get

$$\rho(x)v'' - \mathrm{div}[\kappa(x)\nabla v] = 0 \text{ in } L^2(0,T;H^{-1}(\Omega)). \tag{4.93}$$

Proceeding analogously (4.14)–(4.18) we get $v'\equiv 0$ in $(\Omega\setminus A)\times(0,T)$. Thus, for $w=v'$, it follows that w satisfies

$$\begin{cases} \rho(x)w'' - \mathrm{div}[\kappa(x)\nabla w] = 0 \text{ in } \Omega\times(0,T), \\ w\equiv 0 \text{ in } (\Omega\setminus A)\times(0,T). \end{cases} \tag{4.94}$$

Since $\omega'=\Omega\setminus A$ satisfies the Geometric Control Condition, from Theorem A.2 it follows that $v'\equiv 0$ in $\Omega\times(0,T)$. Using this fact and (4.93) we obtain $v\equiv 0$ in $\Omega\times(0,T)$.

S2. $\alpha = 0$. As in (4.35)–(4.36), we have

$$f(s) = f'(0)s + R(s), \tag{4.95}$$

where

$$|R(s)| \le c|s|^2 + c|s|^p, \quad \forall s \in \mathbb{R}. \tag{4.96}$$

From (4.96), the fact $\alpha = 0$, and since $\{v_n\}$ is bounded in $L^\infty(0,T;H_0^1(\Omega))$, we obtain

$$\frac{R(\alpha_k v_k(t))}{\alpha_k} \to 0 \text{ in } L^\infty(0,T;L^1(\Omega)). \tag{4.97}$$

Passing $(4.81)_1$ to the limit when $k \to \infty$, and using (4.85), (4.95) e (4.97), it follows that

$$\rho(x)v'' - \operatorname{div}[\kappa(x)\nabla v] + f'(0)v = 0 \text{ in } L^2(0,T;H^{-1}(\Omega)). \tag{4.98}$$

Let us define

$$\tilde{z}_k \; = \; \kappa(x)v_k + a(x)\int_0^\infty g(s)\zeta_k^t(s)\,ds, \quad k \in \mathbb{N}.$$

Proceeding as in (4.75)–(4.76), we get $v' \equiv 0$ in $(\Omega \setminus A) \times (0,T)$. From this fact, (4.98) and Theorem A.1, we conclude that $v' \equiv 0$ in $\Omega \times (0,T)$. Besides, by following the same arguments as in (4.39)–(4.41), we get $v \equiv 0$ in $\Omega \times (0,T)$.

From the above considerations and (4.90)–(4.92) we obtain

$$v_k \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty(0,T;H_0^1(\Omega)), \tag{4.99}$$

$$v_k' \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{4.100}$$

$$v_k \to 0 \text{ in } L^2(0,T;L^2(\Omega)). \tag{4.101}$$

Now, let $\theta \in C_0^\infty(0,T)$ and $\varphi \in C_0^\infty(\Omega)$ such that $\operatorname{supp}\varphi \subset (\Omega \setminus A)$. Since $\operatorname{supp}\varphi \subset (\Omega \setminus A)$, we deduce

$$\int_\Omega (\varphi(x) + |\nabla\varphi(x)|)|\zeta_k^t|^2\,dx \le c_\varphi \int_\Omega a(x)|\nabla\zeta_k^t|^2\,dx. \tag{4.102}$$

Consider $\phi_k(x,t) = \int_0^\infty g(s)\zeta_k^t(x,s)\,ds$. Multiplying the first equation in (4.81) by $\theta\varphi(x)\phi_k$ and integrating by parts on $\Omega \times (0,T)$ we get

$$\begin{aligned}
&-\int_0^T\int_\Omega \rho(x)v_k'\phi_k'\varphi(x)\theta(t)\,dxdt - \int_0^T\int_\Omega \rho(x)v_k'\phi_k\varphi(x)\theta'(t)\,dxdt\\
&+\int_0^T\int_\Omega \kappa(x)\nabla v_k\cdot\nabla\phi_k\,\varphi(x)\theta(t)\,dxdt + \int_0^T\int_\Omega \kappa(x)\nabla v_k\cdot\nabla\varphi(x)\phi_k\theta(t)\,dxdt\\
&+\int_0^T\int_0^\infty g(s)\int_\Omega a(x)\nabla\zeta_k^t(s)\cdot\nabla\varphi(x)\phi_k\theta(t)\,dxdsdt\\
&+\int_0^T\int_0^\infty g(s)\int_\Omega a(x)\nabla\zeta_k^t\cdot\nabla\phi_k\,\varphi(x)\theta(t)\,dxdsdt\\
&+\int_0^T\int_\Omega \frac{f(\alpha_k v_k)}{\alpha_k}\phi_k\varphi(x)\theta(t)dxdt\\
&= J_{1k} + J_{2k} + \cdots + J_{6k} + J_{7k} = 0.
\end{aligned} \tag{4.103}$$

As in (4.47)–(4.48) we have that $\lim_{k\to\infty} J_{2k} = \cdots = \lim_{k\to\infty} J_{7k} = 0$. Consequently, $\lim_{k\to\infty} J_{1k} = 0$ and

$$\lim_{k\to\infty}\int_0^T\int_\Omega \rho(x)\theta(t)\,\varphi(x)|v_k'|^2\,dxdt = 0. \tag{4.104}$$

Also, multiplying the first equation in (4.81) by $\theta(t)\varphi(x)v_k$ and integrating by parts in $\Omega\times(0,T)$, we conclude that

$$\begin{aligned} &-\int_0^T\theta(t)\int_\Omega\varphi(x)\rho(x)|v_k'(t)|^2\,dxdt-\int_0^T\theta'(t)\int_\Omega\varphi(x)\rho(x)v_k'(t)v_k(t)\\ &+\int_0^T\theta(t)\int_\Omega\varphi(x)\kappa(x)|\nabla v_k(t)|^2\,dxdt+\int_0^T\int_\Omega\kappa(x)\nabla v_k\cdot\nabla\varphi(x)v_k\theta(t)dxdt\\ &+\int_0^T\int_0^\infty g(s)\int_\Omega a(x)\nabla\zeta_k^t(s)\cdot\nabla\varphi(x)v_k\theta(t)dxdsdt\\ &+\int_0^T\theta(t)\int_\Omega\left(\int_0^\infty g(s)a(x)\nabla\zeta_k^t(s)\,ds\right)\cdot\nabla v_k(t)\varphi(x)\,dxdt\\ &+\int_0^T\int_\Omega\frac{f(\alpha_kv_k(t))}{\alpha_k}v_k(t)\varphi(x)\theta(t)dxdt=0.\end{aligned}\tag{4.105}$$

Combining (4.85), (4.99)–(4.101) and (4.104) with (4.105) we obtain

$$\lim_{k\to\infty}\int_0^T\theta(t)\int_\Omega\varphi(x)\,\kappa(x)|\nabla v_k(t)|^2dx\,dt=0.\tag{4.106}$$

Moreover, from (4.104) and (4.106), defining $\psi(x,t)=\theta(t)\varphi(x)$ with $\theta(t)\in C_0^\infty(0,T)$ and $\varphi(x)\in C_0^\infty(\Omega)$ with $\operatorname{supp}\varphi\subset(\Omega\backslash A)$, we get

$$\lim_{k\to\infty}\int_0^T\int_\Omega\psi(x,t)\left[\rho(x)|v_k'|^2+\rho(x)|\nabla v_k|^2\right]\,dx\,dt=0.$$

Observe that if we consider $\varepsilon>0$ sufficiently small such that $0\le\theta\le1$, $\theta=1$ in $(\varepsilon,T-\varepsilon)$ and $\operatorname{supp}\theta\subset(0,T)$, the previous limit yields

$$\lim_{k\to\infty}\partial_tv_k=0\text{ in }L^2((0,T)\times(\Omega\backslash A))\text{ e }\lim_{k\to\infty}\nabla v_k=0\ \text{ in }\ [L^2((0,T)\times(\Omega\backslash A))]^n.\tag{4.107}$$

At this moment, we consider again the microlocal defect measure μ associated with $\{v_k\}$ in $H^1((0,T)\times(\Omega\setminus A))$. Thus, (4.107) and Remark A.5 imply that $\mu=0$ in $(0,T)\times(\Omega\backslash A)$, that is, $\operatorname{supp}\mu\subset A$.

On the other hand, from (4.81) one has

$$\rho(x)v_k''-\operatorname{div}[\kappa(x)\nabla v_k]=\int_0^\infty g(s)\operatorname{div}[\sqrt{a}\nabla\zeta_k^t(s)]ds-\frac{f(\alpha_kv_k)}{\alpha_k}\text{ in }\Omega\times(0,T).\tag{4.108}$$

Since the convergence (4.85) implies

$$\lim_{k\to\infty}\int_0^\infty g(s)\sqrt{a}\frac{\partial}{\partial x_i}\zeta_k^t(s)ds=0\text{ in }L^2(\Omega\times(0,T)),$$

for each $i=1,\dots,n$, then

$$\lim_{k\to\infty}\frac{\partial}{\partial t}\int_0^\infty g(s)\operatorname{div}[\sqrt{a}\nabla\zeta_k^t(s)]ds=0\text{ in }H^{-1}(\Omega\times(0,T)).$$

Also, using similar arguments as in (4.57) we have

$$\lim_{k\to\infty}\frac{\partial}{\partial t}\frac{f(\alpha_kv_k)}{\alpha_k}=0\quad\text{in}\quad H^{-1}_{loc}(\Omega\times(0,T)).$$

From (4.108) one sees

$$\Box v_k'=\frac{\partial}{\partial t}\left(\rho(x)v_k''-\operatorname{div}[\kappa(x)\nabla v_k]\right)\to0\text{ in }H^{-1}_{loc}(\Omega\times(0,T)).\tag{4.109}$$

Thus, Theorem A.6 along with (4.109) imply

$$\operatorname{supp}\mu \subset \left\{(t,x,\tau,\xi) : \tau^2 = \frac{\kappa(x)}{\rho(x)}|\xi|^2\right\}.$$

Moreover, from Theorem A.9, suppμ is the union of curves which are bicharacteristics of the principal symbol $p(t,x,\tau,\xi) = \tau^2 - \dfrac{\kappa(x)}{\rho(x)}|\xi|^2$, see again the definition and results in Appendix A.3. Since $T > T_0$, every characteristic ray enters the region $\omega' = \Omega\backslash A$ before time T, implying that $\mu = 0$ everywhere in Ω. Consequently,

$$\lim_{k\to\infty} \partial_t v_k = 0 \text{ in } L^2((0,T)\times\Omega). \tag{4.110}$$

Now we are going to prove

$$\lim_{k\to\infty} \nabla v_k = 0 \text{ in } [L^2(\Omega\times(0,T))]^n. \tag{4.111}$$

Indeed, multiplying $(4.81)_1$ by θv_k, where $\theta \in C_0^\infty(0,T)$ is such that $0 \le \theta \le 1$ and $\theta = 1$ in $(\varepsilon, T-\varepsilon)$ for some arbitrary $\varepsilon \in (0,T)$, and integrating on $\Omega\times(0,T)$ we have

$$\begin{aligned}
&-\int_0^T \theta(t)\int_\Omega \rho(x)|v_k'(t)|^2\,dxdt - \int_0^T \theta'(t)\int_\Omega \rho(x)v_k'(t)v_k(t)\,dxdt \\
&+\int_0^T\int_0^\infty g(s)\int_\Omega a(x)\nabla\zeta_k^t(s)\cdot\nabla v_k\theta(t)dxdsdt \\
&+\int_0^T\int_\Omega \frac{f(\alpha_k v_k)}{\alpha_k}v_k\theta(t)dxdt = 0.
\end{aligned} \tag{4.112}$$

Combining (4.85), (4.99)–(4.101), (4.110) and (4.112) we get

$$\lim_{k\to\infty}\int_0^T\int_\Omega \theta(t)\kappa(x)|\nabla v_k|^2dxdt = 0,$$

and since

$$\int_\varepsilon^{T-\varepsilon}\int_\Omega \kappa(x)|\nabla v_k|^2dxdt \le \int_0^T\int_\Omega \theta(t)\kappa(x)|\nabla v_k|^2dxdt, \tag{4.113}$$

with arbitrary $\varepsilon > 0$, we infer that (4.111) holds true.

Similar to (4.63), it follows that

$$\lim_{k\to\infty}\int_0^T\int_\Omega \frac{F(\alpha_k v_k(t))}{\alpha_k}dxdt = 0, \tag{4.114}$$

and the convergences (4.85), (4.110), (4.111) and (4.114) lead to

$$0 = \lim_{k\to\infty}\int_0^T E_{v_k,\zeta_k}(t)\,dt \ge T\lim_{k\to\infty} E_{v_k,\zeta_k}(T),$$

which gives the desired contradiction limit (4.89).

This completes the proof of Proposition 3.5-**II**. □

4.3. Proof of the main result

We are now ready to prove Theorem 3.3. We first deal with part **I** and the second item **II** follows verbatim by using the same arguments and neglecting the frictional damping.

4.3.1. Proof of Theorem 3.3-I

Initially, we note that from Proposition 3.5 and the semigroup property for the solution of problem (1.1), there exists a constant $C = C(R,T) > 0$ such that, for each $m = 0, 1, \ldots$,

$$E_{u,\eta}(mT) \le C \int_{mT}^{(m+1)T} \left\{ \int_0^\infty g(s)\|\sqrt{a}\nabla\eta^t\|^2 ds + \int_\Omega b(x)[|u_t(x,t)|^2 + |h(u_t(x,t))|^2]\, dx \right\} dt. \tag{4.115}$$

In what follows, we are going to estimate each term on the right side of (4.115).

1st. Frictional term. According to Remark 2.1-3 there exists a convex strictly increasing function $H : [0,\infty) \to [0,\infty)$ that vanishes at the origin and such that

$$s^2 + [h(s)]^2 \le H^{-1}(sh(s)), \text{ for all } |s| \le 1.$$

Now, let us consider the following sets

$$D_{1,m} = \{(x,t) \in \Omega \times (mT,(m+1)T); |u_t(x,t)| \le 1\} \text{ and } D_{2,m} = \Omega \times (mT,(m+1)T) \setminus D_{1,m}.$$

For each $m = 0, 1, 2, \ldots$, we have

$$|u_t(x,t)|^2 + |h(u_t(x,t))|^2 \le H^{-1}(u_t(x,t)h(u_t(x,t))), \quad \forall\, (x,t) \in D_{1,m},$$

and

$$|u_t(x,t)|^2 + |h(u_t(x,t))|^2 \le \left(\frac{1}{M_1} + M_2\right) u_t(x,t)h(u_t(x,t)), \quad \forall\, (x,t) \in D_{2,m}.$$

Thus,

$$\begin{aligned}\int_{mT}^{(m+1)T}\int_\Omega b(x)[|u_t|^2 + |h(u_t)|^2]dxdt \quad &\le \quad \int_m^{(m+1)T}\int_\Omega b(x)H^{-1}(h(u_t)u_t)dxdt \\ &+ \quad \left(\frac{1}{M_1} + M_2\right)\int_{mT}^{(m+1)T}\int_\Omega b(x)h(u_t)u_t dxdt.\end{aligned}$$

Let $\widetilde{H} : [0,\infty) \to [0,\infty)$ be given by $\widetilde{H}(s) = T\|b\|_{L^1(\Omega)} H\left(\frac{1}{T\|b\|_{L^1(\Omega)}} s\right)$. From Jensen's inequality and denoting $D(t) = -E'(t)$ we get

$$\begin{aligned}&\int_{mT}^{(m+1)T}\int_\Omega b(x)[|u_t|^2 + |h(u_t)|^2]dxdt \\ &\qquad \le \left(\widetilde{H}^{-1} + \left(\frac{1}{M_1} + M_2\right) Id\right)\left[\int_{mT}^{(m+1)T}\int_\Omega b(x)h(u_t)u_t dxdt\right] \\ &\qquad \le \left(\widetilde{H}^{-1} + \left(\frac{1}{M_1} + M_2\right) Id\right)\left[\int_{mT}^{(m+1)T} D(t)dt\right].\end{aligned} \tag{4.116}$$

2nd. Memory term. Since

$$\int_{mT}^{(m+1)T}\int_0^\infty g(s)\|\sqrt{a}\nabla\eta^t(s)\|^2 dsdt = \int_{mT}^{(m+1)T}\int_0^\infty (-g'(s))^{\frac{\alpha_0}{p_2}} \frac{g(s)}{(-g'(s))^{\frac{\alpha_0}{p_2}}}\|\sqrt{a}\nabla\eta^t(s)\|^2 dsdt,$$

then from Jensen's inequality and $0 < \dfrac{\alpha_0}{p_2} < 1$, we deduce

$$\int_{mT}^{(m+1)T}\int_0^\infty (-g'(s))^{\frac{\alpha_0}{p_2}} \frac{g(s)}{(-g'(s))^{\frac{\alpha_0}{p_2}}}\|\sqrt{a}\nabla\eta^t(s)\|^2 dsdt$$

$$\le \int_{mT}^{(m+1)T}\left[\int_0^\infty \frac{g(s)}{(-g'(s))^{\frac{\alpha_0}{p_2}}}\|\sqrt{a}\nabla\eta^t(s)\|^2 ds\right]^{1-\frac{\alpha_0}{p_2}}\left[\int_0^\infty (-g'(s))\frac{g(s)}{(-g'(s))^{\frac{\alpha_0}{p_2}}}\|\sqrt{a}\nabla\eta^t(s)\|^2 ds\right]^{\frac{\alpha_0}{p_2}} dt.$$

In addition, from condition (2.6) in assumption ($\mathbf{G_1}$), we deduce

$$\frac{g(s)}{(-g'(s))^{\frac{\alpha_0}{p_2}}} \leq \frac{1}{C_2^{\frac{\alpha_0}{p_2}}} g(s)^{1-\alpha_0}, \quad \forall s \geq 0, \tag{4.117}$$

and then

$$\begin{aligned}
&\int_{mT}^{(m+1)T} \int_0^\infty g(s)\|\sqrt{a}\nabla\eta^t(s)\|^2 ds dt \\
&\leq \int_{mT}^{(m+1)T} \left[\int_0^\infty \frac{g(s)^{1-\alpha_0}}{C_2^{\frac{\alpha_0}{p_2}}}\|\sqrt{a}\nabla\eta^t(s)\|^2 ds\right]^{1-\frac{\alpha_0}{p_2}} \left[\int_0^\infty (-g'(s))\frac{g(s)^{1-\alpha_0}}{C_2^{\frac{\alpha_0}{p_2}}}\|\sqrt{a}\nabla\eta^t(s)\|^2 ds\right]^{\frac{\alpha_0}{p_2}} dt \\
&\leq \frac{g(0)^{(1-\alpha_0)\frac{\alpha_0}{p_2}}}{C_2^{\frac{\alpha_0}{p_2}}} \left(\sup_{t\geq 0}\int_0^\infty g(s)^{1-\alpha_0}\|\sqrt{a}\nabla\eta^t(s)\|^2 ds\right)^{1-\frac{\alpha_0}{p_2}} \int_{mT}^{(m+1)T} D(t)^{\frac{\alpha_0}{p_2}} dt.
\end{aligned}$$

Now, we claim that

$$\left(\sup_{t\geq 0}\int_0^\infty g(s)^{1-\alpha_0}\|\sqrt{a}\nabla\eta^t(s)\|^2 ds\right) < \infty. \tag{4.118}$$

This is the precise moment where we use the boundedness condition $\sup_{\tau<0}\|\sqrt{a}\nabla u_0(\tau)\| < R$ and the assumption ($\mathbf{G_2}$), see (2.7). Indeed, from this and recalling (1.2), we have for all $t > s$ that

$$\begin{aligned}
\|\sqrt{a}\nabla\eta^t(s)\|^2 &\leq 2\|\sqrt{a}\|^2_{L^\infty(\Omega)}(\|\nabla u(t)\|^2 + \|\nabla u(t-s)\|^2) \\
&\leq \frac{2}{l^2}\|\sqrt{a}\|^2_{L^\infty(\Omega)}(\|u(t)\|_1^2 + \|u(t-s)\|_1^2) \\
&\leq \frac{8}{l^2}\|\sqrt{a}\|^2_{L^\infty(\Omega)}E_{u,\eta}(0) \\
&\leq \frac{8}{l^2}\|\sqrt{a}\|^2_{L^\infty(\Omega)}R.
\end{aligned}$$

Besides, for all $t \leq s$ we infer

$$\begin{aligned}
\|\sqrt{a}\nabla\eta^t(s)\|^2 &\leq 2\|\sqrt{a}\|^2_{L^\infty(\Omega)}\|\nabla u(t)\|^2 + 2\|\sqrt{a}\nabla u_0(t-s)\|^2 \\
&\leq \frac{4}{l^2}\|\sqrt{a}\|^2_{L^\infty(\Omega)}E_{u,\eta}(0) + 2\sup_{\tau<0}\|\sqrt{a}\nabla u_0(\tau)\|^2 \\
&\leq \frac{4}{l^2}\|\sqrt{a}\|^2_{L^\infty(\Omega)}R + 2R^2.
\end{aligned}$$

Thus, in view of (2.7), we obtain

$$\sup_{t\geq 0}\int_0^\infty g(s)^{1-\alpha_0}\|\sqrt{a}\nabla\eta^t(s)\|^2 ds \leq \left(\frac{12}{l^2}\|\sqrt{a}\|^2_{L^\infty(\Omega)}R + 2R^2\right)\int_0^\infty g(s)^{1-\alpha_0} ds < \infty,$$

which proves (4.118).

Now, using Jensen's inequality again, we deduce

$$\int_{mT}^{(m+1)T} D(t)^{\frac{\alpha_0}{p_2}} dt \leq T\left[\frac{1}{T}\int_{mT}^{(m+1)T} D(t)dt\right]^{\frac{\alpha_0}{p_2}},$$

and then

$$\int_{mT}^{(m+1)T}\int_0^\infty g(s)\|\sqrt{a}\nabla\eta^t(s)\|^2dsdt$$
$$\leq \frac{g(0)^{(1-\alpha_0)\frac{\alpha_0}{p_2}}}{C_2^{\frac{\alpha_0}{p_2}}}\left(\sup_{t\geq 0}\int_0^\infty g(s)^{1-\alpha_0}\|\sqrt{a}\nabla\eta^t(s)\|^2ds\right)^{1-\frac{\alpha_0}{p_2}}\left[\frac{1}{T}\int_{mT}^{(m+1)T}D(t)dt\right]^{\frac{\alpha_0}{p_2}}.$$

Defining $J:[0,\infty)\to[0,\infty)$ by

$$J(s)=\frac{g(0)^{(1-\alpha_0)\frac{\alpha_0}{p_2}}}{C_2^{\frac{\alpha_0}{p_2}}}\left(\sup_{t\geq 0}\int_0^\infty g(r)^{1-\alpha_0}\|\sqrt{a}\nabla\eta^t(r)\|^2dr+1\right)^{1-\frac{\alpha_0}{p_2}}T\left(\frac{s}{T}\right)^{\frac{\alpha_0}{p_2}}$$

which is a strictly increasing positive function such that $J(0)=0$, we get

$$\int_{mT}^{(m+1)T}\int_0^\infty g(s)\|\sqrt{a}\nabla\eta^t(s)\|^2dsdt\leq J\left(\int_{mT}^{(m+1)T}D(t)dt\right). \tag{4.119}$$

Therefore, from (4.115), (4.116) and (4.119), we obtain

$$E_{u,\eta}(mT)\leq C\left[J+\widetilde{H}^{-1}+\left(\frac{1}{M_1}+M_2\right)Id\right]\int_{mT}^{(m+1)T}D(t)dt,$$

and setting

$$H_3=C\left[J+\widetilde{H}^{-1}+\left(\frac{1}{M_1}+M_2\right)Id\right], \tag{4.120}$$

we have

$$E_{u,\eta}(mT)\leq H_3\left(\int_{mT}^{(m+1)T}D(t)dt\right).$$

Using that $E_{u,\eta}$ is non-increasing, it follows that

$$E_{u,\eta}((m+1)T)\leq H_3\left(\int_{mT}^{(m+1)T}D(t)dt\right),$$

and recalling that $D(t)=-E'(t)$, we arrive at

$$E_{u,\eta}((m+1)T)+H_3^{-1}(E_{u,\eta}((m+1)T))\leq E_{u,\eta}(mT),\quad \forall T>T_0. \tag{4.121}$$

Therefore, applying [37, Lemma 3.3] with

$$s_m=E_{u,\eta}(mT),\quad s_0=E_{u,\eta}(0),\quad p_1=H_3^{-1}\ \text{ with } H_3 \text{ given in (4.120)},$$

we conclude that $E_{u,\eta}(t)$ satisfies (3.5), where $S(t)$ is the solution of (3.6) with $q_1(s)=s-(Id+p_1)^{-1}(s)$. This completes the proof of Theorem 3.3-**I**. □

4.3.2. Proof of Theorem 3.3-II

In this case, we remark that (4.115) holds true with $b\equiv 0$ and proceeding analogously as in the estimates for the memory term, we achieve (4.121) with H_3 set in (4.120) just given by $H_3=CJ$ and $p_2=H_3^{-1}$. Hence, the same conclusion can be done with no change, namely, one can conclude that $E_{u,\eta}(t)$ satisfies (3.7), where $S(t)$ is the solution of (3.8) with $q_2(s)=s-(Id+p_2)^{-1}(s)$. This finishes the proof of Theorem 3.3-**II**. □

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Appendix. Auxiliary results

In order to make this paper more self-contained, we collect some important and useful results, which have been crucial in the conclusion of our proofs.

A.1. Two essential results in the literature

We start with an important unique continuation result. It can be found in Koch and Tataru [40].

Theorem A.1. *Let P be a second order hyperbolic operator with coefficients of class C^2. Let Γ be a smooth surface strongly pseudoconvex with respect to P. Then, the unique continuation of $P+V$ in Γ is valid for all potential $V \in L^{\frac{(n+1)}{2}}$, where n is the dimension of $\mathbb{R}^n$.*

When the Geometric Control Condition (GCC)) is taken into account, then problem (1.1) can be stabilized by only using the memory term, that is, $b \equiv 0$. To this end, an important result is presented by Burq and Gérard [41].

Theorem A.2. *Consider $\rho, \kappa \in C^\infty(\Omega)$ and let ω' be a subset of a given set $\Omega \subset \mathbb{R}^n$ which satisfies the GCC. Besides that, let $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $u \in C^0(\mathbb{R}; L^2(\Omega))$ satisfying that $\rho(x)u \in C^1(\mathbb{R}; H^{-1}(\Omega))$ is the ultra-weak solution of*

$$\begin{cases} \rho(x)u'' - div[\kappa(x)\nabla u] = 0 \quad in \quad \Omega \times (0,\infty), \\ u = 0 \quad in \quad \partial\Omega \times (0,\infty), \\ u(0) = u_0 \in L^2(\Omega);\ \rho(x)u'(0) = u_1 \in H^{-1}(\Omega). \end{cases} \tag{A.1}$$

Then,

$$\|u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{H^{-1}(\Omega)} \le C \int_0^T \int_{\omega'} |u(x,t)|^2 dxdt, \tag{A.2}$$

for all $T > T_0$.

A.2. A short review on microlocal analysis

In what follows, we are going to present some results in microlocal local analysis develop in [39,41]. We also refer to [1] (see Appendix therein) for a more complete review of the results to be presented below and the explanation of how to apply such results in the present framework.

Theorem A.3. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^2_{loc}(\Omega)$ such that it converges weakly to zero in $L^2_{loc}(\Omega)$. Then, there exist a subsequence $\{u_{\varphi(n)}\}$ and a positive Radon measure μ on $T^1\Omega := \Omega \times S^{n-1}$*[3] *such that for all pseudo-differential operator A of order 0 on Ω which admits a principal symbol $\sigma_0(A) \ge 0$ (i.e. $A \in \Psi^0_c(\Omega)$ and, in addition, $\sigma_0(A) \ge 0$) and for all $\chi \in C^\infty_0(\Omega)$ such that $\chi\sigma_0(A) = \sigma_0(A)$, one has*

$$\big(A(\chi u_{\varphi(n)}), \chi u_{\varphi(n)}\big)_{L^2} \underset{n\to+\infty}{\longrightarrow} \int_{\Omega\times S^{n-1}} \sigma_0(A)(x,\xi)\, d\mu(x,\xi). \tag{A.3}$$

[3] The notation S^d stands for the d-dimensional unitary sphere.

Definition A.4. Under the hypotheses of Theorem A.3 μ is called the Microlocal Defect Measure (in short, MDM) of the sequence $\{u_{\varphi(n)}\}_{n\in\mathbb{N}}$.

Remark A.5. Theorem A.3 assures that for all bounded sequence $\{u_n\}_{n\in\mathbb{N}} \in L^2_{loc}(\Omega)$ such that it converges weakly to zero in $L^2_{loc}(\Omega)$, there exists a subsequence $\{u_{\varphi(n)}\}$ admitting a microlocal defect measure. We observe that from (A.3), in the particular case when $A = f \in C_0^\infty(\Omega)$, it follows that

$$\int_{\Omega\times S^{n-1}} f(x)|u_{\varphi_n}(x)|^2\,dx \underset{n\to+\infty}{\longrightarrow} \int_{\Omega\times S^{n-1}} f(x)\,d\mu(x,\xi). \tag{A.4}$$

Then, u_{φ_n} converges to zero if, and only if, $\mu = 0$.

Theorem A.6. *Let P be a differential operator of order m on Ω and let $\{u_n\}$ be a bounded sequence of $L^2_{loc}(\Omega)$ which converges weakly to 0 and admits a m.d.m. μ. The following statements are equivalents:*

> *(i)* $Pu_n \underset{n\to+\infty}{\longrightarrow} 0$ *strongly in* $H^{-m}_{loc}(\Omega)$ $(m>0)$.
>
> *(ii)* $supp(\mu) \subset \{(x,\xi)\in\Omega\times S^{n-1} : \sigma_m(P)(x,\xi)=0\}$.

Remark A.7. Let P be a differential operator on Ω of order m with smooth coefficients and let $\{u_n\}_n$ be a bounded sequence of $L^2_{loc}(\Omega)$ with a microlocal defect measure μ. Then, Theorem A.6 states that $Pu_n \underset{n\to+\infty}{\longrightarrow} 0$ strongly in $H^{-m}_{loc}(\Omega)$ $(m>0)$ if and only if μ is supported by the characteristic set of P, that is, $\sigma_m(P)\mu = 0$. Hence, Theorem A.6 provides a localization of the support of μ under a very weak assumption on $\{Pu_n\}$. The next theorem shows that, under a slightly stronger assumption, the Hamiltonian H_p of the principal symbol of P satisfies an integral equation on $\Omega\times S^{n-1}$, namely:

$$\int_{\Omega\times S^{d-1}} H_p a\,d\mu = 0, \quad \text{where } p=\sigma_m(P),$$

for all $a \in C^\infty(\Omega\times(\mathbb{R}^n\backslash\{0\}))$ homogeneous of degree $1-m$ in the second variable and with compact support in the first one.

Theorem A.8. *Let P be a differential operator of order m on Ω, verifying $P^* = P$, and let $\{u_n\}$ be a bounded sequence in $L^2_{loc}(\Omega)$ which converges weakly to 0 and it admits a m.d.m. μ. Let us assume that $Pu_n \underset{n\to+\infty}{\longrightarrow} 0$ strongly in $H^{1-m}_{loc}(\Omega)$. Then, for all homogeneous function $a \in C^\infty(\Omega\times\mathbb{R}^n\backslash\{0\})$ of degree $1-m$ in the second variable and with compact support in the first one, we have*

$$\int_{\Omega\times S^{n-1}} \{a,p\}(x,\xi)\,d\mu(x,\xi) = 0. \tag{A.5}$$

Theorem A.9. *Let P be a self-adjoint differential operator of order m on Ω which admits a principal symbol p. Let $\{u_n\}$ be a bounded sequence in $L^2_{loc}(\Omega)$ which converges weakly to zero, with a microlocal defect measure μ. Let us assume that Pu_n converges to 0 in $H^{-(m-1)}_{loc}(\Omega)$. Then the support of μ, $supp(\mu)$, is a union of curves like $s\in I\mapsto \left(x(s), \frac{\xi(s)}{|\xi(s)|}\right)$, where $s\in I\mapsto (x(s),\xi(s))$ is a bicharacteristic of p.*

A.3. Principal symbol of the wave operator

This final section is devoted to the explanation of how to apply the above abstract results to characterize the bicharacteristics of the principal symbol of the wave operator related to our problem. The next statements follow the same lines as done in [1, Section 5].

Let us consider the wave operator

$$\rho(x)\partial_t^2 - \sum_{j=1}^{d} \partial_{x_j}\left[\kappa(x)\partial_{x_j}\right].$$

Under the notation $D_j = \frac{1}{i}\partial_j$, we write

$$P(t,x,D_t,D_x) = -\rho(x)D_t^2 + \sum_{j=1}^{d} D_{x_j}[\kappa(x)D_{x_j}], \quad D_x = (D_{x_1},\dots,D_{x_d}),$$

whose *principal symbol* $p(t,x,\tau,\xi)$ is given by

$$p(t,x,\tau,\xi) = -\rho(x)\tau^2 + \kappa(x)\,\xi\cdot\xi, \quad \xi = (\xi_1,\dots,\xi_d), \tag{A.6}$$

where $t\in\mathbb{R},\ \ x\in\Omega\subset\mathbb{R}^n,\ \ (\tau,\xi)\in\mathbb{R}\times\mathbb{R}^n$, and $\kappa(x) := 1-g_0a(x)$ is the C^∞ function on Ω verifying

$$1 \ge \kappa(x) \ge 1 - g_0\|a\|_\infty := l > 0,$$

since we are assuming that $l := 1-g_0\|a\|_\infty > 0$, where $g_0 := \int_0^\infty g(s)\,ds$.

Let us describe the *bicharacteristics* of p. They do not change if we multiply p by a non-null function

$$\tilde{p}(t,x,\tau,\xi) = \frac{1}{2}\left(\frac{\kappa(x)}{\rho(x)}\,\xi\cdot\xi - \tau^2\right). \tag{A.7}$$

From this, we have

$$\begin{cases} \dot{t} = \dfrac{\partial\tilde{p}}{\partial\tau} = -\tau, \\ \dot{x} = \dfrac{\partial\tilde{p}}{\partial\xi} = \dfrac{\kappa(x)}{\rho(x)}\xi, \\ \dot{\tau} = -\dfrac{\partial\tilde{p}}{\partial t} = 0, \\ \dot{\xi} = -\dfrac{\partial\tilde{p}}{\partial x} = -\dfrac{1}{2}\nabla\left(\dfrac{\kappa}{\rho}\right)(x)(\xi\cdot\xi). \end{cases} \tag{A.8}$$

Introducing the function $G(x) := \frac{\rho(x)}{\kappa(x)}$, the above equations become

$$\xi = G(x)\dot{x}, \tag{A.9}$$

which implies

$$\begin{aligned} \dot{\xi} = (G(x)\dot{x})^{\cdot} &= -\frac{1}{2}\nabla\left(\frac{\kappa}{\rho}\right)(x)\,\langle G(x)\dot{x}, G(x)\dot{x}\rangle \\ &= -\frac{1}{2}\nabla\left(\frac{\kappa}{\rho}\right)(x)\,G^2(x)\,\langle \dot{x},\dot{x}\rangle \\ &= -\frac{1}{2}\nabla\left[G(x)\right]^{-1}\,G^2(x)\,\dot{x}\cdot\dot{x} \\ &= \frac{1}{2}\nabla G(x)\,\dot{x}\cdot\dot{x}. \end{aligned} \tag{A.10}$$

where we have used the notation $\langle x,\xi\rangle = x\cdot\xi$. Once $\tilde{p}$ is null on each of its bicharacteristic curves, from (A.7) we deduce that

$$\kappa(x)\,\xi\cdot\xi = \tau^2 = \text{ constant on the curve}, \tag{A.11}$$

On the other hand, equalities (A.8) and (A.9) yield

$$G(x)\dot{x}\cdot\dot{x} = \frac{\kappa(x)}{\rho(x)}\,\xi\cdot\xi. \tag{A.12}$$

Combining (A.11) and (A.12) we deduce

$$G(x)\dot{x} \cdot \dot{x} = \kappa(x)\, \xi \cdot \xi = \tau^2 = \text{ constant on the curve,} \tag{A.13}$$

that is, the quantity $G(x)\,\dot{x} \cdot \dot{x}$ is preserved under the flow. From (A.10) and (A.13) we have

$$\frac{d}{ds}\frac{G(x)\dot{x}}{\sqrt{G(x)\,\dot{x} \cdot \dot{x}}} = \frac{1}{2}\frac{\nabla G(x)\,\dot{x} \cdot \dot{x}}{\sqrt{G(x)\,\dot{x} \cdot \dot{x}}}, \tag{A.14}$$

and setting

$$L(x, \dot{x}) := \sqrt{G(x)\,\dot{x} \cdot \dot{x}},$$

the last identity yields

$$\frac{d}{ds}\frac{\partial}{\partial \dot{x}} L(x, \dot{x}) = \frac{\partial}{\partial x} L(x, \dot{x}),$$

which is the Euler–Lagrange equation associated with L, namely, the geodesic equation for the metric G of Ω. Conversely, if $\alpha \mapsto x(\alpha)$ is a geodesic for the metric G on Ω, and parameterizing the curve x by the curvilinear abscissa σ defined by

$$\frac{d\sigma}{d\alpha} = \sqrt{G(x(\alpha))\,\dot{x}(\alpha) \cdot \dot{x}(\alpha)}, \tag{A.15}$$

Eq. (A.14) becomes

$$\frac{d}{d\sigma}\left(G(x)\frac{dx}{d\sigma}\right) = \frac{1}{2}\nabla G(x)\,\frac{dx}{d\sigma} \cdot \frac{dx}{d\sigma}, \tag{A.16}$$

with

$$G(x)\,\frac{dx}{d\sigma} \cdot \frac{dx}{d\sigma} = 1.$$

If we set, for instance, $s = -\frac{\sigma}{\tau}$, we obtain (A.8), (A.9) and (A.10). In addition, we note that $\frac{dt}{d\sigma} = 1$. In conclusion, we have proved the following result.

Proposition A.10. *Unless a change of variables is needed, the bicharacteristics of (A.6) are curves of the form*

$$t \mapsto \left(t, x(t), \tau, -\tau\left(\frac{\kappa(x(t))}{\rho(x(t))}\right)^{-1} \dot{x}(t)\right),$$

where $t \mapsto x(t)$ is a geodesic of the metric $G = \left(\frac{\kappa}{\rho}\right)^{-1}$ on Ω, parameterized by the curvilinear abscissa. □

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