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# Solvability for perturbations of a class of real vector fields on the two-torus

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ABSTRACT

Let $L = \partial_t + a(x)\partial_x$ be a real vector field defined on the two-dimensional torus $\mathbb{T}^2$, where $a$ is a real-valued and smooth function on $\mathbb{T}^1$. We deal with the global solvability of equations in the form $Lu + pu = f$, where $p, f \in \mathcal{C}^\infty(\mathbb{T}^2)$. Solvability to the equation $Lu = f$ is well-understood. We show that a perturbation of zero order may affect the global solvability of $L$; we may maintain, gain or lose solvability by adding a perturbation. This phenomenon is linked to the order of vanishing of the coefficient $a$ of $L$. We obtained results in the class of smooth functions on $\mathbb{T}^2$ and, also, in the space of Schwartz distributions $\mathcal{D}'(\mathbb{T}^2)$.


## 1. Introduction

Let $\mathcal{L}$ be a nonsingular real vector field defined on a smooth $n$-dimensional manifold $M$.

We are interested in solving equations in the form

$$\mathcal{L}u + pu = f, \tag{1.1}$$

where $p$, $f$ and $u$ are complex-valued smooth functions defined on $M$. The space of such functions will be denoted by $\mathcal{C}^\infty(M)$.

It follows from tubular flow theorem that locally (1.1) always has smooth solutions.

The solvability of (1.1) is still an interesting problem if we seek for global solutions.

For $p \in \mathcal{C}^\infty(M)$, we define $\mathcal{L}_p : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ by $\mathcal{L}_p u = \mathcal{L}u + pu$.

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In the case where $M$ is noncompact, paracompact, Malgrange in [15] and also Duistermaat and Hörmander in [10] presented necessary and sufficient conditions to obtain $\mathcal{L}_p(\mathcal{C}^\infty(M)) = \mathcal{C}^\infty(M)$.

Malgrange showed essentially that $\mathcal{L}_p\mathcal{C}^\infty(M) = \mathcal{C}^\infty(M)$ is equivalent to the following geometric condition:

(GC) (i) No complete integral curve of $\mathcal{L}$ is relatively compact;

(ii) For every compact subset $K$ of $M$ there exists a compact subset $K'$ of $M$ such that every compact interval on an integral curve with end points in $K$ is contained in $K'$.

Duistermaat and Hörmander proved that (GC) is equivalent to

(iii) There exists a manifold $M_0$, an open neighborhood $M_1$ of $M_0 \times \{0\}$ in $M_0 \times \mathbb{R}$ which is convex in the $\mathbb{R}$ direction, and a diffeomorphism $M \to M_1$ which carries $\mathcal{L}$ into the operator $\partial/\partial t$, where points in $M_1$ are denoted by $(x, t)$.

Comparing with local solvability, under (GC) the zero order term in $\mathcal{L}_p$ does not play a relevant role in the global solvability.

There is no general result about global solvability of $\mathcal{L}_p$ in the case where $M$ is compact.

We will address global solvability issues for zero order perturbations of certain real vector fields on the 2-dimensional torus, $\mathbb{T}^2 \simeq \mathbb{R}^2/2\pi\mathbb{Z}^2$.

Denoting the coordinates in $\mathbb{T}^2$ by $(x, t) \in \mathbb{T}^1 \times \mathbb{T}^1$, we consider the real vector field given by

$$L_0 = \partial_t + a(x)\partial_x, \tag{1.2}$$

where $a$ is a real-valued and smooth function on $\mathbb{T}^1$.

In [6], Bergamasco and Petronilho characterized the closedness of the range (global solvability) of $L_0$. By Theorem 1.1 in [6], $L_0$ is globally solvable if and only if one of the following conditions holds:

- $a$ vanishes identically;
- $a$ never vanishes and $(2\pi)^{-1}\int_0^{2\pi}(1/a)$ is either rational or non-Liouville irrational;
- $\emptyset \neq a^{-1}(0) \neq \mathbb{T}^1$ and $a$ vanishes only of finite order.

One of the motivations for this work was a question posed by Adalberto Bergamasco and José Ruidival dos Santos Filho to the second author concerning some results in the paper [6]. As mentioned in [6] without a detailed proof, the operator $\partial_t + a(x)\partial_x + a'(x)$ is globally solvable on $\mathcal{C}^\infty(\mathbb{T}^2)$.

Our purpose here is to study the influence of a perturbation of $L_0$ by a zero order term in the existence of global solutions. We consider differential operators

$$L_p = \partial_t + a(x)\partial_x + p(x), \tag{1.3}$$

defined on $\mathbb{T}^2$, where $a$ is a real-valued and smooth function on $\mathbb{T}^1$, and $p$ is a complex-valued and smooth function on $\mathbb{T}^1$.

As in [6], we say that $L_p$ is globally solvable on $\mathcal{C}^\infty(\mathbb{T}^2)$ if its range $L_p\mathcal{C}^\infty(\mathbb{T}^2)$ is a closed subspace of $\mathcal{C}^\infty(\mathbb{T}^2)$. By standard arguments of Functional Analysis, $\mathcal{L}_p$ is globally solvable if and only if $L_p\mathcal{C}^\infty(\mathbb{T}^2) = (\ker {}^tL_p)^\circ$, where ${}^tL_p$ denotes the transpose operator and $(\ker {}^tL_p)^\circ$ is the set of functions $\phi \in C^\infty(\mathbb{T}^2)$ such that $\mu(\phi) = 0$, for all $\mu \in \ker {}^tL_p \subset \mathcal{D}'(\mathbb{T}^2)$.

We will also consider the global solvability problem in the space of Schwartz distributions $\mathcal{D}'(\mathbb{T}^2)$. We say that $L_p$ is globally solvable on $\mathcal{D}'(\mathbb{T}^2)$ if $L_p\mathcal{D}'(\mathbb{T}^2) = {}^\circ(\ker L_{a'-p})$, in which $L_{a'-p} : \mathcal{C}^\infty(\mathbb{T}^2) \to \mathcal{C}^\infty(\mathbb{T}^2)$ is given by $L_{a'-p} = \partial_t + a(x)\partial_x + a'(x) - p(x)$ and ${}^\circ(\ker L_{a'-p})$ is the space of distributions $\nu \in \mathcal{D}'(\mathbb{T}^2)$ such that $\langle \nu, \phi \rangle = 0$, for all $\phi \in \ker L_{a'-p}$.

In contrast to the previously mentioned results of Malgrange, Duistermaat and Hörmander, in our context the zero order perturbation $p$ has a direct bearing on the closed range property for $L_p$.

Our first contribution is to show that we may lose, maintain, or gain solvability when we add a perturbation of zero order in the operator $L_0$. This is the subject of Section 2, where the global solvability of $L_p$ is characterized in the case where the coefficient $a$ never vanishes.

In Section 3 we show that $L_p$ is globally solvable, provided that $a^{-1}(0) \neq \emptyset$ and $a$ vanishes only of finite order. This is our main result in this article. When $\Re p$ vanishes as much as $a$ at the points in $a^{-1}(0)$, then we may use the approach in [6]; however, the proof of certain results needs a careful treatment (compare, for instance, the proof of Proposition 3.1 with the proof of Lemma 2.1 in [6]). When the difference between the order of vanishing of $a$ and the order of vanishing of $\Re p$, at some point in $a^{-1}(0)$, is greater than one, then we use certain techniques which appear in [4] and [11]. The biggest novelty appears when there exist points in $a^{-1}(0)$, where the difference between the order of vanishing of $a$ and the order of vanishing of $\Re p$ is one. This is a critical case which is related to other critical cases that appear in [2], [4], and [7]. To treat these critical cases, we need an improvement of techniques of [6] and we also need a new approach, which is presented in Case 5, in the proof of Theorem 3.4.

We also notice in Section 3 that the global solvability may become stronger on $\mathcal{C}^\infty(\mathbb{T}^2)$, by the action of the perturbation, i.e., $L_p\mathcal{C}^\infty(\mathbb{T}^2)$ may have finite codimension, while $L_0\mathcal{C}^\infty(\mathbb{T}^2)$ has infinite codimension. On the other hand, we show that the surjectivity of $L_0 : \mathcal{D}'(\mathbb{T}^2) \to \mathcal{D}'(\mathbb{T}^2)$ may be affected, i.e., $L_p\mathcal{D}'(\mathbb{T}^2) \subsetneq \mathcal{D}'(\mathbb{T}^2)$ for certain perturbations $p$. Hence, the solvability on $\mathcal{D}'(\mathbb{T}^2)$ may become weaker by the action of the perturbation.

Finally, in Section 4 we give further results considering the existence of points at which the coefficient $a$ is flat. In the case where $a$ vanishes identically we show that the zero order perturbation $p$ may destroy the global solvability of $L_0$ (see Theorem 4.2).

The questions addressed here are within the spirit of those contained in the articles [1,5,7–9,12–14,16].

## 2. Coefficient $a$ never vanishing

When $a$ never vanishes, it follows from [6] that $L_0 = \partial_t + a(x)\partial_x$ is globally solvable if and only if the mean $(1/2\pi)\int_0^{2\pi}(1/a)$ is either a rational or a non-Liouville irrational number. Recall that an irrational number $\alpha$ is said to be a Liouville number if there exists a sequence $(p_n, q_n) \in \mathbb{Z} \times \mathbb{N}$ such that $q_n \to \infty$ and $|p_n + \alpha q_n| < q_n^n$, for all $n \in \mathbb{N}$.

We will see that the perturbations $L_p = \partial_t + a(x)\partial_x + p(x)$ may maintain, lose or gain solvability.

In order to present the main result of this section, we denote by $\theta_0$ the mean of a function $\theta \in \mathcal{C}^\infty(\mathbb{T}^1)$, that is, $\theta_0 = (1/2\pi)\int_0^{2\pi}\theta(s)ds$.

**Theorem 2.1.** *Let $L_p$ be the operator given by (1.3). Suppose that $a$ never vanishes. In this case, $L_p$ is globally solvable (on either $\mathcal{C}^\infty(\mathbb{T}^2)$ or $\mathcal{D}'(\mathbb{T}^2)$) if and only if there exist positive constants $C$ and $\gamma$ such that*

$$|j + k(1/a)_0 + (\Im p/a)_0 - i(\Re p/a)_0| \geq C(|j| + |k|)^{-\gamma},$$

*for all $(j,k) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $j + k(1/a)_0 + (\Im p/a)_0 - i(\Re p/a)_0 \neq 0$.*

Before writing the proof, we would like to discuss certain implications of the above result.

Next example shows that the perturbation $p$ may maintain the global solvability of $L_0$.

**Example 2.2.** If $(1/a)_0$ is a rational number, then $L_0$ is globally solvable and any perturbation $L_p$ is still globally solvable (the a priori estimate presented in Theorem 2.1 is satisfied). For instance, $\partial_t + (1+\pi)(1+\cos^2(x))^{-1}\partial_x + p(x)$ is globally solvable for any $p \in \mathcal{C}^\infty(\mathbb{T}^1)$.

The same situation occurs (both $L_0$ and any perturbation $L_p$ are globally solvable) if $(1/a)_0$ is a non-Liouville irrational number and $(\Im p/a)_0$ is a rational number. For instance, $\partial_t - \sqrt{2}(2-\sin(x))^{-1}\partial_x$ and $\partial_t - \sqrt{2}(2-\sin(x))^{-1}\partial_x + i(\pi\sqrt{2})^{-1}\sin(x)$ are globally solvable.

In certain cases, we may have a non-globally solvable vector field $L_0$ with a perturbation $L_p$ which is globally solvable.

**Example 2.3.** When $(\Re p/a)_0 \neq 0$, then Theorem 2.1 implies that any perturbation $L_p$ is globally solvable. Hence, if $(1/a)_0$ is either rational or a non-Liouville irrational number, then the perturbation $L_p$ maintains the global solvability of $L_0$. On the other hand, if $(1/a)_0$ is a Liouville irrational number, then $L_0$ is not globally solvable, but $L_p$ is globally solvable. It follows that we may obtain a globally solvable operator by considering a perturbation of zero order of the non-globally solvable vector field $L_0$. This situation also occurs when $(\Re p/a)_0 = 0$. For instance, if $(\Re p_0/a) = 0$, $(\Im p/a)_0 = -1/2$, and $(1/a)_0 = -\beta$, in which $\beta$ is the Liouville number constructed in Example 4.4 in [1], then $L_0$ is not globally solvable, while $L_p$ is globally solvable. In particular, $\partial_t - 2\beta^{-1}(2-\sin(x))^{-1}\partial_x$ is not globally solvable, while $\partial_t - 2\beta^{-1}(2-\sin(x))^{-1}\partial_x + i(2\beta)^{-1}$ is globally solvable.

May we obtain a non-globally solvable operator by considering a perturbation of zero order of a globally solvable vector field $L_0$? The answer is positive. As we have already noticed, to find operators satisfying this property, $p$ must satisfy $(\Re p/a)_0 = 0$ and $(\Im p/a)_0$ must be an irrational number. In addition, if $(1/a)_0$ is an irrational number, then the a priori estimate presented in Theorem 2.1, also characterizes the global hypoellipticity of the operator $\partial_t + (1/a)_0\partial_x + i(\Im p/a)_0$, which is an issue treated in [1]. Applying Proposition 3.5 in [1], we see that the set of functions $p$ such that $L_p$ is not globally solvable contains a dense $\mathcal{G}_\delta$. Hence, in the sense of Baire category, there are many perturbations (of a globally solvable vector field) which are not globally solvable.

We now proceed to prove Theorem 2.1: Since $a$ never vanishes, the global solvability of $L_p$ is equivalent to the global solvability of $(1/a)L_p$. On the other hand, in the new coordinates $X = x$ and $T = t + (1/a)_0 - \int_0^x (1/a)$, operator $(1/a)L_p$ becomes $\partial_X + (1/a)_0\partial_T + (p/a)$. The global solvability of this last operator is equivalent to the global solvability of $\partial_T + \alpha\partial_X + \alpha(p/a)$, in which $\alpha = (1/a)_0^{-1}$. The proof now follows by applying Propositions 2.4 and 2.5 below.

**Proposition 2.4.** *The operator $L = \partial_t + \lambda\partial_x + \rho(x)$, where $\lambda \in \mathbb{R} \setminus \{0\}$ and $\rho \in \mathcal{C}^\infty(\mathbb{T}^1)$, is globally solvable on $\mathcal{C}^\infty(\mathbb{T}^2)$ if and only if there exist positive constants $C$ and $\gamma$ such that*

$$|k + j\lambda + (\Im\rho)_0 - i(\Re\rho)_0| \geq C(|j| + |k|)^{-\gamma}, \tag{2.1}$$

*for all $(j,k) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $k + j\lambda + (\Im\rho)_0 - i(\Re\rho)_0 \neq 0$.*

**Proof.** For each $k \in \mathbb{Z}$, define

$$c_k \doteq \frac{1}{2\pi}\int_0^{2\pi} \frac{\Re\rho(s) + i(k + \Im\rho(s))}{\lambda} ds.$$

Applying Lemma 3.1 of [3], it follows that (2.1) is equivalent to the following condition: there exist positive constants $C$ and $\gamma$ such that

$$|1 - \exp\{-2\pi c_k\}| \geq C(|k| + 1)^{-\gamma},$$

for all $k \in \mathbb{Z}$ such that $c_k \notin i\mathbb{Z}$.

Necessity:

If (2.1) fails to hold, then there exists a sequence $(k_n) \subset \mathbb{Z}$ such that $|k_{n+1}| > |k_n| > n$, $c_{k_n} \notin i\mathbb{Z}$, and $0 < |1 - \exp\{-2\pi c_{k_n}\}| < (1 + |k_n|)^{-n}$, for all $n \in \mathbb{N}$.

Let $\widehat{f}(x, k_n)$ be the $2\pi$-periodic extension of

$$\lambda(1 + |k_n|)^{-n/2}\phi(x) \exp\left\{ \int_x^\pi \frac{\Re\rho(s) + i(k_n + \Im\rho(s))}{\lambda} ds \right\}, \; t \in [0, 2\pi],$$

in which $\phi \in \mathcal{C}_c^\infty((\pi/2) - \epsilon, (\pi/2) + \epsilon)$, $0 \le \phi(x) \le 1$, $\phi \equiv 1$ on $[(\pi - \epsilon)/2, (\pi + \epsilon)/2]$, and $\epsilon > 0$ is small enough so that $((\pi/2) - \epsilon, (\pi/2) + \epsilon) \subset (0, \pi)$.

We claim that the function $f$, given by

$$f(x, t) = \sum_{n=1}^\infty \widehat{f}(x, k_n) \exp\{ik_n t\},$$

belongs to $(\ker {}^tL)^\circ \setminus L\mathcal{C}^\infty(\mathbb{T}^2)$.

Notice that the term $(1 + |k_n|)^{-n/2}$ yields the rapid decaying of $\widehat{f}(\cdot, k_n)$. Hence $f \in \mathcal{C}^\infty(\mathbb{T}^2)$. In addition, if $\mu \in \ker {}^tL$, then $\widehat{\mu}(x, -k_n) = 0$, since $c_{k_n} \notin i\mathbb{Z}$. Therefore, $f \in (\ker {}^tL)^\circ$.

To complete the proof of necessity, we will show that $f$ cannot belong to $L\mathcal{C}^\infty(\mathbb{T}^2)$.

Suppose that there exists $u \in \mathcal{C}^\infty(\mathbb{T}^2)$ solution to $Lu = f$. By using partial Fourier series in the variable $t$ we obtain

$$\partial_x\widehat{u}(x, k_n) + \frac{1}{\lambda}(ik_n + p(x))\widehat{u}(x, k_n) = \widehat{f}(x, k_n)/\lambda, \; x \in \mathbb{T}^1, \; n \in \mathbb{N}.$$

Since $c_{k_n} \notin i\mathbb{Z}$, each of the above equations has a unique solution in $\mathcal{C}^\infty(\mathbb{T}^1)$, which may be written as

$$\widehat{u}(x, k_n) = C_n \int_0^{2\pi} \exp\left\{ -\int_{x-y}^x \frac{\Re\rho(s) + i(k_n + \Im\rho(s))}{\lambda} ds \right\} \lambda^{-1}\widehat{f}(x - y, k_n) dy, \tag{2.2}$$

with $C_n = (1 - \exp\{-2\pi c_{k_n}\})^{-1}$.

The definition of $\widehat{f}(x, k_n)$ implies that, for $x \in [\pi, 2\pi]$, we have

$$C_n^{-1} \exp\left\{ \int_\pi^x \frac{\Re\rho(s) + i(k_n + \Im\rho(s))}{\lambda} ds \right\} \widehat{u}(x, k_n) \ge \epsilon(1 + |k_n|)^{-n/2}. \tag{2.3}$$

Since $|C_n| \ge (1 + |k_n|)^n$, we obtain

$$|\widehat{u}(\pi, k_n)| \ge \epsilon(1 + |k_n|)^{n/2}, \; n \in \mathbb{N},$$

which is a contradiction, since $\widehat{u}(\pi, k_n)$ decays rapidly.

Sufficiency:

By assuming (2.1) we must prove that for each $f \in (\ker {}^tL)^\circ \subset \mathcal{C}^\infty(\mathbb{T}^2)$, there exists $u \in \mathcal{C}^\infty(\mathbb{T}^2)$ such that $Lu = f$.

Partial Fourier series in the variable $t$ leads us to the equations

$$\partial_x\widehat{u}(x, k) + \frac{1}{\lambda}(ik + p(x))\widehat{u}(x, k_n) = \widehat{f}(x, k)/\lambda, \; x \in \mathbb{T}^1, \; k \in \mathbb{Z}.$$

We obtain solutions to these equations defining

$$\widehat{u}(x,k) = \int_0^x \exp\left\{ - \int_y^x \frac{\Re\rho(s) + i(k + \Im\rho(s))}{\lambda} ds \right\} \lambda^{-1} \widehat{f}(y,k) dy, \tag{2.4}$$

if $k$ is such that $c_k \in i\mathbb{Z}$, and, in the other case, we define

$$\widehat{u}(x,k) = C_k \int_0^{2\pi} \exp\left\{ - \int_{x-y}^x \frac{\Re\rho(s) + i(k + \Im\rho(s))}{\lambda} ds \right\} \lambda^{-1} \widehat{f}(x-y,k) dy, \tag{2.5}$$

in which $C_k = (1 - \exp\{-2\pi c_k\})^{-1}$.

Solutions given by (2.4) are $2\pi$-periodic, since $f \in (\ker {}^tL)^\circ$ and the distributions

$$\exp\left\{ \int_0^x \frac{\Re\rho(s) + i(k + \Im\rho(s))}{\lambda} ds \right\} \exp\{-ikt\}$$

belong to $\ker {}^tL$, whenever $c_k \in i\mathbb{Z}$. In addition, the rapid decaying of $\widehat{f}(\cdot,k)$ implies the rapid decaying of the sequence of solutions given by (2.4).

Finally, condition (2.1) implies that there exist positive constants $C$ and $\gamma$ such that $|1 - \exp\{-2\pi c_k\}| \geq C(1+|k|)^{-\gamma}$, for all $k \in \mathbb{Z}$ such that $c_k \notin i\mathbb{Z}$; hence the rapid decaying of $\widehat{f}(\cdot,k)$ yield the rapid decaying of the sequence given by (2.5). $\square$

**Proposition 2.5.** *The operator $L = \partial_t + \lambda\partial_x + \rho(x)$, where $\lambda \in \mathbb{R} \setminus \{0\}$ and $\rho \in \mathcal{C}^\infty(\mathbb{T}^1)$, is globally solvable on $\mathcal{D}'(\mathbb{T}^2)$ if and only if condition (2.1) holds.*

**Proof.** Notice that the operator $P : \mathcal{C}^\infty(\mathbb{T}^2) \to \mathcal{C}^\infty(\mathbb{T}^2)$, given by $P = \partial_t + \lambda\partial_x - \rho(x)$, satisfies ${}^tP = -L$. Since the global solvability of $P$ on $\mathcal{C}^\infty(\mathbb{T}^2)$ implies the global solvability of $L$ on $\mathcal{D}'(\mathbb{T}^2)$, the sufficiency of condition (2.1) follows from Proposition 2.4.

On the other hand, if condition (2.1) fails to hold, then we define $f \in C^\infty(\mathbb{T}^2)$ by proceeding as in the proof of necessity in Proposition 2.4. We claim that this function $f$ belongs to ${}^\circ(\ker P) \setminus L\mathcal{D}'(\mathbb{T}^2)$, which means that $L$ is not globally solvable on $\mathcal{D}'(\mathbb{T}^2)$. Notice that, if $u \in C^\infty(\mathbb{T}^2)$ and $Pu = 0$, then $\widehat{u}(\cdot, -k_n)$ vanishes identically, since $c_{k_n} \notin i\mathbb{Z}$. Hence $f \in {}^\circ(\ker P)$. Finally, if $\mu \in \mathcal{D}'(\mathbb{T}^2)$ and $L\mu = f$, then $\widehat{\mu}(\cdot, k_n)$ belongs to $C^\infty(\mathbb{T}^1)$ and it is given by (2.2). Hence estimate (2.3) holds true. By taking $\psi \in \mathcal{C}^\infty(\mathbb{T}^1, \mathbb{R})$ such that $\psi \geq 0$, $\mathrm{supp}\psi \subset [\pi, 2\pi]$, and $\int_\pi^{2\pi} \psi = 1$, it follows that

$$\left| \left\langle \widehat{\mu}(x,k_n), C_n^{-1} \exp\left\{ \int_\pi^x \frac{\Re\rho(s) + i(k_n + \Im\rho(s))}{\lambda}(s) ds \right\} \psi(x) \right\rangle \right| \geq \epsilon(1+|k_n|)^{-n/2}, \text{ for all } n \in \mathbb{N}. \tag{2.6}$$

Since $\widehat{\mu}(\cdot, k_n)$ increases slowly, we obtain constants $K > 0$ and $M \in \mathbb{Z}_+$, which do not depend on $n$, such that

$$\left| \left\langle \widehat{\mu}(x,k_n), C_n^{-1} \exp\left\{ \int_\pi^x \frac{\Re\rho(s) + i(k_n + \Im\rho(s))}{\lambda}(s) ds \right\} \psi(x) \right\rangle \right| \leq |C_n|^{-1} K(1+|k_n|)^M, \text{ for all } n \in \mathbb{N}. \tag{2.7}$$

The estimates (2.6) and (2.7) imply that

$$K \geq \epsilon |C_n| (1+|k_n|)^{-(n/2)-M} \geq \epsilon(1+|k_n|)^{(n/2)-M},$$

for all $n \in \mathbb{N}$, which is a contradiction. $\square$

**Remark 2.6.** When $\lambda = 0$, we have operators of the type $\partial_t + \rho(x)$. The global solvability of these operators is characterized in Section 4. See Theorem 4.2 and Theorem 4.3.

## 3. Coefficient $a$ with only zeros of finite order

Throughout this section, we assume that $a^{-1}(0) \neq \emptyset$ and that $a$ vanishes only of finite order. Thus, we may write $a^{-1}(0) = \{x_1 < \cdots < x_N\}$, and $x_{N+1} = x_1 + 2\pi$. In this case, we denote by $n_j$ the order of vanishing of $a$ at $x_j$, $j = 1, \ldots, N$.

According to [6], under these assumptions the operator $L_0$ is globally solvable on both $\mathcal{C}^\infty(\mathbb{T}^2)$ and $\mathcal{D}'(\mathbb{T}^2)$. Moreover, $L_0$ is surjective on $\mathcal{D}'(\mathbb{T}^2)$. Hence $L_0\mathcal{C}^\infty(\mathbb{T}^2) = (\ker {}^tL_0)^\circ$ and $L_0\mathcal{D}'(\mathbb{T}^2) = \mathcal{D}'(\mathbb{T}^2)$.

By using partial Fourier series in the variable $t$, we may describe the functions $p = p(x)$ which belongs to $L_0\mathcal{C}^\infty(\mathbb{T}^2)$. Note that, if a distribution $\mu = \sum_{k\in\mathbb{Z}} \widehat{\mu}(x,k)\exp\{ikt\}$ belongs to $\ker {}^tL_0$, then the division theorem implies that

$$\widehat{\mu}(x,0) = \frac{C_0}{a} + \sum_{j=1}^{N}\sum_{\ell=0}^{n_j-1} C_{j\ell}\delta^{(\ell)}(x - x_j),$$

where $C_0$ and $C_{j\ell}$ are constants. Conversely, for any constants the distribution

$$\left(\frac{C_0}{a} + \sum_{j=1}^{N}\sum_{\ell=0}^{n_j-1} C_{j\ell}\delta^{(\ell)}(x - x_j)\right) \otimes 1_t$$

belongs to $\ker {}^tL_0$. Therefore, $p = p(x)$ belongs to $L_0\mathcal{C}^\infty(\mathbb{T}^2) = (\ker {}^tL_0)^\circ$ if and only if $p$ vanishes at each $x_j$, with order of vanishing greater than $n_j - 1$, for each $j = 1, \ldots, N$, and moreover, $\int_0^{2\pi}(p/a)(x)dx = 0$.

If $L_0$ is globally solvable on $\mathcal{C}^\infty(\mathbb{T}^2)$ (respectively $\mathcal{D}'(\mathbb{T}^2)$) and $p \in L_0\mathcal{C}^\infty(\mathbb{T}^2)$, then simple computations show that $L_p$ is still globally solvable on $\mathcal{C}^\infty(\mathbb{T}^2)$ (respectively $\mathcal{D}'(\mathbb{T}^2)$).

Also, according to [6], $L_{a'}$ is globally solvable on $\mathcal{C}^\infty(\mathbb{T}^2)$. Note that $a' \notin L_0\mathcal{C}^\infty(\mathbb{T}^2)$.

Our main result in this section will imply that, for any $p \in \mathcal{C}^\infty(\mathbb{T}^1_x)$, the operator $L_p$ is globally solvable on both $\mathcal{C}^\infty(\mathbb{T}^2)$ and $\mathcal{D}'(\mathbb{T}^2)$.

We first present a result concerning solvability modulo functions which are flat at $a^{-1}(0) \times \mathbb{T}^1$.

**Proposition 3.1.** *Let $L_p$ be given by (1.3). Suppose that $a$ vanishes only of finite order and $a^{-1}(0) \neq \emptyset$. Given $f \in (\ker {}^tL_p)^\circ$, there exists $u \in \mathcal{C}^\infty(\mathbb{T}^2)$ such that $L_pu - f$ is flat at $a^{-1}(0) \times \mathbb{T}^1$.*

**Proof.** By using cutoff functions we see that it is enough to work on a small neighborhood $(x_0-\epsilon, x_0+\epsilon)\times\mathbb{T}^1$, with $x_0 \in a^{-1}(0)$.

Given $f, u \in \mathcal{C}^\infty((x_0 - \epsilon, x_0 + \epsilon) \times \mathbb{T}^1)$, we use formal Taylor series to write

$$u(x,t) \simeq \sum_{j=0}^{\infty} u_j(t)(x - x_0)^j,\ u_j(t) = \frac{1}{j!}\partial_x^j u(x_0,t),$$

$$f(x,t) \simeq \sum_{j=0}^{\infty} f_j(t)(x - x_0)^j,\ f_j(t) = \frac{1}{j!}\partial_x^j f(x_0,t),$$

$$a(x) \simeq \sum_{j=0}^{\infty} a_j(x - x_0)^j,\ a_j = \frac{1}{j!}a^{(j)}(x_0),$$

and

$$p(x) \simeq \sum_{j=0}^{\infty} p_j(x-x_0)^j, \ p_j = \frac{1}{j!}p^{(j)}(x_0).$$

The formal Taylor series of $L_p u - f$ is

$$L_p u - f \simeq \sum_{j=0}^{\infty} \left( u_j' + \sum_{k=0}^{j} [(j+1-k)a_k u_{j+1-k} + p_k u_{j-k}] - f_j \right) (x-x_0)^j.$$

It follows that $L_p u - f$ is flat at $\{x_0\} \times \mathbb{T}^1$ if and only if

$$u_j' + \sum_{k=0}^{j} [(j+1-k)a_k u_{j+1-k} + p_k u_{j-k}] = f_j, \ j = 0, 1, 2, \dots. \tag{3.1}$$

After finding the sequence of solutions $(u_j)_{j\in\mathbb{Z}_+}$, the required function $u$ is obtained by using Borel's Lemma. Hence the proof reduces to solve the equations (3.1). In order to solve them (we want solutions in $\mathcal{C}^\infty(\mathbb{T}^1_t)$), we denote by $n$ ($1 \le n < \infty$) the order of vanishing of $a$ at $x_0$ and we split the proof into three cases:

**Case 1:** Assume that $n = 1$. Then the equations (3.1) reduce to

$$u_0' + p_0 u_0 = f_0, \tag{3.2}$$

$$u_1' + (a_1 + p_0)u_1 = f_1 - p_1 u_0, \tag{3.3}$$

$$u_j' + (ja_1 + p_0)u_j = f_j - \sum_{k=1}^{j} p_k u_{j-k} - \sum_{k=2}^{j} (j+1-k)a_k u_{j+1-k}, \tag{3.4}$$

for all $j > 1$.

All these equations may be solved, with unique solutions, if $p_0 \notin (-a_1)\mathbb{Z}_+ + i\mathbb{Z}$.

If $p_0 = -im$, $m \in \mathbb{Z}$, then $\delta(x-x_0) \otimes \exp\{-imt\}$ belongs to $\ker {}^tL_p$. Since $f \in (\ker {}^tL_p)^\circ$, it follows that $\widehat{f}_0(m) = (2\pi)^{-1}\langle \delta(x-x_0) \otimes \exp\{-imt\}, f\rangle = 0$. Hence equation (3.2) has a solution. All the other equations, given by (3.3) and (3.4), have unique solutions, since $a_1 \neq 0$.

If $p_0 = -a_1 - im$, ($m \in \mathbb{Z}$), then equation (3.2) has a unique solution $u_0$ and the next equation is $u_1' - imu_1 = f_1 - p_1 u_0$, which has solution if and only if $0 = \widehat{f}_1(m) - p_1\widehat{u_0}(m) = \widehat{f}_1(m) + p_1\widehat{f}_0(m)/a_1$. Since $[-a_1\delta'(x-x_0) + p_1\delta(x-x_0)] \otimes \exp\{-imt\} \in \ker {}^tL_p$, it follows that $a_1\widehat{f}_1(m) + p_1\widehat{f}_0(m) = 0$ and then we can find a solution $u_1$ to (3.3). The other equations, given by (3.4), have unique solutions $u_j$, $j \ge 2$.

In the case $p_0 = -2a_1 - im$, ($m \in \mathbb{Z}$), the first two equations have unique solutions $u_0$ and $u_1$, which satisfy

$$\widehat{u_0}(m) = -\widehat{f}_0(m)/2a_1 \text{ and } \widehat{u_1}(m) = -\frac{1}{a_1}\left(\widehat{f}_1(m) + \frac{p_1\widehat{f}_0(m)}{2a_1}\right).$$

The next equation is

$$u_2' + (p_0 + 2a_1)u_2 = f_2 - (p_1 + a_2)u_1 - p_2 u_0,$$

which has solutions if and only if

$$0 = \widehat{f}_2(m) + \frac{(p_1 + a_2)}{a_1}\widehat{f}_1(m) + \left(\frac{(p_1 + a_2)p_1}{2a_1^2} + \frac{p_2}{2a_1}\right)\widehat{f}_0(m). \tag{3.5}$$

Setting

$$\nu(x) = \frac{1}{2}\delta''(x - x_0) - \frac{(p_1 + a_2)}{a_1}\delta'(x - x_0) + \left(\frac{(p_1 + a_2)p_1}{2a_1^2} + \frac{p_2}{2a_1}\right)\delta(x - x_0),$$

direct computations show that $\nu(x) \otimes \exp\{-imt\} \in \ker {}^tL_p$. Since $f \in (\ker {}^tL_p)^\circ$, we obtain $0 = \langle \nu(x) \otimes \exp\{-imt\}, f\rangle$. This last identity is equivalent to (3.5). Hence we can find a solution $u_2$.

Again, all the next equations, given by (3.4), have unique solutions $u_j$, $j \geq 3$.

More general, assume that $p_0 = -j_0a_1 - im$ ($j_0 \in \mathbb{N}$, $j_0 \geq 2$, and $m \in \mathbb{Z}$). For $k = 1, \ldots, j_0$, we set

$$C_k = \sum_{\ell=1}^{k} \sum_{\substack{\alpha_\ell \in \mathbb{N}^\ell \\ |\alpha_\ell| = k}} \frac{\prod_{o=1}^{\ell}(p_{\alpha_{o\ell}} + (j_0 - \alpha_{1\ell} - \cdots - \alpha_{o\ell})a_{\alpha_{o\ell}+1})}{a_1^\ell \alpha_{1\ell}(\alpha_{1\ell} + \alpha_{2\ell}) \cdots (\alpha_{1\ell} + \cdots + \alpha_{\ell\ell})},$$

where $\alpha_\ell = (\alpha_{1\ell}, \ldots, \alpha_{\ell\ell})$ is a multi-index in $\mathbb{N}^\ell$ and $|\alpha_\ell| = \alpha_{1\ell} + \cdots + \alpha_{\ell\ell}$. Note that the equation

$$u'_{j_0} + (j_0a_1 + p_0)u_{j_0} = f_{j_0} - \sum_{k=1}^{j_0} p_k u_{j_0-k} - \sum_{k=2}^{j_0}(j_0 + 1 - k)a_k u_{j_0+1-k}$$

has a solution $u_{j_0}$ if and only if

$$0 = \widehat{f_{j_0}}(m) - \sum_{k=1}^{j_0} p_k \widehat{u_{j_0-k}}(m) - \sum_{k=2}^{j_0}(j_0 + 1 - k)a_k \widehat{u_{j_0+1-k}}(m), \tag{3.6}$$

while all the other equations, given by (3.2)–(3.4) (with $j \neq j_0$), have unique solutions. By using the previous equations (indices $j = 0, 1, \ldots, j_0 - 1$) we see that condition (3.6) is equivalent to

$$0 = \sum_{k=0}^{j_0} C_k \widehat{f_{j_0-k}}(m), \tag{3.7}$$

in which $C_0 \doteq 1$.

Define

$$\mu \doteq \left(\sum_{k=0}^{j_0} \tilde{C}_k \delta^{(j_0-k)}(x - x_0)\right) \otimes \exp\{-imt\},$$

with $\tilde{C}_k = (-1)^{j_0-k}\frac{C_k}{(j_0-k)!}$, $k = 0, 1, \ldots, j_0$.

If $\mu$ belongs to $\ker {}^tL_p$, then $0 = \langle \mu, f\rangle$, since $f \in (\ker {}^tL_p)^\circ$. On the other hand, by the above expression which defines $\mu$, we see that the condition $0 = \langle \mu, f\rangle$ is equivalent to (3.7). Hence the proof reduces to show that $\mu \in \ker {}^tL_p$.

By using induction we may verify the identity

$$ka_1C_k = \sum_{\ell=0}^{k-1}(p_{k-\ell} + (j_0 - k)a_{k-\ell+1})C_\ell, \ 1 \leq k \leq j_0. \tag{3.8}$$

The distribution $\mu$ belongs to $\ker {}^tL_p$ if and only if

$$\frac{d}{dx}\left(a(x)\sum_{k=0}^{j_0}\tilde{C}_k\delta^{(j_0-k)}(x-x_0)\right)-(im+p)\sum_{k=0}^{j_0}\tilde{C}_k\delta^{(j_0-k)}(x-x_0)=0.$$

Above identity is equivalent to

$$\begin{aligned}0=&\sum_{\ell=1}^{j_0}\left(\sum_{k=0}^{j_0-\ell}(-1)^{j_0-k+\ell-1}\tilde{C}_k\binom{j_0-k}{\ell-1}a^{(j_0-k-\ell+1)}(x_0)\right)\delta^{(\ell)}(x-x_0)\\&+\sum_{\ell=0}^{j_0}\left(\sum_{k=0}^{j_0-\ell}(-1)^{j_0-k+\ell-1}\tilde{C}_k\binom{j_0-k}{\ell}(im+p)^{(j_0-k-\ell)}(x_0)\right)\delta^{(\ell)}(x-x_0).\end{aligned}$$

Since the derivatives of $\delta(x-x_0)$ are linearly independent, it follows that the above identity becomes equivalent to (3.8).

Therefore, $\mu\in\ker {}^tL_p$.

**Case 2:** Assume that $1<n<\infty$ and $p_j=0$, for all $j\geq 1$. Then the equations (3.1) become

$$u_j'+p_0u_j=f_j,\ j=1,\dots,n-1, \tag{3.9}$$

$$u_j'+p_0u_j=f_j-\sum_{k=n}^{j}(j+1-k)a_ku_{j+1-k},\ j\geq n. \tag{3.10}$$

If $p_0\notin i\mathbb{Z}$, then each equation in either (3.9) or (3.10) has a unique solution. We then solve recursively these equations.

If $p_0=-im$, $m\in\mathbb{Z}$, then $\delta^{(j)}(x-x_0)\otimes\exp\{-imt\}\in\ker {}^tL_p$, $j=0,\dots,n-1$, and we may choose a solution $u_j$, which is not unique, to $u_j'+p_0u_j=f_j$. After this, we proceed to solve equations (3.10). In order to find a solution $u_n$ to

$$u_n'+p_0u_n=f_n-a_nu_1,$$

we must adjust the solution $u_1$ so that

$$\widehat{u_1}(m)=\frac{\widehat{f_j}(m)}{a_n}.$$

Similarly, to find a solution $u_j$ $(j>n)$ to (3.10), we must adjust $u_{j+1-n}$ setting

$$\widehat{u_{j+1-n}}(m)=\frac{1}{(j+1-n)a_n}\left(\widehat{f_j}(m)-\sum_{k=n+1}^{j}(j+1-k)a_k\widehat{u_{j+1-k}}(m)\right).$$

**Case 3:** Suppose now that $1<n<\infty$ and there exists $j\geq 1$ such that $p_j\neq 0$. Set $j_0=1$ if $p_1\neq 0$. In the other case, let $j_0$ be the smallest of the indices $1<j<\infty$ such that $p_j\neq 0$ and $p_\ell=0$, $\ell=1,\dots,j-1$.

When $j_0\geq n$, the equations (3.1) become

$$u_j'+p_0u_j=f_j,\ j=0,\dots,n-1, \tag{3.11}$$

$$u_j'+p_0u_j=f_j-\sum_{k=n}^{j}(j+1-k)a_ku_{j+1-k},\ n\leq j<j_0, \tag{3.12}$$

$$u'_j + p_0 u_j = f_j - \sum_{k=j_0}^{j} p_k u_{j-k} - \sum_{k=n}^{j} (j+1-k) a_k u_{j+1-k},\ j \geq j_0. \tag{3.13}$$

Notice that equations (3.12) do not appear when $j_0 = n$.

Applying arguments similar to those used above, we see that we may solve recursively equations (3.11), (3.12) and (3.13).

When $1 \leq j_0 < n$, the equations (3.1) reduce to

$$u'_j + p_0 u_j = f_j,\ j = 0, \dots, j_0 - 1, \tag{3.14}$$

$$u'_j + p_0 u_j = f_j - \sum_{k=j_0}^{j} p_k u_{j-k},\ j_0 \leq j < n, \tag{3.15}$$

$$u'_j + p_0 u_j = f_j - \sum_{k=j_0}^{j} p_k u_{j-k} - \sum_{k=n}^{j} (j+1-k) a_k u_{j+1-k},\ j \geq n. \tag{3.16}$$

Again, if either $j_0 < n-1$ or $p_0 \notin i\mathbb{Z}$, then we may solve recursively these equations without further difficulties.

When $j_0 = n-1$ and $p_0 = -im$, $m \in \mathbb{Z}$, then we do not find further difficulties to solve the equations provided that $p_{j_0} \notin -a_n \mathbb{N}$.

The last case to be checked is $j_0 = n-1$, $p_0 = -im$, $m \in \mathbb{Z}$, and $p_{j_0} = -\ell a_n$, $\ell \in \mathbb{N}$. For instance, if $\ell = 1$, then we may solve the first $j_0$ equations in (3.14), since $\delta^{(j)}(x - x_0) \otimes \exp\{-imt\}$ belongs to $\ker {}^t L_p$, for $j = 0, \dots, j_0 - 1$. Since $j_0 = n-1$, the next equation is $u'_{n-1} + p_0 u_{n-1} = f_{n-1} - p_{j_0} u_0$ (see (3.15)). In order to solve this equation, we must choose the solution $u_0$ such that $\widehat{u_0}(m) = \widehat{f_{n-1}}(m)/p_{j_0}$. The next equation is $u'_n + p_0 u_n = f_n - p_n u_0$ (see (3.16)). This equation has a solution if and only if $p_{j_0} \widehat{f_n}(m) - p_n \widehat{f_{n-1}}(m) = 0$. This last identity follows from the fact that the distribution

$$\left( \frac{p_{j_0}}{n!} \delta^{(n)}(x - x_0) + \frac{p_n}{(n-1)!} \delta^{(n-1)}(x - x_0) \right) \otimes \exp\{-imt\}$$

belongs to $\ker {}^t L_p$. Hence we have solutions to the equation $u'_n + p_0 u_n = f_n - p_n u_0$. The other equations to be solved are given by (3.16), with $j \geq n+1$. Each of these equations has a solution. Indeed, since $p_{j_0} + (j+1-n) a_n \neq 0$, for all $j \geq n+1$, we may choose the previous solution $u_{j-j_0}$ so that

$$\widehat{f_j}(m) - \sum_{k=j_0}^{j} p_k \widehat{u_{j-k}}(m) - \sum_{k=n}^{j} (j+1-k) a_k \widehat{u_{j+1-k}}(m) = 0.$$

Notice that, to find solutions $u_{n+1}, u_{n+2}, \dots$, we must adjust $u_2, u_3, \dots$.

Suppose now that $j_0 = n-1$, $p_0 = -im$, $m \in \mathbb{Z}$, and $p_{j_0} = -\ell a_n$, $\ell \in \mathbb{N}$, $\ell \geq 2$. Again, we solve the first $j_0$ equations in (3.14), since $\delta^{(j)}(x - x_0) \otimes \exp\{-imt\}$ belongs to $\ker {}^t L_p$, for $j = 0, \dots, j_0 - 1$. The next equation is $u'_{n-1} + p_0 u_{n-1} = f_{n-1} - p_{j_0} u_0$, which we may solve by choosing the solution $u_0$ such that $\widehat{u_0}(m) = \widehat{f_{n-1}}(m)/p_{j_0}$. Similarly, we may recursively find solutions $u_n, \dots, u_{n+\ell-2}$, to the respective equation in (3.16), provided that we adjust the previous solutions $u_1, \dots, u_{\ell-1}$ by choosing them so that

$$\widehat{u_k}(m) = \frac{1}{p_{j_0} + k a_n} \left( \widehat{f_{j_0+k}}(m) - \sum_{j=1}^{k} [p_{j_0+j} + (k-j) a_{j_0+j+1}] \widehat{u_{k-j}}(m) \right),$$

$k = 1, \dots, \ell - 1$. The next equation is

$$u'_{n+\ell-1} + p_0 u_{n+\ell-1} = f_{n+\ell-1} - (p_{j_0+1} + (\ell-1)a_{j_0+2})u_{\ell-1} - \cdots - (p_{j_0+\ell-1} + a_{j_0+\ell})u_1 - p_{j_0+\ell}u_0.$$

This equation has solutions if and only if

$$0 = \widehat{f_{j_0+\ell}}(m) - \sum_{k=0}^{\ell-1}(p_{j_0+\ell-k} + ka_{j_0+\ell-k+1})\widehat{u_k}(m). \tag{3.17}$$

If there exists a solution $u_{n+\ell-1}$, then all the other equations given by (3.16) (with $j \geq n+\ell$) may be solved recursively by adjusting previous solutions, with a procedure analogous to the one which gives the solutions $u_n, \ldots, u_{n+\ell-2}$.

Hence the proof reduces to verify (3.17).

Define $F_0 = 1$ and for $k = 1, \ldots, \ell$, define

$$F_k = \sum_{j=1}^{k}(-1)^j \sum_{\substack{\alpha_j \in \mathbb{N}^j \\ |\alpha_j|=k}} \frac{\prod_{o=1}^{j}(p_{j_0+\alpha_{oj}} + (\ell - \alpha_{1j} - \cdots - \alpha_{oj})a_{j_0+\alpha_{oj}+1})}{\prod_{o=1}^{j}(p_{j_0} + (\ell - \alpha_{1j} - \cdots - \alpha_{oj})a_n)},$$

in which $\alpha_j = (\alpha_{1j}, \ldots, \alpha_{jj}) \in \mathbb{N}^j$ and $|\alpha_j| = \alpha_{1j} + \cdots + \alpha_{jj}$.

An induction process shows that

$$-(p_{j_0} + (\ell-k)a_n)F_k = \sum_{j=0}^{k-1}(p_{j_0+k-j} + (\ell-k)a_{j_0+k-j+1})F_j, \tag{3.18}$$

for $k = 1, \ldots, \ell$.

With notation above, (3.17) is equivalent to

$$0 = \sum_{k=0}^{\ell} F_k \widehat{f_{j_0+\ell-k}}(m).$$

Since $f \in (\ker {}^tL_p)^\circ$, the proof will be completed by showing that the distribution

$$\left(\sum_{k=0}^{\ell} \tilde{F}_k \delta^{(j_0+\ell-k)}(x-x_0)\right) \otimes \exp\{-imt\}$$

belongs to $\ker {}^tL_p$, in which $\tilde{F}_k = (-1)^{j_0+\ell-k}\frac{F_k}{(j_0+\ell-k)!}$, $k = 0, \ldots, \ell$.

By the expression of ${}^tL_p$, it follows that this distribution belongs to $\ker {}^tL_p$ if and only if

$$\frac{d}{dx}\left(a\sum_{k=0}^{\ell} \tilde{F}_k \delta^{(j_0+\ell-k)}(x-x_0)\right) - (im+p)\sum_{k=0}^{\ell} \tilde{F}_k \delta^{(j_0+\ell-k)}(x-x_0) = 0. \tag{3.19}$$

By using the linear dependence of the derivatives of $\delta(x-x_0)$, we may see that identity (3.19) is equivalent to the following identities

$$0 = \sum_{k=0}^{\ell} \tilde{F}_k \binom{j_0+\ell}{0}(-1)^{j_0+\ell-k}(im+p)^{(j_0+\ell-k)}(x_0),$$

$$0=\sum_{k=0}^{\ell-\omega}\tilde{F}_k(-1)^{j_0+\ell-k+\omega-1}\binom{j_0+\ell-k}{\omega-1}a^{(j_0+\ell-k-\omega+1)}(x_0)$$
$$+\sum_{k=0}^{\ell-\omega}\tilde{F}_k(-1)^{j_0+\ell-k+\omega-1}\binom{j_0+\ell-k}{\omega}(im+p)^{(j_0+\ell-k-\omega)}(x_0),$$

$\omega=1,\dots,\ell$.

Finally, we use (3.18) in order to verify the validity of the above identities.

This completes the proof. □

**Remark 3.2.** Given $f\in(\ker{}^tL_p)^\circ$, by Proposition 3.1 it follows that there exists $u\in\mathcal{C}^\infty(\mathbb{T}^2)$ such that $L_pu-f$ is flat at $a^{-1}(0)\times\mathbb{T}^1$. Notice that $L_pu-f\in(\ker{}^tL_p)^\circ$. Hence, if there exists $v\in\mathcal{C}^\infty(\mathbb{T}^2)$ such that $L_pv=L_pu-f$, then $L_p(u-v)=f$. In other words, Proposition 3.1 allows us to assume that the right-hand side of the equation $L_pu=f$ is flat at $a^{-1}(0)\times\mathbb{T}^1$.

Before solving the equation $L_pu=f$ for $f$ flat, we need a technical result, which will be used to produce certain distributions in the kernel of the transpose operator.

**Lemma 3.3.** *Let $a\in\mathcal{C}^\infty(\mathbb{T}^1,\mathbb{R})$ and assume that $a$ vanishes of finite order $n\geq1$ at $x_1$. Let $k\in\mathbb{Z}$, $m_0\in\mathbb{Z}_+$ and $p\in\mathcal{C}^\infty(\mathbb{T}^1)$. Denote by $q$ the order of vanishing of $p+ik$ at $x_1$ ($q=0$ if $p(x_1)+ik\neq0$). Suppose that $0\leq q\leq n-1$ and whenever $q=n-1$ we have $(\Im p+k)^{(q)}(x_1)\neq0$ or $\Re p^{(q)}(x_1)+(\ell/(q+1))a^{(n)}(x_1)\neq0$, for all $\ell=0,\dots,m_0$.*

*Under the above assumptions, given*

$$\nu=\sum_{\ell=0}^{m_0}c_\ell\delta^{(\ell)}(x-x_1),\ c_\ell\in\mathbb{C},$$

*there exists*

$$\mu=\sum_{\ell=q}^{m_0+q}d_\ell\delta^{(\ell)}(x-x_1),\ d_\ell\in\mathbb{C},$$

*such that $-\partial_x(a\mu)+(p+ik)\mu=\nu$.*

**Proof.** Given

$$\mu=\sum_{\ell=q}^{m_0+q}d_\ell\delta^{(\ell)}(x-x_1),$$

we may write

$$-\partial_x(a\mu)+(p+ik)\mu=\sum_{\ell=0}^{m_0}D_\ell\delta^{(\ell)}(x-x_1),$$

in which the constants $D_\ell$ are given as follows: denoting $C_{j,\ell}=(-1)^j\binom{\ell}{j}$, we have

$$D_0=\sum_{j=q}^{m_0+q}C_{j,j}d_j(p+ik)^{(j)}(x_1),$$

and (when $m_0 > 0$)

$$D_\ell = \sum_{j=\ell+q}^{m_0+q} d_j [C_{j-\ell,j}(p+ik)^{(j-\ell)}(x_1) - C_{j-\ell+1,j}a^{(j-\ell+1)}(x_1)],$$

for $\ell = 1, \dots, m_0$.

In order to have $-\partial_x(a\mu) + (p+ik)\mu = \nu$, we must choose $d_\ell$ so that $D_\ell = c_\ell$, for $\ell = 0, \dots, m_0$.

If $m_0 = 0$, then it is enough to choose

$$d_q = \frac{c_0}{(-1)^q (p+ik)^{(q)}(x_1)}.$$

If $m_0 = 1$, then it is enough to choose

$$d_{q+1} = \frac{c_1}{C_{q,q+1}(p+ik)^q(x_1) - C_{q+1,q+1}a^{(q+1)}(x_1)}$$

and

$$d_q = \frac{c_0 - C_{q+1,q+1}d_{q+1}(p+ik)^{(q+1)}(x_1)}{C_{q,q}(p+ik)^{(q)}(x_1)}.$$

Finally, if $m_0 \geq 2$, then we recursively choose

$$d_{m_0+q} = \frac{c_{m_0}}{C_{q,m_0+q}(p+ik)^{(q)}(x_1) - C_{q+1,m_0+q}a^{(q+1)}(x_1)},$$

$$d_{\ell+q} = \frac{c_\ell}{C_{q,\ell+q}(p+ik)^{(q)}(x_1) - C_{q+1,\ell+q}a^{(q+1)}(x_1)} + \sum_{j=\ell+q+1}^{m_0+q} \frac{d_j[C_{j-\ell+1,j}a^{(j-\ell+1)}(x_1) - C_{j-\ell,j}(p+ik)^{(j-\ell)}(x_1)]}{C_{q,\ell+q}(p+ik)^{(q)}(x_1) - C_{q+1,\ell+q}a^{(q+1)}(x_1)},$$

for $\ell = m_0 - 1, m_0 - 2, \dots, 1$, and

$$d_q = \frac{c_0 - \sum_{j=q+1}^{m_0+q} C_{j,j}d_j(p+ik)^{(j)}(x_1)}{(-1)^q(p+ik)^{(q)}(x_1)}. \quad \square$$

We are now in position to state and prove our main result.

**Theorem 3.4.** *If $a^{-1}(0) \neq \emptyset$ and $a$ vanishes only of finite order, then the operator $L_p$ given by (1.3) is globally solvable on $\mathcal{C}^\infty(\mathbb{T}^2)$.*

**Proof.** Write $a^{-1}(0) = \{x_1 < \cdots < x_N\}$, $x_{N+1} = x_1 + 2\pi$, and denote by $n_j$ the order of vanishing of $a$ at $x_j$, $j = 1, \dots, N$.

Given $f \in (\ker {}^tL_p)^\circ$, we must show that there exists $u$ such that $L_p u = f$. By Proposition 3.1, we may assume that $f$ is flat at $a^{-1}(0) \times \mathbb{T}^1$ (see Remark 3.2).

By using partial Fourier series in the variable $t$, we are led to the equations

$$a(x)\partial_x \widehat{u}(x,k) + (ik + p(x))\widehat{u}(x,k) = \widehat{f}(x,k),\ x \in \mathbb{T}^1,\ k \in \mathbb{Z}. \tag{3.20}$$

We will first solve these equations and after we will show that the sequence of solutions decays rapidly. Suppose that $ik + p$ vanishes at $a^{-1}(0)$ as much as $a$.

If the mean $((ik+p)/a)_0 = (2\pi)^{-1}\int_0^{2\pi}((ik+p)/a)(x)dx$ belongs to $i\mathbb{Z}$, then a solution to (3.20) is given by

$$\widehat{u}(x,k) = \exp\left\{-\int_0^x \frac{ik+p}{a}(s)ds\right\}\int_0^x \frac{\widehat{f}(y,k)}{a(y)}\exp\left\{\int_0^y \frac{ik+p}{a}(s)ds\right\}dy.$$

In order to see that $\widehat{u}(x,k) \in \mathcal{C}^\infty(\mathbb{T}^1)$, it is enough to show that

$$\int_0^x \frac{\widehat{f}(y,k)}{a(y)}\exp\left\{\int_0^y \frac{ik+p}{a}(s)ds\right\}dy = 0.$$

This last identity follows from the following two facts: $f \in (\ker {}^tL_p)^\circ$ and

$$\left[\exp\left\{\int_0^x ((ik+p)/a)(s)ds\right\}/a(x)\right] \otimes \exp\{-ikt\} \in \ker {}^tL_p \subset \mathcal{D}'(\mathbb{T}^1).$$

If the mean $((ik+p)/a)_0$ does not belong to $i\mathbb{Z}$, then the unique solution to (3.20) is given by

$$\widehat{u}(x,k) = C\exp\left\{-\int_0^x \frac{ik+p}{a}(s)ds\right\} + \int_0^x \frac{\widehat{f}(y,k)}{a(y)}\exp\left\{-\int_y^x \frac{ik+p}{a}(s)ds\right\}dy,$$

where

$$C = \frac{1}{\left(\exp\left\{\int_0^{2\pi}\frac{ik+p}{a}(s)ds\right\}-1\right)}\int_0^{2\pi}\frac{\widehat{f}(y,k)}{a(y)}\exp\left\{\int_0^y \frac{ik+p}{a}(s)ds\right\}dy.$$

Note that $ik+p$ does not vanish, for $|k|$ large enough. Hence, there is no need to discuss the rapid decaying of the solutions given above.

Suppose now that $ik+p$ vanishes less than $a$ at some point in $a^{-1}(0) = \{x_1 < \ldots < x_N\}$.

In this case, in order to solve the equations (3.20), it is enough to work on intervals of the form $(x_j, x_{j+\sigma})$, in which $ik+p(x)$ vanishes less than $a$ on both $x_j$ and $x_{j+\sigma}$. In addition, either $\sigma = 1$ or $\sigma > 1$ and $ik+p(x)$ vanishes as much as $a$ at the intermediate zeros $x_{j+1},\ldots,x_{j+\sigma-1}$. Here $\sigma \le N$ and we set $x_{N+1} = x_1+2\pi, x_{N+2} = x_2+2\pi,\ldots,x_{N+j} = x_j+2\pi$.

On an interval $(x_j, x_{j+\sigma})$ as described above, we have

$$\partial_x\widehat{u}(x,k) + \frac{ik+p(x)}{a(x)}\widehat{u}(x,k) = \frac{\widehat{f}(x,k)}{a(x)}, \tag{3.21}$$

in which the right-hand side is smooth on $[x_j, x_{j+\sigma}]$ and flat at the points $x_j,\ldots,x_{j+\sigma}$. We will seek a solution $\widehat{u}(x,k)$ to (3.21) which is smooth on $(x_j, x_{j+\sigma})$ and flat at $\{x_j, x_{j+\sigma}\}$, so that we may join this solution with solutions on other intervals with the same type, in order to obtain a solution to (3.20) on $\mathbb{T}^1$.

In addition, if the sequence $\{\widehat{u}(x,k)\}_{k\in\mathbb{Z}}$ decays rapidly on each $[x_j, x_{j+\sigma}]$, then it decays rapidly on $\mathbb{T}^1$ and

$$u(x,t) = \sum_{k\in\mathbb{Z}}\widehat{u}(x,k)\exp\{ikt\}$$

will be a solution to $L_p u = f$.

By the comments above, it is enough to work only on intervals of the type $(x_j, x_{j+\sigma})$.

We now split the construction of the solutions $\widehat{u}(x,k)$ on $(x_j, x_{j+\sigma})$ into five cases:

**Case 1:** $\Re p$ vanishes as much as $a$ at both $x_j$ and $x_{j+\sigma}$, and, consequently, $\Im p + k$ either does not vanish or it vanishes less than $a$ at both $x_j$ and $x_{j+\sigma}$.

In this case, for $\eta \in (x_j, x_{j+\sigma})$, the functions

$$E_k(x) = \exp\left\{-\int_{\eta}^{x} \frac{\Re p(s) + i(k + \Im p(s))}{a(s)} ds\right\}$$

and

$$1/E_k(x) = \exp\left\{\int_{\eta}^{x} \frac{\Re p(s) + i(k + \Im p(s))}{a(s)} ds\right\}$$

are bounded on $(x_j, x_{j+\sigma})$. Hence, the expression

$$\widehat{u}(x,k) = E_k(x) \int_{x_j}^{x} \frac{\widehat{f}(y,k)}{a(y)E_k(y)} dy \tag{3.22}$$

defines a smooth function on $(x_j, x_{j+\sigma})$ which is flat at $x_j$ and solve (3.21) on $[x_j, x_{j+\sigma})$.

The next step is to show that $\widehat{u}(x,k)$ is also flat at $x_{j+\sigma}$. This is obtained by showing that

$$\begin{aligned} 0 &= \int_{x_j}^{x_{j+\sigma}} \frac{\widehat{f}(y,k)}{a(y)E_k(y)} dy \\ &= \int_{x_j}^{x_{j+\sigma}} \frac{\widehat{f}(y,k)}{a(y)} \exp\left\{\int_{\eta}^{y} \frac{ik + p(s)}{a(s)} ds\right\} dy. \end{aligned}$$

As in [6], we will see that the above identity follows from the existence of certain distributions in $\ker {}^t L_p$.

Notice that

$$\psi_k(x) = \begin{cases} 1/E_k(x), & x \in (x_j, x_{j+\sigma}) \\ 0, & x \in \mathbb{T}^1 \setminus (x_j, x_{j+\sigma}) \end{cases}$$

belongs to $\mathcal{D}'(\mathbb{T}^1)$, since $1/E_k(x)$ is bounded. By the division theorem, there exists $\psi_k/a \in \mathcal{D}'(\mathbb{T}^1)$, whose order is at most $m$, the maximum of the order of vanishing of $a$ on $[x_j, x_{j+\sigma}]$. Hence $\omega_k = -\partial_x(\psi_k) + (p + ik)(\psi_k/a)$ is a distribution of order at most $m+1$. In addition, $\operatorname{supp} \omega_k \subset \{x_j, x_{j+\sigma}\}$. It follows that

$$\omega_k = \sum_{\ell=0}^{m+1} c_{1\ell}\delta^{(\ell)}(x - x_j) + \sum_{\ell=0}^{m+1} c_{2\ell}\delta^{(\ell)}(x - x_{j+\sigma}),$$

where $c_{1\ell}$ and $c_{2\ell}$ are constants. By Lemma 3.3, there is

$$\nu_k = \sum_{\ell=0}^{m+1+q_j} d_{1\ell}\delta^{(\ell)}(x - x_j) + \sum_{\ell=0}^{m+1+q_{j+\sigma}} d_{2\ell}\delta^{(\ell)}(x - x_{j+\sigma})$$

(in which $q_\ell$ is the order of vanishing of $\Im p + k$ at $x_\ell$, $\ell = j, j+\sigma$), such that $\omega_k = -\partial_x(a\nu_k) + (p+ik)\nu_k$. Hence $[(\psi_k/a) - \nu_k] \otimes \exp\{-ikt\}$ belongs to $\ker {}^tL_p$. Consequently,

$$0 = \langle (\psi_k/a) - \nu_k, \widehat{f}(\cdot,k)\rangle = \int_{x_j}^{x_{j+\sigma}} \frac{\widehat{f}(y,k)}{a(y)} \exp\left\{ \int_{\eta}^{y} \frac{ik + p(s)}{a(s)} ds \right\} dy,$$

since $f \in (\ker {}^tL_p)^\circ$ and $f$ is flat at $\{x_j, x_{j+\sigma}\}$.

Therefore, $\widehat{u}(x,k)$, given by (3.22), is a solution to (3.21), which is smooth on $[x_j, x_{j+\sigma}]$ and flat at the extremes.

**Case 2:** Suppose that the order of vanishing of $a$ is greater than the order of vanishing of $\Re p$ plus one, at both $x_j$ and $x_{j+\sigma}$. In particular, $\Re p$ does not vanish or it vanishes of finite order.

If $(\Re p/a)(x) > 0$ on small intervals $(x_j, x_j + \epsilon)$ and $(x_{j+\sigma} - \epsilon, x_{j+\sigma})$, we set

$$\begin{aligned}\widehat{u}(x,k) =& E_k(x) \int_{x_j}^{x} \frac{\widehat{f}(y,k)}{a(y)E_k(y)} dy \\ =& \int_{x_j}^{x} \frac{\widehat{f}(y,k)}{a(y)} \exp\left\{ -\int_{y}^{x} \frac{p+ik}{a}(s) \right\} dy, \;\; x \in (x_j, x_{j+\sigma}).\end{aligned}$$

Notice that $\widehat{u}(\cdot,k)$ is well defined on $(x_j, x_{j+\sigma})$, since $(\Re p/a)(x) > 0$ on $(x_j, x_j+\epsilon)$. In addition, $\widehat{u}(\cdot,k) \in \mathcal{C}^\infty([x_j, x_{j+\sigma}))$ and it is flat at $x_j$. We now proceed to show that $\widehat{u}(\cdot,k)$ is also flat at $x_{j+\sigma}$. It is enough to show that $\widehat{u}(x_{j+\sigma}+h,k) = O(|h|^n)$, for all $n \in \mathbb{Z}_+$. For $h < 0$ sufficiently small, we have

$$\begin{aligned}\widehat{u}(x_{j+\sigma}+h,k) = & \int_{x_j}^{x_{j+\sigma}+2h} \frac{\widehat{f}(y,k)}{a(y)} \exp\left\{ -\int_{y}^{x_{j+\sigma}+h} \frac{p+ik}{a}(s) \right\} dy \\ & + \int_{x_{j+\sigma}+2h}^{x_{j+\sigma}+h} \frac{\widehat{f}(y,k)}{a(y)} \exp\left\{ -\int_{y}^{x_{j+\sigma}+h} \frac{p+ik}{a}(s) \right\} dy \\ & \doteq I_1 + I_2.\end{aligned}$$

We have

$$|I_1| \le C_h \left\| \frac{f}{a} \right\|_\infty \int_{x_j}^{x_{j+\sigma}+2h} \exp\left\{ -\int_{x_{j+\sigma}+2h}^{x_{j+\sigma}+h} \frac{\Re p}{a}(s) \right\} dy,$$

in which

$$C_h = \sup_{x_j \le y \le x_{j+\sigma}+2h} \left( \exp\left\{ -\int_{y}^{x_{j+\sigma}+2h} \frac{\Re p}{a}(s) \right\} \right);$$

notice that $C_h$ is a positive constant that does not depend on $k$.

Since $(\Re p/a)(x) > 0$ on $(x_{j+\sigma} - \epsilon, x_{j+\sigma})$, it follows that the constants $C_h$ are bounded. In addition, on $(x_{j+\sigma} - \epsilon, x_{j+\sigma})$ we may write $(\Re p/a)(s) = (x_{j+\sigma} - s)^{-\rho}\beta(s)$, in which $\rho \geq 2$ and $0 < r \leq \beta(s) \leq M$. We then obtain

$$\left| \int_{x_j}^{x_{j+\sigma}+2h} \frac{\widehat{f}(y,k)}{a(y)} \exp\left\{ -\int_{y}^{x_{j+\sigma}+h} \frac{p+ik}{a}(s) \right\} dy \right| \leq C \left\| \frac{f}{a} \right\|_{\infty} |x_{j+\sigma} - x_j| \exp\left\{ \frac{-r}{\rho - 1} \left( \frac{2^{\rho-1} - 1}{2^{\rho}|h|^{\rho-1}} \right) \right\} = O(|h|^n),$$

for all $n \in \mathbb{Z}_+$.

We can estimate $|I_2|$ similarly, since $f$ is flat at $x_{j+\sigma}$.

In the sequel we treat the other possibilities to the sign of $\Re p/a$.

If $(\Re p/a)(x) < 0$ on small intervals $(x_j, x_j + \epsilon)$ and $(x_{j+\sigma} - \epsilon, x_{j+\sigma})$, we pick $\eta \in (x_j, x_{j+\sigma})$ and we set

$$\widehat{u}(x,k) = -\int_{x}^{x_{j+\sigma}} \frac{\widehat{f}(y,k)}{a(y)} \exp\left\{ -\int_{y}^{x} \frac{p+ik}{a}(s) \right\} dy, \ x \in (x_j, x_{j+\sigma}). \tag{3.23}$$

As before, this solution $\widehat{u}(\cdot,k)$ is smooth on $[x_j, x_{j+\sigma}]$ and it is flat at the extremes.

If $(\Re p/a)(x) < 0$ on a small interval $(x_j, x_j + \epsilon)$ and $(\Re p/a)(x) > 0$ on $(x_{j+\sigma} - \epsilon, x_{j+\sigma})$, then we set

$$\widehat{u}(x,k) = \int_{\eta}^{x} \frac{\widehat{f}(y,k)}{a(y)} \exp\left\{ -\int_{y}^{x} \frac{p+ik}{a}(s) \right\} dy, \ x \in (x_j, x_{j+\sigma}).$$

Proceeding as in the first situation treated in this Case 2, we may verify that this solution $\widehat{u}(\cdot,k)$ is smooth on $[x_j, x_{j+\sigma}]$ and it is flat at the extremes.

If $(\Re p/a)(x) > 0$ on a small interval $(x_j, x_j + \epsilon)$ and $(\Re p/a)(x) < 0$ on $(x_{j+\sigma} - \epsilon, x_{j+\sigma})$, then we set

$$\begin{aligned} \widehat{u}(x,k) =& E_k(x) \int_{x_j}^{x} \frac{\widehat{f}(y,k)}{a(y)E_k(y)} dy \\ =& \int_{x_j}^{x} \frac{\widehat{f}(y,k)}{a(y)} \exp\left\{ -\int_{y}^{x} \frac{p+ik}{a}(s) \right\} dy, \ x \in (x_j, x_{j+\sigma}). \end{aligned}$$

As before, the second identity above may be used in order to show that this solution $\widehat{u}(\cdot,k)$ is smooth on $[x_j, x_{j+\sigma})$ and it is flat at $x_j$. Moreover, proceeding as in the Case 1, via Lemma 3.3 we see that this solution is also flat at $x_{j+\sigma}$.

**Case 3:** The order of vanishing of $\Re p$ is equal to the order of vanishing of $a'$, at both $x_j$ and $x_{j+\sigma}$.

In this case, we pick $\phi(x) = [1 - \cos(x - x_j)]^M [1 - \cos(x - x_{j+\sigma})]^M$, $M \in \mathbb{N}$, which vanishes only at $\{x_j, x_{j+\sigma}\}$, with finite order of vanishing. Picking $M$ large enough, it follows that $\phi(x)/E_k(x)$ is bounded on $(x_j, x_{j+\sigma})$. Hence,

$$\psi_k(x) = \begin{cases} \phi(x)/E_k(x), & x \in (x_j, x_{j+\sigma}) \\ 0, & x \in \mathbb{T}^1 \setminus (x_j, x_{j+\sigma}) \end{cases}$$

belongs to $\mathcal{D}'(\mathbb{T}^1)$. By the division theorem, there exists $\psi_k/(a\phi) \in \mathcal{D}'(\mathbb{T}^1)$, whose order is at most $2M+m$, in which $m$ is the maximum of the order of vanishing of $a$ on $[x_j, x_{j+\sigma}]$. Hence

$$\omega_k = -\partial_x(a\psi_k/(a\phi)) + (p+ik)(\psi_k/(a\phi))$$

is a distribution of order at most $2M+m+1$, which is equal to

$$-\partial_x(1/E_k) + (p+ik)(1/(E_k a)) = 0 \text{ on } (x_j, x_{j+\sigma}).$$

Proceeding as in the Case 1, we want to apply Lemma 3.3 in order to yield the existence of $\nu_k \in \mathcal{D}'(\mathbb{T}^1)$, which is a linear combination of derivatives of $\delta(x-x_j)$ and $\delta(x-x_{j+\sigma})$, such that $\omega_k = -\partial_x(a\nu_k)+(p+ik)\nu_k$. This is possible assuming that two additional properties occur:

- if $\Im p+k$ vanishes at $x_j$ of order at least $n_j$, then $\Re p^{(n_j-1)}(x_j) + (\ell/n_j)a^{(n_j)}(x_j) \neq 0$, for $\ell = 1, ..., 2M+m+1$.
- if $\Im p+k$ vanishes at $x_{j+\sigma}$ of order at least $n_{j+\sigma}$, then $\Re p^{(n_{j+\sigma}-1)}(x_{j+\sigma}) + (\ell/n_{j+\sigma})a^{(n_{j+\sigma})}(x_{j+\sigma}) \neq 0$, for $\ell = 1, ..., 2M+m+1$.

Before proceeding, we mention that the situation in which the extremes do not satisfy the above properties will be treated in Case 5.

Under the assumptions made in this Case 3, we have

$$\begin{aligned} 0 =& \langle(\psi_k/(a\phi)) - \nu_k, \widehat{f}(\cdot,k)\rangle \\ =& \langle\psi_k, [\widehat{f}(\cdot,k)/(a\phi)]\rangle \\ =& \int_{x_j}^{x_{j+\sigma}} \frac{\widehat{f}(y,k)}{a(y)} \exp\left\{ \int_{\eta}^{y} \frac{ik+p(s)}{a(s)} ds \right\} dy. \end{aligned} \tag{3.24}$$

Therefore, the expression

$$\widehat{u}(x,k) = E_k(x) \int_{x_j}^{x} \frac{\widehat{f}(y,k)}{a(y)E_k(y)} dy, \;\; x \in (x_j, x_{j+\sigma})$$

defines a solution to (3.21), which is flat at $\{x_j, x_{j+\sigma}\}$. Indeed, since $\phi(x)/E_k(x)$ is bounded on $(x_j, x_{j+\sigma})$ and $\widehat{f}(x,k)$ is flat at $x_j$, it follows that

$$\lim_{x\to x_j^+} \frac{\widehat{f}(x,k)}{a(x)E_k(x)} = \lim_{x\to x_j^+} \frac{\widehat{f}(x,k)}{a(x)\phi(x)} \frac{\phi(x)}{E_k(x)} = 0.$$

A similar procedure shows that all the right-hand side derivatives of the function $\widehat{f}(x,k)/[a(x)E_k(x)]$ are zero at $x_j$. Hence,

$$x \mapsto \int_{x_j}^{x} \frac{\widehat{f}(y,k)}{a(y)E_k(y)} dy$$

is well defined, smooth on $[x_j, x_{j+\sigma})$ and flat at $x_j$. Similarly, it follows that $\widehat{u}(x,k)$ is well defined, smooth on $[x_j, x_{j+\sigma})$ and flat at $x_j$. Finally, by using that $\widehat{f}(x,k)$ is flat at $x_{j+\sigma}$ and using (3.24) we see that $\widehat{u}(x,k)$ is also flat at $x_{j+\sigma}$.

**Case 4:** Suppose $x_j$ as in Case 1 (or 3) and $x_{j+\sigma}$ as in Case 2. We then define $\widehat{u}(x,k)$ on $(x_j, x_{j+\sigma})$ by using formula (3.22). If $(\Re p/a) > 0$ on a small interval $(x_{j+\sigma} - \epsilon, x_{j+\sigma})$, then we proceed as in Case 1 (or as in the end of Case 3), to see that this solution is flat at $x_j$, and we proceed as in the first part of the Case 2 in order to show that this solution is flat at $x_{j+\sigma}$. On the other hand, if $(\Re p/a) < 0$ on a small interval $(x_{j+\sigma} - \epsilon, x_{j+\sigma})$, then we argue as in Case 1 (or 3), via Lemma 3.3, in order to verify that we may rewrite $\widehat{u}(x,k)$ as in (3.23), from which follows that this solution is flat at $\{x_j, x_{j+\sigma}\}$.

A similar procedure constructs a solution $\widehat{u}(x,k)$ when $x_j$ is as in Case 2 and $x_{j+\sigma}$ is as in Case 1 (or 3). Indeed, it is enough to consider $\widehat{u}(x,k)$ given by (3.23).

When $x_j$ is as in Case 1 and $x_{j+\sigma}$ is as in Case 3 or $x_j$ is as in Case 3 and $x_{j+\sigma}$ is as in Case 1, then we proceed as in Case 3 in order to present a smooth solution $\widehat{u}(x,k)$ on $(x_j, x_{j+\sigma})$ which is flat at $\{x_j, x_{j+\sigma}\}$.

**Case 5:** The order of vanishing of $\Re p$ is equal to the order of vanishing of $a'$ at some $x_j$, to say $x_{j_1}$. Moreover, $\Im p + k$ vanishes at $x_{j_1}$ of order at least $n_{j_1}$, and there exists $\ell_1 \in \mathbb{N}$ such that $\Re p^{(n_{j_1}-1)}(x_{j_1}) + (\ell_1/n_{j_1})a^{(n_{j_1})}(x_{j_1}) = 0$. In this case, we look to a previous interval of the type $(x_{j_0}, x_{j_0+\sigma_0})$, such that $x_{j_0} < x_{j_1}$ and $x_{j_0+\sigma_0} = x_{j_1}$. The choice of a solution to (3.20) on $(x_{j_1}, x_{j_1+\sigma_1})$ must be adjusted depending on the fixed solution to (3.20) on $(x_{j_0}, x_{j_0+\sigma_0})$. This approach leads us to the following two situations:

$(i)$ solving (3.20) on a chain of intervals

$$(x_{j_0}, x_{j_0+\sigma_0}] \cup [x_{j_1}, x_{j_1+\sigma_1}) \cup \cdots \cup [x_{j_n}, x_{j_n+\sigma_n}),$$

in which $1 \leq n \leq N$, and $x_{j_m+\sigma_m} = x_{j_{m+1}}$, for $m = 0, \dots, n$; in addition, the extremes $x_{j_0}$ and $x_{j_n+\sigma_n}$ satisfy the assumptions in Lemma 3.3, that is, they are of the types treated in Cases 1–4. For $m = 1, \dots, n$, the order of vanishing of $\Re p$ is equal to the order of vanishing of $a'$ at $x_{j_m}$. Moreover, $\Im p + k$ vanishes at $x_{j_m}$ of order at least $n_{j_m}$, and there exists $\ell_m \in \mathbb{N}$ such that $\Re p^{(n_{j_m}-1)}(x_{j_m}) + (\ell_m/n_{j_m})a^{(n_{j_m})}(x_{j_m}) = 0$.

$(ii)$ solving (3.20) on a chain of intervals

$$(x_{j_0}, x_{j_0+\sigma_0}] \cup [x_{j_1}, x_{j_1+\sigma_1}) \cup \cdots \cup [x_{j_n}, x_{j_n+\sigma_n}),$$

as in the previous item. However, now $x_{j_0} + 2\pi = x_{j_n+\sigma_n}$ and for all $m = 0, \dots, n, n+1$, the order of vanishing of $\Re p$ is equal to the order of vanishing of $a'$ at $x_{j_m}$. Moreover, $\Im p + k$ vanishes at $x_{j_m}$ of order at least $n_{j_m}$, and there exists $\ell_m \in \mathbb{N}$ such that $\Re p^{(n_{j_m}-1)}(x_{j_m}) + (\ell_m/n_{j_m})a^{(n_{j_m})}(x_{j_m}) = 0$.

For each $m = 0, \dots, n$, we fix $\eta_m \in (x_{j_m}, x_{j_m+\sigma_m})$, and we set

$$E_{m,k}(x) = \exp\left\{ -\int_{\eta_m}^{x} \frac{p+ik}{a}(s)ds \right\} \quad x \in (x_{j_m}, x_{j_m+\sigma_m}).$$

We now split the construction of a solution to (3.20) on $(x_{j_0}, x_{j_n+\sigma_n})$ into five subcases, which take into account different situations concerning the extremes $x_{j_0}$ and $x_{j_n+\sigma_n}$. The Subcases 5.1–5.4 treat the situation $(i)$, while Subcase 5.5 treats the situation $(ii)$.

**Subcase 5.1:** The extremes $x_{j_0}$ and $x_{j_n+\sigma_n}$ satisfy the assumptions in Case 1, i.e., $\Re p$ vanishes as much as $a$ and $\Im p + k$ vanishes less than $a$.

We consider

$$\widehat{u}(x,k) = E_{m,k}(x) \left( C_m + \int_{x_{j_m}}^{x} \frac{\widehat{f}(y,k)}{a(y)E_{m,k}(y)} dy \right), \tag{3.25}$$

for $x \in (x_{j_m}, x_{j_m+\sigma_m})$, and $m = 0, \ldots, n$.

We set $C_0 = 0$ and the other constants $C_1, \ldots, C_n$ must be adjusted in order to obtain a smooth solution $\widehat{u}(\cdot, k)$ to (3.20) on $(x_{j_0}, x_{j_n} + \sigma_n)$. This is obtained (see the Appendix A) by setting

$$C_1 = \frac{\partial_-^{\ell_1} E_{0,k}(x_{j_1})}{\partial_+^{\ell_1} E_{1,k}(x_{j_1})} \int_{x_{j_0}}^{x_{j_1}} \frac{\widehat{f}(y,k)}{a(y)E_{0,k}(y)} dy,$$

$$C_2 = \frac{\partial_-^{\ell_2} E_{1,k}(x_{j_2})}{\partial_+^{\ell_2} E_{2,k}(x_{j_2})} \left( C_1 + \int_{x_{j_1}}^{x_{j_2}} \frac{\widehat{f}(y,k)}{a(y)E_{1,k}(y)} dy \right),$$

and, recursively, for all $m = 2, \ldots, n$,

$$C_m = \frac{\partial_-^{\ell_m} E_{m-1,k}(x_{j_m})}{\partial_+^{\ell_m} E_{m,k}(x_{j_m})} \left( C_{m-1} + \int_{x_{j_{m-1}}}^{x_{j_m}} \frac{\widehat{f}(y,k)}{a(y)E_{m-1,k}(y)} dy \right). \tag{3.26}$$

Since $C_0 = 0$, $\widehat{u}(x, k)$ is flat at $x_{j_0}$ (as in Case 1). We then proceed to verify that this solution is also flat at $x_{j_n+\sigma_n}$.

It is enough to show that

$$C_n + \int_{x_{j_n}}^{x_{j_n+\sigma_n}} \frac{\widehat{f}(y,k)}{a(y)E_{n,k}(y)} dy = 0. \tag{3.27}$$

Setting

$$D_m = \frac{\partial_-^{\ell_1} E_{0,k}(x_{j_1}) \cdots \partial_-^{\ell_m} E_{m-1,k}(x_{j_m})}{\partial_+^{\ell_1} E_{1,k}(x_{j_1}) \cdots \partial_+^{\ell_m} E_{m,k}(x_{j_m})}, \quad m = 1, \ldots, n,$$

and denoting the characteristic function of the interval $(x_{j_m}, x_{j_{m+1}})$ by $\chi_m$, $m = 0, \ldots, n$, it follows that the function

$$G_k(x) = \chi_0(x)E_{0,k}(x) + \sum_{m=1}^{n} D_m \chi_m(x) E_{m,k}(x) \tag{3.28}$$

is smooth on $(x_{j_0}, x_{j_n+\sigma_n})$ (we may verify this by using identity (A.1) in the Appendix A) and vanishes only at the points $x_{j_m}$, where the order of vanishing is $\ell_m$, $m = 1, \ldots, n$. In addition, $a(x)G_k'(x) = -[p(x) + ik]G_k(x)$, for all $x \in (x_{j_0}, x_{j_n+\sigma_n})$.

Consider $\mu_k$ a distribution in $\mathcal{D}'(\mathbb{T}^1)$ which is given as follows: on $(x_{j_0}, x_{j_1})$, $\mu_k = 1/E_{k,0}$; on $(x_{j_n}, x_{j_n} + \sigma_n)$, $\mu_k = 1/(D_n E_{n,k})$; on $(x_{j_0}, x_{j_n+\sigma_n})$, $\mu_k = 1/G_k$; finally $\mu_k = 0$ on $\mathbb{T}^1 \setminus [x_{j_0}, x_{j_n+\sigma_n}]$.

The distribution $\mu_k/a$ belongs to $\mathcal{D}'(\mathbb{T}^1)$; it is supported on $[x_{j_0}, x_{j_n+\sigma_n}]$ and satisfies

$$0 = \partial_x(aG_k(\mu_k/a)) = G_k[\partial_x(a(\mu_k/a)) - (p + ik)(\mu_k/a)]$$

on $(x_{j_0}, x_{j_n+\sigma_n})$. Hence

$$\omega_k \doteq -\partial_x(a(\mu_k/a)) + (p + ik)(\mu_k/a)$$

is a linear combination of derivatives of the distributions $\delta(x - x_{j_m})$, $m = 0, \ldots, n+1$, and the order of the derivatives at $x_{j_m}$ is lesser than $\ell_m$, for all $m = 1, \ldots, n$. We then apply Lemma 3.3 in order to obtain a linear combination of derivatives of delta, $\nu_k$, such that

$$-\partial_x(a(\mu_k/a - \nu_k)) + (p + ik)(\mu_k/a - \nu_k) = 0.$$

Hence

$$0 = \langle \mu_k/a, \widehat{f}(\cdot, k)\rangle = \langle \mu_k, \widehat{f}(\cdot, k)/a\rangle = \int_{x_{j_0}}^{x_{j_n+\sigma_n}} \frac{\widehat{f}(y,k)}{a(y)G_k(y)} dy = 0. \tag{3.29}$$

In order to see that (3.29) is equivalent to (3.27), set

$$F_m = \frac{\partial_-^{\ell_m} E_{m-1,k}(x_{j_m})}{\partial_+^{\ell_m} E_{m,k}(x_{j_m})},\ m = 1, 2, \ldots, n.$$

Identity (3.27) is

$$0 = D_n \int_{x_{j_0}}^{x_{j_1}} \frac{\widehat{f}(y,k)}{a(y)E_{0,k}(y)} dy + F_2 F_3 \cdots F_n \int_{x_{j_1}}^{x_{j_2}} \frac{\widehat{f}(y,k)}{a(y)E_{1,k}(y)} dy +$$
$$\cdots + F_n \int_{x_{j_{n-1}}}^{x_{j_n}} \frac{\widehat{f}(y,k)}{a(y)E_{n-1,k}(y)} dy + \int_{x_{j_n}}^{x_{j_{n+1}}} \frac{\widehat{f}(y,k)}{a(y)E_{n,k}(y)} dy.$$

Multiplying by $1/D_n$, we obtain

$$0 = \int_{x_{j_0}}^{x_{j_1}} \frac{\widehat{f}(y,k)}{a(y)E_{0,k}(y)} dy + (1/D_1) \int_{x_{j_1}}^{x_{j_2}} \frac{\widehat{f}(y,k)}{a(y)E_{1,k}(y)} dy +$$
$$\cdots + (1/D_{n-1}) \int_{x_{j_{n-1}}}^{x_{j_n}} \frac{\widehat{f}(y,k)}{a(y)E_{n-1,k}(y)} dy + (1/D_n) \int_{x_{j_n}}^{x_{j_{n+1}}} \frac{\widehat{f}(y,k)}{a(y)E_{n,k}(y)} dy,$$

which is (3.29).

Therefore, we have constructed, via formula (3.25), a smooth solution to (3.20) on the interval $(x_{j_0}, x_{j_n+\sigma_n})$, which is flat at the extremes.

**Subcase 5.2:** Suppose that the extremes $x_{j_0}$ and $x_{j_n+\sigma_n}$ satisfy the assumptions in Case 2, i.e., the order of vanishing of $a$ is greater than the order of vanishing of $\Re p$ plus one, at both $x_{j_0}$ and $x_{j_n+\sigma_n}$.

If for some $\epsilon > 0$ $\Re p/a > 0$ on $(x_{j_0}, x_{j_0} + \epsilon)$ and $\Re p/a < 0$ on $(x_{j_{n+1}} - \epsilon, x_{j_{n+1}})$, then we may repeat the approach in Subcase 5.1.

Assume that $\Re p/a > 0$ on $(x_{j_0}, x_{j_0}+\epsilon)$ and $\Re p/a > 0$ on $(x_{j_{n+1}}-\epsilon, x_{j_{n+1}})$. Set $C_0 = 0$ and $C_1, \ldots, C_n$ given as in Subcase 5.1. With this choice, (3.25) defines a smooth solution to (3.20) on the interval $(x_{j_0}, x_{j_n+\sigma_n})$, which is flat at $x_{j_0}$. As in Case 2, a direct computation (without exhibiting distributions in the kernel of the transpose operator) shows that this solution is also flat at $x_{j_{n+1}}$.

Assume now that $\Re p/a < 0$ on $(x_{j_0}, x_{j_0} + \epsilon)$ and $\Re p/a < 0$ on $(x_{j_{n+1}} - \epsilon, x_{j_{n+1}})$. We consider

$$\widehat{u}(x,k) = E_{m,k}(x)\left(C_m - \int_x^{x_{j_{m+1}}} \frac{\widehat{f}(y,k)}{a(y)E_{m,k}(y)}dy\right),$$

for $x \in (x_{j_m}, x_{j_{m+\sigma_m}})$, and $m = 0, \ldots, n$. Setting $C_n = 0$, it follows that $\widehat{u}(\cdot, k)$ is smooth on $(x_{j_n}, x_{j_{n+1}}]$ and flat at $x_{j_{n+1}}$. Similar to the previous subcase, in order to obtain a smooth solution on $(x_{j_0}, x_{j_{n+1}}]$, we choose

$$C_{n-1} = -\frac{\partial_+^{\ell_n} E_{n,k}(x_{j_n})}{\partial_-^{\ell_n} E_{n-1,k}(x_{j_n})} \int_{x_{j_n}}^{x_{j_{n+1}}} \frac{\widehat{f}(y,k)}{a(y)E_{n,k}(y)}dy,$$

$$C_{n-2} = \frac{\partial_+^{\ell_{n-1}} E_{n-1,k}(x_{j_{n-1}})}{\partial_-^{\ell_{n-1}} E_{n-2,k}(x_{j_{n-1}})}\left(C_{n-1} - \int_{x_{j_{n-1}}}^{x_{j_n}} \frac{\widehat{f}(y,k)}{a(y)E_{n-1,k}(y)}dy\right),$$

and, recursively, for all $m = n-2, \ldots, 0$,

$$C_m = \frac{\partial_+^{\ell_{m+1}} E_{m+1,k}(x_{j_{m+1}})}{\partial_-^{\ell_{m+1}} E_{m,k}(x_{j_{m+1}})}\left(C_{m+1} - \int_{x_{j_{m+1}}}^{x_{j_{m+2}}} \frac{\widehat{f}(y,k)}{a(y)E_{m+1,k}(y)}dy\right).$$

Again, as in Case 2, a direct computation (without exhibiting distributions in the kernel of the transpose operator) shows that this solution is also flat at $x_{j_0}$.

We complete this Subcase 5.2 by considering the situation in which $\Re p/a < 0$ on $(x_{j_0}, x_{j_0} + \epsilon)$ and $\Re p/a > 0$ on $(x_{j_{n+1}} - \epsilon, x_{j_{n+1}})$. In this case, we may construct a solution $\widehat{u}(\cdot, k)$ with the following procedure: Set $C_0 = C_1 = 0$ and define

$$\widehat{u}(x,k) = -E_{0,k}(x)\int_x^{x_{j_1}} \frac{\widehat{f}(y,k)}{a(y)E_{0,k}(y)}dy,$$

for $x \in (x_{j_0}, x_{j_1})$; define

$$\widehat{u}(x,k) = E_{1,k}(x)\int_{x_{j_1}}^{x} \frac{\widehat{f}(y,k)}{a(y)E_{1,k}(y)}dy,$$

for $x \in (x_{j_1}, x_{j_2})$. For $m = 2, \ldots, n$ and for $x \in (x_{j_m}, x_{j_{m+1}})$, we set

$$\widehat{u}(x,k) = E_{m,k}(x)\left(C_m + \int_{x_{j_m}}^{x} \frac{\widehat{f}(y,k)}{a(y)E_{m,k}(y)}dy\right),$$

in which $C_2, \ldots, C_n$ are given as in Subcase 5.1. As before, this procedure gives us a solution $\widehat{u}(\cdot, k)$ to (3.20), which is smooth on $[x_{j_0}, x_{j_{n+1}}]$ and flat at $\{x_{j_0}, x_{j_{n+1}}\}$.

**Subcase 5.3:** We now treat the case in which the extremes $x_{j_0}$ and $x_{j_n+\sigma_n}$ satisfy the assumptions in Case 3, i.e., the order of vanishing of $\Re p$ is equal to the order of vanishing of $a'$, at both $x_{j_0}$ and $x_{j_n+\sigma_n}$. In addition, for $m=0,n+1$, either $\Im p+k$ vanishes less than $a$ at $x_{j_m}$ or $\Re p^{(n_{j_m}-1)}(x_{j_m})+(\ell/n_{j_m})a^{(n_{j_m})}(x_{j_m})\neq 0$, for all $\ell\in\mathbb{N}$.

The construction of a solution $\widehat{u}(\cdot,k)$ to (3.20) on $(x_{j_0},x_{j_{n+\sigma_n}})$ is analogous to the one in subcase 5.1. That is, on each interval $(x_{j_m},x_{j_{m+1}})$, $m=0,\dots,n$, it is given by the formula (3.25), with $C_0=0$ and $C_m$ satisfies (3.26), for $m=1,\dots,n$. This gives a smooth solution on $[x_{j_0},x_{j_{n+\sigma_n}})$, which is flat at $x_{j_0}$ (as explained in the end of Case 3). We now proceed to show that this solution is flat at $x_{j_{n+\sigma_n}}$.

Pick $\phi(x)=[1-\cos(x-x_{j_0})]^M[1-\cos(x-x_{j_{n+\sigma_n}})]^M$, $M\in\mathbb{N}$, which vanishes only at $\{x_{j_0},x_{j_{n+\sigma_n}}\}$, with finite order of vanishing. Moreover, if $M$ is large enough, then $\phi/E_{0,k}$ is bounded on a small interval $(x_{j_0},x_{j_0}+\epsilon)$ and $\phi/E_{n,k}$ is bounded on a small interval $(x_{j_{n+\sigma_n}}-\epsilon,x_{j_{n+\sigma_n}})$.

Take $\mu_k$ a distribution in $\mathcal{D}'(\mathbb{T}^1)$ which satisfies: on the open interval $(x_{j_0},x_{j_1})$, $\mu_k=\phi/E_{k,0}$; on $(x_{j_n},x_{j_n}+\sigma_n)$, $\mu_k=\phi/(D_nE_{n,k})$; on $(x_{j_0},x_{j_{n+\sigma_n}})$, $\mu_k=\phi/G_k$, with $G_k$ given by (3.28); finally $\mu_k=0$ on $\mathbb{T}^1\setminus[x_{j_0},x_{j_{n+\sigma_n}}]$.

By using the division theorem, there exists $\omega_k=\mu_k/(a\phi)\in\mathcal{D}'(\mathbb{T}^1)$, which is supported on $[x_{j_0},x_{j_{n+\sigma_n}}]$ and $aG_k\omega_k=1$ on $(x_{j_0},x_{j_{n+\sigma_n}})$. As before, on $(x_{j_0},x_{j_{n+\sigma_n}})$ we have

$$0=\partial_x(aG_k\omega_k)=G_k(\partial_x(a\omega_k)-(p+ik)\omega_k).$$

It follows that $\partial_x(a\omega_k)-(p+ik)\omega_k$ is a linear combination of derivatives of $\delta(x-x_{j_m})$, $m=0,\dots,n+1$. By Lemma 3.3, we obtain $\nu_k$ such that $-\partial_x(a(\omega_k-\nu_k))+(p+ik)(\omega_k-\nu_k)=0$. Hence

$$0=\langle\omega_k,\widehat{f}(\cdot,k)\rangle=\langle\mu_k,\widehat{f}(\cdot,k)/(a\phi)\rangle=\int_{x_{j_0}}^{x_{j_{n+\sigma_n}}}\frac{\widehat{f}(y,k)}{a(y)G_k(y)}dy.$$

As in subcase 5.1, this implies that $\widehat{u}(\cdot,k)$ is also flat at $x_{j_{n+\sigma_n}}$.

**Subcase 5.4:** If the extreme $x_{j_0}$ belongs to one of the situations treated in Subcases 5.1–5.3, and the other extreme $x_{j_{n+\sigma_n}}$ belongs to another situation treated in Subcases 5.1–5.3, we may combine the previous techniques in order to construct a solution $\widehat{u}(\cdot,k)$ to (3.20), which is smooth on $(x_{j_0},x_{j_{n+\sigma_n}})$ and flat at $\{x_{j_0},x_{j_{n+\sigma_n}}\}$. The arguments are similar to the ones in Case 4.

**Subcase 5.5:** To complete the Case 5, we now turn our attention to the other situation, when $x_{j_0}+2\pi=x_{j_{n+\sigma_n}}$ and for all $m=0,\dots,n,n+1$, the order of vanishing of $\Re p$ is equal to the order of vanishing of $a'$ at $x_{j_m}$. Moreover, $\Im p(x_{j_m})+k=0$ and $\Im p+k$ vanishes at $x_{j_m}$ of order at least $n_{j_m}$, and there exists $\ell_m\in\mathbb{N}$ such that $\Re p^{(n_{j_m}-1)}(x_{j_m})+(\ell_m/n_{j_m})a^{(n_{j_m})}(x_{j_m})=0$. In other words, now the extremes $x_{j_0}$ and $x_{j_{n+\sigma_n}}$ are of the same type as the middle zeros $x_{j_1},\dots,x_{j_n}$.

If $\partial_-^{\ell_{n+1}}E_{n,k}(x_{j_{n+1}})D_n\neq\partial_+^{\ell_0}E_{0,k}(x_{j_0})$, then we must choose $C_0$ satisfying

$$\left[\frac{\partial_+^{\ell_0}E_{0,k}(x_{j_0})}{\partial_-^{\ell_{n+1}}E_{n,k}(x_{j_{n+1}})}-D_n\right]C_0=D_n\int_{x_{j_0}}^{x_{j_1}}\frac{\widehat{f}(y,k)}{a(y)E_{0,k}(y)}dy+$$
$$F_2F_3\cdots F_n\int_{x_{j_1}}^{x_{j_2}}\frac{\widehat{f}(y,k)}{a(y)E_{1,k}(y)}dy+\cdots+$$
$$F_n\int_{x_{j_{n-1}}}^{x_{j_n}}\frac{\widehat{f}(y,k)}{a(y)E_{n-1,k}(y)}dy+\int_{x_{j_n}}^{x_{j_{n+1}}}\frac{\widehat{f}(y,k)}{a(y)E_{n,k}(y)}dy.$$

With this choice, formula (3.25) defines a smooth solution to (3.20) on $\mathbb{T}^1$.

On the other hand, if $\partial_-^{\ell_{n+1}} E_{n,k}(x_{j_{n+1}}) D_n = \partial_+^{\ell_0} E_{0,k}(x_{j_0})$, then $G_k(x)$ given in (3.28) belongs to $\mathcal{C}^\infty(\mathbb{T}^1)$. We then set $C_0 = 0$ and the same argument used in subcase 5.1, via Lemma 3.3, shows that the formula (3.25) defines a smooth solution to (3.20) on $\mathbb{T}^1$.

We now proceed to conclude our proof. After Cases 1–5, by joining the pieces of solutions, we are able to construct a smooth solution $\widehat{u}(x,k)$ to (3.20) on $\mathbb{T}^1$. We must show that this sequence of solutions decays rapidly. It is enough to show the decay on each one of its pieces.

First, note that the situation treated in Case 5 occurs for at most finitely many indices $k \in \mathbb{Z}$. Hence, to show the rapid decaying, there is no need to consider the solutions produced in this case.

As in [6], the solutions constructed in Case 1 decay rapidly on the intervals $(x_j, x_{j+\sigma})$, since the real part of $E_k(x)$ does not depend on $k$, and then it does not affect the rapid decaying.

Likewise, we see that the solutions constructed in Case 3 decay rapidly on the intervals $(x_j, x_{j+\sigma})$.

Arguing as in [4], we see that the solutions constructed in Case 2 also decay rapidly on the intervals $(x_j, x_{j+\sigma})$.

Since in Case 4 the solutions have the same formulas as in Cases $1-3$, it follows that they also decay rapidly on $(x_j, x_{j+\sigma})$.

Therefore, the sequence of smooth solutions $\widehat{u}(\cdot,k)$ decays rapidly. Consequently,

$$u(x,t) = \sum_{k\in\mathbb{Z}} \widehat{u}(x,k)\exp\{ikt\}$$

belongs to $\mathcal{C}^\infty(\mathbb{T}^2)$ and satisfies $L_p u = f$. This completes the proof of Theorem 3.4. $\square$

It is curious that the perturbation of zero order $p$ may turn the global solvability of $L_0 : \mathcal{C}^\infty(\mathbb{T}^2) \to \mathcal{C}^\infty(\mathbb{T}^2)$ stronger. For instance, under the assumptions in Theorem 3.4, the vector field $L_0$ has closed range with infinite codimension.

In contrast, for certain functions $p$, the operator $L_p : \mathcal{C}^\infty(\mathbb{T}^2) \to \mathcal{C}^\infty(\mathbb{T}^2)$ may have a range with finite codimension.

Consider the operator $\mathcal{L} : \mathcal{C}^\infty(\mathbb{T}^2) \to \mathcal{C}^\infty(\mathbb{T}^2)$ given by

$$\mathcal{L} = \partial_t + \cos^2(x)q(x)\partial_x + q(x),$$

in which $q$ is real-valued and never vanishes. We claim that this operator is surjective. Indeed, if $\mu \in \ker {}^t\mathcal{L}$, then on $(\pi/2, 3\pi/2)$ we have

$$a(x)\widehat{\mu}(x,k) = C\exp\left\{\int_\pi^x \frac{1}{\cos^2}\right\}\exp\left\{-ik\int_\pi^x \frac{1}{q\cos^2}\right\},$$

in which $a(x) = q(x)\cos^2(x)$. Since the function

$$\exp\left\{-\int_\pi^x \frac{1}{\cos^2}\right\}, \quad x \in (\pi/2, 3\pi/2),$$

is flat at $x = 3\pi/2$, it follows that $C = 0$. Similarly, we have $a(x)\widehat{\mu}(x,k) = 0$ on $(3\pi/2, 5\pi/2)$. Hence $\widehat{\mu}(x,k)$ is a linear combination of derivatives of deltas with centers at $\pi/2$, $3\pi/2$ and $5\pi/2$. Finally, by using the equation $(a(x)\widehat{\mu}(x,k))' + (ik - q(x))\widehat{\mu}(x,k) = 0$ and calculations in the proof of Lemma 3.3), we can show that this linear combination must be identically zero. Therefore, $\ker {}^t\mathcal{L} = \{0\}$. By Theorem 3.4, we have $\mathcal{L}\mathcal{C}^\infty(\mathbb{T}^2) = (\ker {}^t\mathcal{L})^\circ = \mathcal{C}^\infty(\mathbb{T}^2)$.

*3.1. Solvability in the space of Schwartz distributions*

We now consider $L_p : \mathcal{D}'(\mathbb{T}^2) \to \mathcal{D}'(\mathbb{T}^2)$.

By Theorem 3.4 we know that $L_{a'-p} : \mathcal{C}^\infty(\mathbb{T}^2) \to \mathcal{C}^\infty(\mathbb{T}^2)$ is globally solvable, and since $-L_p = {}^tL_{a'-p}$, it follows that $L_p$ is globally solvable on $\mathcal{D}'(\mathbb{T}^2)$. Hence, we have proved:

**Proposition 3.5.** *If $a^{-1}(0) \neq \emptyset$ and $a$ vanishes only of finite order, then the operator $L_p$ given by (1.3) is globally solvable on $\mathcal{D}'(\mathbb{T}^2)$.*

In which follows, we study the surjectivity of the operator $L_p : \mathcal{D}'(\mathbb{T}^2) \to \mathcal{D}'(\mathbb{T}^2)$.

According to [6], the operator $L_0$ is surjective on $\mathcal{D}'(\mathbb{T}^2)$ whenever $a^{-1}(0) \neq \emptyset$ and $a$ vanishes only of finite order. This property may be affected by the perturbations $p$. Since $\mathcal{C}^\infty(\mathbb{T}^2)$ is reflexive and by using Hanh-Banach Theorem, it follows that $L_p$ is surjective on $\mathcal{D}'(\mathbb{T}^2)$ if and only if $L_{a'-p}$ is injective on $\mathcal{C}^\infty(\mathbb{T}^2)$. In particular, we see that $L_{a'}$ is not surjective on $\mathcal{D}'(\mathbb{T}^2)$, since any constant belongs to $\ker L_0$. On the other hand, $L_{-a'}$ is surjective on $\mathcal{D}'(\mathbb{T}^2)$. Indeed, if a smooth function $u$ belongs to $\ker L_{2a'}$, then for each $k \in \mathbb{Z}$ and $j \in \{1, \dots, N\}$, there exists a constant $C_{jk}$ such that

$$\widehat{u}(x,k) = C_{jk} \exp\left\{ -\int_{\eta_j}^{x} \frac{ik + 2a'}{a}(s)ds \right\},\ x \in (x_j, x_{j+1}).$$

Hence, $|\widehat{u}(x,k)| = |C_{jk}| a(\eta_j)^2/a(x)^2$, which implies that $C_{jk} = 0$.

After presenting the two examples above, we now proceed to treat the general situations.

Suppose that $\Re p - a'$ vanishes as much as $a$ at the points in $a^{-1}(0) = \{x_1 < x_2 < \cdots < x_N\}$. If for all $k \in \mathbb{Z}$ and each $j \in \{1, \dots, N\}$, the function $\Im p - k$ vanishes less than $a$ at $x_j$ (respectively $x_{j+1}$), then $L_p$ is surjective on $\mathcal{D}'(\mathbb{T}^2)$. Indeed, for $u \in \ker L_{a'-p}$ we have

$$\widehat{u}(x,k) = C_{jk} \exp\left\{ \int_{\eta_j}^{x} \frac{\Re p - a'}{a}(s)ds \right\} \exp\left\{ i\int_{\eta_j}^{x} \frac{\Im p - k}{a}(s)ds \right\}, \tag{3.30}$$

whenever $x \in (x_j, x_{j+1})$. Hence, if $C_{jk} \neq 0$, then the limit $\lim_{x \to x_j^+} \widehat{u}(x,k)$ (respectively $\lim_{x \to x_{j+1}^-} \widehat{u}(x,k)$) does not exist. It follows that $\ker L_{a'-p} = \{0\}$. Therefore, $L_p$ is surjective.

Assume now that $\Re p - a'$ vanishes as much as $a$ at the points in $a^{-1}(0)$ and there exist $k_0 \in \mathbb{Z}$ and $j_0 \in \{1, \dots, N\}$ such that the function $\Im p - k_0$ vanishes as much as $a$ at both $x_{j_0}$ and $x_{j_0+1}$. By analysing formula (3.30) we see that $L_p$ is not surjective if and only if $\Im p - k_0$ vanishes as much as $a$ at each $x_j$ and

$$\int_0^{2\pi} \frac{ik_0 + a' - p}{a}(s)ds \in i\mathbb{Z}.$$

From now on, we assume that there exists a point in $a^{-1}(0)$ at which $\Re p - a'$ vanishes less than $a$. We then consider intervals of the type $(x_j, x_{j+\sigma})$ such that on $(x_j, x_{j+\sigma}) \cap a^{-1}(0)$ the function $ik + a' - p$ vanishes as much as $a$ and it vanishes less than $a$ at the extremes $x_j$ and $x_{j+\sigma}$. In order to see if there exists $u \in \ker L_{a'-p}$ such that $\widehat{u}(x,k)$ does not vanish identically on $(x_j, x_{j+\sigma})$, we may assume (by the comments above) that $\Re p - a'$ vanishes less than $a$ at the extremes.

Assume that there exists an interval $(x_j, x_{j+\sigma})$ such that at the extremes, $x_j$ and $x_{j+\sigma}$, the order of vanishing of $a$ is greater than the order of vanishing of $\Re p - a'$ plus one. In addition, if $(\Re p - a')/a$ is positive near $x_j$ and it is negative near $x_{j+\sigma}$, then $L_p$ is not surjective on $\mathcal{D}'(\mathbb{T}^2)$. Indeed, in order to exhibit a smooth function in $\ker L_{a'-p} \setminus \{0\}$, it is enough to consider

$$u(x,t) = \widehat{u}(x,k)\exp\{ikt\}$$

such that $\widehat{u}(\cdot,k)$ is given by

$$\begin{aligned}\widehat{u}(x,k) &= \exp\left\{-\int_{\eta_j}^{x} \frac{ik+a'-p}{a}(s)ds\right\} \\ &= \exp\left\{\int_{\eta_j}^{x} \frac{\Re p - a'}{a}(s)ds\right\} \exp\left\{i\int_{\eta_j}^{x} \frac{\Im p - k}{a}(s)ds\right\},\end{aligned}$$

$x \in (x_j, x_{j+\sigma})$, and $\widehat{u}(x,k) = 0$, if $x \in \mathbb{T}^1 \setminus (x_j, x_{j+\sigma})$.

In the presence of a point in $a^{-1}(0)$ at which the order of vanishing of $\Re p - a'$ is equal to the order of vanishing of $a'$, we consider the chains

$$(x_{j_0}, x_{j_0+\sigma_0}] \cup [x_{j_1}, x_{j_1+\sigma_1}) \cup \cdots \cup [x_{j_n}, x_{j_n+\sigma_n}) \tag{3.31}$$

with the properties:

- $x_{j_m+\sigma_m} = x_{j_{m+1}}$, $m = 0, \ldots, n-1$,
- on each $(x_{j_m}, x_{j_m} + \sigma_m) \cap a^{-1}(0)$ the function $a$ vanishes less than $ik + a' - p$,
- the order of vanishing of $\Re p - a'$ is equal to the order of vanishing of $a'$ at $x_{j_m}$, $m = 1, \ldots, n$.

If there exists $m \in \{1, \ldots, n\}$ such that either $\Im p - k$ vanishes less than $a$ at $x_{j_m}$ or

$$n_{j_m}(\Re p - a')^{(n_{j_m}-1)}(x_{j_m})/a^{(n_{j_m})}(x_{j_m}) \notin \mathbb{Z}_+,$$

then similar computations to the ones performed in the Appendix A allow us to conclude that, for every smooth function $u \in \ker L_{a'-p}$, its partial Fourier coefficient $\widehat{u}(\cdot,k)$ must vanish identically on $(x_{j_0}, x_{j_n+\sigma_n})$.

Suppose that for each $m \in \{1, \ldots, n\}$, $\Im p - k$ vanishes as much as $a$ at $x_{j_m}$ and $(\Re p - a')^{(n_{j_m}-1)}(x_{j_m}) = \ell_m a^{(n_{j_m})}(x_{j_m})/n_{j_m}$, for some $\ell_m \in \mathbb{N}$. In addition, if at both $x_{j_0}$ and $x_{j_n+\sigma_n}$ the order of vanishing of $a$ is greater than the order of vanishing of $\Re p - a'$ plus one, then we may proceed as in the proof of Theorem 3.4 (Case 5) to show that there exists a smooth function on $\ker L_{a'-p}$ whose support is $[x_{j_0}, x_{j_n+\sigma_n}]$ whenever $(\Re p - a')/a$ is positive near $x_{j_0}$ and it is negative near $x_{j_n+\sigma_n}$.

If $x_{j_n+\sigma_n} = x_{j_0} + 2\pi$ and $\Im p - k$ vanishes as much as $a$ at $x_{j_0}$, and $\Re p - a'$ vanishes at $x_{j_0}$ of the order of $a'$, then we again follow the lines in the proof of Theorem 3.4 (Case 5) in order to show that $L_p$ is not surjective on $\mathcal{D}'(\mathbb{T}^2)$ ($\ker L_{a'-p} \neq \{0\}$) if and only if

$$-n_{j_0}(\Re p - a')^{(n_{j_0}-1)}(x_{j_0})/a^{(n_{j_0})} \in \mathbb{Z}_+$$

and $\partial_-^{\ell_{n+1}} E_{n,k}(x_{j_{n+1}}) D_n = \partial_+^{\ell_0} E_{0,k}(x_{j_0})$, where here

$$E_{m,k}(x) = \exp\left\{-\int_{\eta_m}^{x} \frac{ik+a'-p}{a}(s)ds\right\},$$

with $\eta_m \in (x_{j_m}, x_{j_m+\sigma_m})$, $m = 0, \ldots, n$.

The discussion above may be used to prove the following:

**Theorem 3.6.** *Let $L_p : \mathcal{D}'(\mathbb{T}^2) \to \mathcal{D}'(\mathbb{T}^2)$ be given by (1.3). Assume that $a$ vanishes only of finite order and that $\emptyset \neq a^{-1}(0) = \{x_1 < \cdots < x_N\}$. Operator $L_p$ is not surjective (i.e. $L_p\mathcal{D}'(\mathbb{T}^2) \subsetneq \mathcal{D}'(\mathbb{T}^2)$) if and only if there exists an integer number $k \in \mathbb{Z}$ such that one of the following situations occurs:*

*(i) $p - a' - ik$ vanishes as much as $a$ at $a^{-1}(0)$ and*

$$\int_0^{2\pi} \frac{p - a' - ik}{a}(s)ds \in i\mathbb{Z}.$$

*(ii) there exists an interval $(x_j, x_{j+\sigma})$ such that at $a^{-1}(0) \cap (x_j, x_{j+\sigma})$ the function $p - a' - ik$ vanishes as much as $a$; at the extremes $x_j$ and $x_{j+\sigma}$ the order of vanishing of $a$ is greater than the order of vanishing of $\Re p - a'$ plus one; in addition, $\Re p - a'/a > 0$ near $x_j$ and $\Re p - a'/a < 0$ near $x_{j+\sigma}$.*

*(iii) there exists a chain*

$$(x_{j_0}, x_{j_0+\sigma_0}] \cup [x_{j_1}, x_{j_1+\sigma_1}) \cup \cdots \cup [x_{j_n}, x_{j_n+\sigma_n}),$$

*as in (3.31), such that the order of vanishing of $a$ at the extremes $x_{j_0}$ and $x_{j_n+\sigma_n}$ is greater than the order of vanishing of $\Re p - a'$ plus one, $(\Re p - a')/a$ is positive near $x_{j_0}$ and it is negative near $x_{j_n+\sigma_n}$, and for all $m = 1, \ldots, n$, $\Im p - k$ vanishes as much as $a$ at $x_{j_m}$ and*

$$n_{j_m}(\Re p - a')^{(n_{j_m}-1)}(x_{j_m})/a^{(n_{j_m})}(x_{j_m}) \in \mathbb{Z}_+.$$

*(iv) there exists a chain as in (3.31), such that $x_{j_0} + 2\pi = x_{j_n+\sigma_n}$ and, for all $m = 0, 1, \ldots, n$, $\Im p - k$ vanishes as much as $a$ at $x_{j_m}$, $\Re p - a'$ vanishes of the order of $a'$, and*

$$n_{j_m}(\Re p - a')^{(n_{j_m}-1)}(x_{j_m})/a^{(n_{j_m})}(x_{j_m}) \in \mathbb{Z}_+.$$

*In addition $\partial_-^{\ell_{n+1}} E_{n,k}(x_{j_{n+1}})D_n = \partial_+^{\ell_0} E_{0,k}(x_{j_0})$, where*

$$D_n = \frac{\partial_-^{\ell_1} E_{0,k}(x_{j_1}) \cdots \partial_-^{\ell_n} E_{n-1,k}(x_{j_n})}{\partial_+^{\ell_1} E_{1,k}(x_{j_1}) \cdots \partial_+^{\ell_n} E_{n,k}(x_{j_n})}$$

*and*

$$E_{m,k}(x) = \exp\left\{ -\int_{\eta_m}^{x} \frac{ik + a' - p}{a}(s)ds \right\},$$

*with $\eta_m \in (x_{j_m}, x_{j_m+\sigma_m})$, $m = 0, \ldots, n$.*

## 4. Further results: coefficient $a$ flat at some point

This section treats cases in which $a$ is flat at some point. We begin by studying the case in which $a$ vanishes identically. As mentioned in [6], the operator $\partial_t$ is globally solvable. As we will see in the sequel, $L_p = \partial_t + p$ may be non-globally solvable. Indeed, if there exists $k_0 \in \mathbb{Z}$ such that $p + ik_0$ is flat at some point but it does not vanish identically, then proceeding as in [6] we can show that $L_p$ is not globally solvable on $\mathcal{C}^\infty(\mathbb{T}^2)$. For instance, if $\mathfrak{F}$ is the set of points at which $p + ik_0$ is flat and picking $x_1 \in \partial\mathfrak{F}$, then

$$f(x,t) = \frac{p + ik_0}{1 - \cos(x - x_1)} \otimes \exp\{ik_0 t\}$$

belongs to $(\ker {}^tL_p)^\circ \setminus L_p\mathcal{C}^\infty(\mathbb{T}^2)$.

Assume now that, for all $k \in \mathbb{Z}$, $p + ik$ vanishes identically or it has only zeros of finite order. We will show that $L_p$ is globally solvable in this case.

Given $f \in (\ker {}^tL_p)^\circ$, we will construct $u \in \mathcal{C}^\infty(\mathbb{T}^2)$ such that $L_p u = f$. For the integers $k \in \mathbb{Z}$ such that $p + ik$ vanishes identically, we have

$$\widehat{f}(x,k) \otimes \exp\{-ikt\} \in \ker {}^tL_p;$$

hence,

$$\int_0^{2\pi} \widehat{f}(x,k)^2 dx = 0.$$

This implies that $\widehat{f}(x,k)$ vanishes identically. In this case, we may set $\widehat{u}(x,k) \equiv 0$ as a solution to

$$(p(x) + ik)\widehat{u}(x,k) = \widehat{f}(x,k). \tag{4.1}$$

We now move on to solve (4.1) to the indices $k$ such that $p + ik$ vanishes only of finite order (or it does not vanish). Since $p$ is bounded, for all but a finite number of indices $k$, the function $p + ik$ never vanishes and we have a unique smooth solution given by $\widehat{u}(x,k) = \widehat{f}(x,k)/(p(x) + ik)$. From this formula we see that the sequence $\widehat{u}(x,k)$ decays rapidly, since $\widehat{f}(x,k)$ decays rapidly.

Finally, let $k_1, \ldots, k_r$ be the integers such that $p + ik_j$ has zeros of finite order. To define a solution $\widehat{u}(x,k_j)$ to

$$(p(x) + ik_j)\widehat{u}(x,k_j) = \widehat{f}(x,k_j),$$

it is enough to show that we may assume that each $\widehat{f}(x,k_j)$ is flat at the finite set of zeros $\cup_{j=1}^r (p+ik_j)^{-1}(0)$. Hence, as in Section 3, we need a reduction modulo flat functions. This is obtained by using cutoff functions and applying the following:

**Lemma 4.1.** *Suppose that $p \in \mathcal{C}^\infty(x_0 - \epsilon, x_0 + \epsilon)$ ($p$ is complex-valued), $a \in \mathcal{C}^\infty((x_0 - \epsilon, x_0 + \epsilon), \mathbb{R})$ and $x_0$ is a zero of $a$ of infinity order. Given $f \in (\ker {}^tL_p)^\circ$, there exists $u \in \mathcal{C}^\infty((x_0 - \epsilon, x_0 + \epsilon) \times \mathbb{T}^1)$ such that $L_p u - f$ is flat at $\{x_0\} \times \mathbb{T}^1$.*

**Proof.** Given $f, u \in \mathcal{C}^\infty((x_0 - \epsilon, x_0 + \epsilon) \times \mathbb{T}^1)$, we use formal Taylor series to write

$$u(x,t) \simeq \sum_{j=0}^{\infty} u_j(t)(x - x_0)^j, \; u_j(t) = \frac{1}{j!}\partial_x^j u(x_0,t),$$

$$f(x,t) \simeq \sum_{j=0}^{\infty} f_j(t)(x - x_0)^j, \; f_j(t) = \frac{1}{j!}\partial_x^j f(x_0,t),$$

$$a(x) \simeq \sum_{j=0}^{\infty} a_j(x - x_0)^j \equiv 0, \text{ since } a_j = \frac{1}{j!}a^{(j)}(x_0),$$

and

$$p(x) \simeq \sum_{j=0}^{\infty} p_j (x-x_0)^j, \; p_j = \frac{1}{j!} p^{(j)}(x_0).$$

The formal Taylor series of $L_p u - f$ is

$$L_p u - f \simeq \sum_{j=0}^{\infty} \left( u_j' + \sum_{k=0}^{j} p_k u_{j-k} - f_j \right) (x-x_0)^j.$$

It follows that $L_p u - f$ is flat at $\{x_0\} \times \mathbb{T}^1$ if and only if

$$u_0' + p_0 u_0 = f_0, \tag{4.2}$$
$$u_1' + p_0 u_1 = f_1 - p_1 u_0, \tag{4.3}$$
$$u_j' + p_0 u_j = f_j - p_1 u_{j-1} - \cdots - p_j u_0, \; j \geq 1. \tag{4.4}$$

After finding the sequence of solutions $(u_j)_{j \in \mathbb{Z}_+}$, the required function $u$ is obtained by employing Borel's Lemma. Hence the proof reduces to solve equations (4.2)–(4.4).

If $p_0 \notin i\mathbb{Z}$, then we may solve (4.2)–(4.4) recursively.

Assume now that $p_0 = -im$, $m \in \mathbb{Z}$, and $p_1 \neq 0$. Then $\delta(x-x_0) \otimes e^{-imt} \in \ker {}^t L_p$ and since $f \in (\ker {}^t L_p)^\circ$, we may find $u_0$ which solves (4.2). Moreover, the solutions are of the form

$$u_0(t) = \sum_{k \in \mathbb{Z}} \widehat{u_0}(k) e^{ikt},$$

where $\widehat{u_0}(m)$ is any complex number and $\widehat{u_0}(k) = (p_0 + ik)^{-1} \widehat{f_0}(k)$, for $k \neq m$.

We then choose $\widehat{u_0}(m) = \widehat{f_1}(m)/p_1$, so that we may solve next equation, (4.3). Again, the coefficient $\widehat{u_1}(m)$ must be $\widehat{u_1}(m) = [\widehat{f_2}(m) - p_2 \widehat{u_0}(m)]/p_1$, so that we may solve next equation.

This recursive procedure allows us to find solutions $u_0, u_1, \ldots, u_j, \ldots$.

Suppose now that $p_0 = -im$, $m \in \mathbb{Z}$ and $p_1 = 0$. Set $j_0$ the smallest index $j > 1$ such that $p_j \neq 0$ and $p_\ell = 0$, $\ell = 1, \ldots, j-1$. If such a smallest index does not exist, then we say $j_0 = \infty$.

Equations (4.2)–(4.4) reduce to

$$u_j' + p_0 u_j = f_j, \; j = 0, \ldots, j_0 - 1 \tag{4.5}$$
$$u_{j_0}' + p_0 u_{j_0} = f_{j_0} - p_{j_0} u_0, \tag{4.6}$$
$$u_j' + p_0 u_j = f_j - p_{j_0} u_{j-j_0} - \cdots - p_j u_0, \; j \geq j_0. \tag{4.7}$$

As before, equations (4.5) may be solved, since $f \in (\ker {}^t L_p)^\circ$ and $\delta^{(\ell)}(x-x_0) \otimes e^{-imt} \in \ker {}^t L_p$, $\ell = 0, \ldots, j_0 - 1$. After this, we solve (4.6) by adjusting $\widehat{u_0}(m)$. Finally, after finding the solutions $u_0, \ldots, u_{j_0+k}$ recursively, we find a solution $u_{j_0+k+1}$ to the next equation by adjusting $u_{k+1}$.

The proof of Lemma 4.1 is completed. □

The discussion above allows to prove the following:

**Theorem 4.2.** *The operator $\partial_t + p$ is globally solvable on $\mathcal{C}^\infty(\mathbb{T}^2)$ if and only if for each $k \in \mathbb{Z}$ the function $p(x) + ik$ is not flat at any point or it vanishes identically.*

Since ${}^t(\partial_t - p) = -(\partial_t + p)$, Theorem 4.2 implies that $\partial_t + p$ is globally solvable on $\mathcal{D}'(\mathbb{T}^2)$, provided that, for each $k \in \mathbb{Z}$, the function $p(x) + ik$ is not flat at any point or it vanishes identically. On the other hand, assume that there exists $k_0$ such that $p + ik_0$ does not vanish identically and it is flat at some point. Proceeding as in [6], we will show that $\partial_t + p$ is not globally solvable on $\mathcal{D}'(\mathbb{T}^2)$. Setting $\mathcal{G} = \mathbb{T}^1 \setminus \mathfrak{F}$, in which $\mathfrak{F}$ denotes the set of points at which $p + ik_0$ is flat, it follows that $\mathcal{G}$ is a non-empty open subset of $\mathbb{T}^1$. If $(\alpha, \beta)$ is a connected component of $\mathcal{G}$ and $\chi_{\alpha,\beta}$ is the characteristic function of $(\alpha, \beta)$, then we can show that the distribution $\chi_{\alpha,\beta}(x) \otimes \exp\{ik_0 t\}$ belongs to ${}^\circ(\ker L_{-p}) \setminus L_p\mathcal{D}'(\mathbb{T}^2)$, where $L_{-p} = \partial_t - p$. Therefore, we obtain the following version of Theorem 4.2.

**Theorem 4.3.** *The operator $\partial_t + p$ is globally solvable on $\mathcal{D}'(\mathbb{T}^2)$ if and only if, for each $k \in \mathbb{Z}$, the function $p(x) + ik$ is not flat at any point or it vanishes identically.*

### *4.1. Assuming that a does not vanish identically and it is flat at some point*

In this case, we stress that the search for solutions of $L_p u = f$ and the description of the distributions in $\ker {}^tL_p$ become quite chaotic, since we may have infinitely many zeros of finite (or infinite) order accumulating near a zero of infinite order of the coefficient $a$. Hence, following our approach, it is more difficult to control, at these points, the interactions between the order of vanishing of the functions $a$ and $p + ik$. Although we do not have a complete answer in this case, we will present a result to shed light to the general problem.

Recall that operator (1.2) is not globally solvable (on neither $\mathcal{C}^\infty(\mathbb{T}^2)$ nor $\mathcal{D}'(\mathbb{T}^2)$) if $a^{-1}(0) \neq \mathbb{T}^1$ and $a$ is flat at some point (see [6]).

Similar to the comment in the previous section, if $p$ is in the range of $L_0$, then $L_p$ is still non-globally solvable (on either $\mathcal{C}^\infty(\mathbb{T}^2)$ or $\mathcal{D}'(\mathbb{T}^2)$). However, now the range of $L_0$ is not well-understood, since $L_0$ is not globally solvable. We only know that $L_0\mathcal{C}^\infty(\mathbb{T}^2)$ is a subspace of $(\ker {}^tL_0)^\circ$.

Next result will cover situations where $p$ is not in the range of $L_0$, but $L_p$ remains non-globally solvable.

**Proposition 4.4.** *Suppose that $a$ does not vanish identically and there exists $k \in \mathbb{Z}$ such that both $a$ and $p + ik$ are flat at a same point. Then operator $L_p$ is not globally solvable on $\mathcal{C}^\infty(\mathbb{T}^2)$.*

**Proof.** Since the global solvability of $L_p : \mathcal{C}^\infty(\mathbb{T}^2) \to \mathcal{C}^\infty(\mathbb{T}^2)$ implies the global solvability of ${}^tL_p : \mathcal{D}'(\mathbb{T}^2) \to \mathcal{D}'(\mathbb{T}^2)$, it is enough to show that the latter is not globally solvable.

Picking $x_0$ at the boundary of the set

$$\{x \in \mathbb{T}^1;\ a \ \text{ and } \ p + ik \ \text{ are flat at } \ x\},$$

we will show that there exist constants $c_0, c_1 \in \mathbb{C}$ such that the distribution $[c_0\delta(x-x_0)+c_1\delta'(x-x_0)]\otimes e^{-ikt}$ belongs to ${}^\circ(\ker L_p) \setminus {}^tL_p\mathcal{D}'(\mathbb{T}^2)$.

Firstly, we assume that there exists a sequence $x_n \to x_0$ such that $a(x_n) \neq 0$, for all $n$. If $u \in \ker L_p$, we have

$$\partial_x \widehat{u}(x_n, k) + \left[\frac{p(x_n) + ik}{a(x_n)}\right] \widehat{u}(x_n, k) = 0.$$

When $\lim_{n\to\infty} \frac{|p(x_n)+ik|}{|a(x_n)|}$ neither exists nor is finite, then $\widehat{u}(x_0, k) = 0$. Hence $\delta(x - x_0) \otimes e^{-ikt}$ belongs to ${}^\circ(\ker L_p)$.

When $\lim_{n\to\infty} \frac{|p(x_n)+ik|}{|a(x_n)|}$ is a real number, then using a subsequence, if necessary, we may assume that there exists $c \in \mathbb{C}$ such that $[\delta'(x - x_0) + c\delta(x - x_0)] \otimes e^{-ikt}$ belongs to ${}^\circ(\ker L_p)$.

If there exists a sequence $x_n \to x_0$ such that $p(x_n) + ik \neq 0$, for all $n$, then a similar argument allows us to conclude that there exists a distribution of the form $[c_0\delta(x - x_0) + c_1\delta'(x - x_0)] \otimes e^{-ikt}$, belonging to ${}^\circ(\ker L_p)$.

Finally, the below result implies that no distribution of the form $[c_0\delta(x - x_0) + c_1\delta'(x - x_0)] \otimes e^{-ikt}$ belongs to the range of ${}^tL_p$. $\square$

**Lemma 4.5.** *If $a$ and $q$ are functions flat at 0, and $r$ is a non-negative integer, then there is no distribution $\mu$ satisfying $a\mu' + q\mu = \sum_{j=0}^{r} c_j\delta^{(r)} + c$ on an interval $(-\epsilon, \epsilon)$, for any $c, c_j \in \mathbb{C}$, $c_r \neq 0$ and $\epsilon > 0$.*

**Proof.** Suppose that there exists $\mu \in \mathcal{D}'(-\epsilon, \epsilon)$ such that

$$a\mu' + q\mu = \sum_{j=0}^{r} c_j\delta^{(r)} + c.$$

Then,

$$\langle a\mu' + q\mu, \phi\rangle = \sum_{j=0}^{r} c_j(-1)^j\phi^{(j)}(0) + c\int_{-\epsilon}^{\epsilon} \phi,$$

for all $\phi \in \mathcal{C}_c^\infty(-\epsilon, \epsilon)$ and, consequently, there is a constant $C > 0$ and a positive integer $m$ such that

$$\left|\sum_{j=0}^{r} c_j(-1)^j\phi^{(j)}(0) + c\int_{-\epsilon/2}^{\epsilon/2} \phi\right| \leq C\sum_{j=0}^{m} \sup_{|x|<\epsilon} |(q\phi)^{(j)}(x) - (a\phi)^{(j+1)}(x)|, \tag{4.8}$$

for all $\phi \in \mathcal{C}_c^\infty(-\epsilon, \epsilon)$ such that $\operatorname{supp}\phi \subset [-\epsilon/2, \epsilon/2]$.

Let $\psi_0 \in \mathcal{C}_c^\infty(-\epsilon/2, \epsilon/2)$ be a non-negative function and identically 1 on a neighborhood of $[-\epsilon/4, \epsilon/4]$. Define $\phi_0 \in \mathcal{C}_c^\infty(-\epsilon/2, \epsilon/2)$ by $\phi_0(x) = x^r\psi_0(x)$.

The derivatives of $\phi_0$ satisfy $\phi_0^{(j)}(0) = 0$ if $j \neq r$, and $\phi_0^{(r)}(0) = r!$.

For $n \in \mathbb{N}$, define $\phi_n(x) = \phi_0(nx)$. Then $\phi_n \in \mathcal{C}_c^\infty(-\epsilon/2n, \epsilon/2n)$ and estimate (4.8) becomes (for each $n \in \mathbb{N}$)

$$\left|c_r(-1)^r r!n^r + \frac{c}{n}\int_{-n\epsilon/2}^{n\epsilon/2} \phi_0\right| \leq C\sum_{j=0}^{m} \sup_{|x|\leq\epsilon/2n} |(q\phi_n)^{(j)}(x) - (a\phi_n)^{(j+1)}(x)|;$$

hence,

$$|c_r|r!n^r/2 \leq C\sum_{j=0}^{m} \sup_{|x|\leq\epsilon/n} |(q\phi_n)^{(j)}(x) - (a\phi_n)^{(j+1)}(x)|, \tag{4.9}$$

for all $n \in \mathbb{N}$ large enough.

By writing $q(x) = x^{m+1}\tilde{q}(x)$, $x \in (-\epsilon, \epsilon)$, we obtain

$$\sum_{j=0}^{m} \sup_{|x|\leq\epsilon/2n} |(q\phi_n)^{(j)}(x)| = \sum_{j=0}^{m} \sup_{|x|\leq\epsilon/2n} |(d/dx)^j[x^{m+1}\tilde{q}(x)\phi_0(nx)]|.$$

Taking $n$ large enough, Leibniz's rule implies

$$\sum_{j=0}^{m} \sup_{|x|\leq\epsilon/2n} |(q\phi_n)^{(j)}(x)| \leq$$

$$\sum_{j=0}^{m}\sum_{\ell=0}^{j}\binom{j}{\ell} \sup_{|x|\leq\epsilon/2n} |(d/dx)^{j-\ell}(x^{m+1})(d/dx)^{\ell}(\tilde{q}(x)\phi_0(nx))| \leq$$

$$(m+1)!\sum_{j=0}^{m}(j!)^2\sum_{\ell=0}^{j}\sum_{k=0}^{\ell} \sup_{|x|\leq\epsilon/2n} \left|x^{m+1-j+\ell}\tilde{q}^{(k)}(x)n^{\ell-k}\phi_0^{(\ell-k)}(nx)\right| \leq$$

$$(m+1)!\sum_{j=0}^{m}(j!)^2 n^{j-m-1}\sum_{\ell=0}^{j}\sum_{k=0}^{\ell} \sup_{|x|\leq\epsilon/2} |\tilde{q}^{(k)}(x)| \left\|\phi_0^{(\ell-k)}\right\| \leq$$

$$n^{-1}(m+1)!(m!)^2\sum_{j=0}^{m}\sum_{\ell=0}^{j}\sum_{k=0}^{\ell} \sup_{|x|\leq\epsilon/2} |\tilde{q}^{(k)}(x)| \left\|\phi_0^{(\ell-k)}\right\|.$$

Writing $a(x) = x^{m+2}\tilde{a}(x)$, we see that a similar estimate is satisfied by the term

$$\sum_{j=0}^{m} \sup_{|x|\leq\epsilon/2n} |(a\phi_n)^{(j+1)}(x)|.$$

Therefore, by (4.9) we obtain $\tilde{C} > 0$ such that

$$|c_r| r! n^{r+1} \leq \tilde{C},$$

for all $n \in \mathbb{N}$ large enough, which is a contradiction. □

**Corollary 4.6.** *Suppose that $a$ does not vanish identically and there exists $k \in \mathbb{Z}$ such that both $a$ and $p + ik$ are flat at a same point. Then operator $L_p$ is not globally solvable on $\mathcal{D}'(\mathbb{T}^2)$.*

The proof is quite similar to the proof of Proposition 4.4 and it will be omitted.

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## Appendix A

Here we perform the computations which show that the solution $\widehat{u}(\cdot, k)$, presented in the proof of Theorem 3.4, subcase 5.1, is smooth on $(x_{j_0}, x_{j_n+\sigma_n})$. Recall that $a$ vanishes only of finite order and $a^{-1}(0)$ is a finite set $\{x_1 < x_2 < \cdots < x_N\}$. In addition, $n_j$ is the order of vanishing of $a$ at $x_j$, and we have a chain of intervals

$$(x_{j_0}, x_{j_0+\sigma_0}] \cup [x_{j_1}, x_{j_1+\sigma_1}) \cup \cdots \cup [x_{j_n}, x_{j_n+\sigma_n}),$$

with $n \leq N$, such that $x_{j_{m+1}} = x_{j_m+\sigma_m}$, $m = 0, \ldots, n-1$, and on each $(x_{j_m}, x_{j_m+\sigma_m})$ the function $p + ik$ ($k \in \mathbb{Z}$) vanishes as much as $a$. We set $x_{j_{n+1}} = x_{j_n+\sigma_n}$. At each $x_{j_m}$, $m = 1, \ldots, n$, the function $\Im p + k$ vanishes as much as $a$, while $\Re p$ vanishes of order $n_{j_m} - 1$ and $\Re p^{(n_{j_m}-1)}(x_{j_m}) + (\ell_m/n_{j_m})a^{(n_{j_m})}(x_{j_m}) = 0$, for some $\ell_m \in \{1, 2, \ldots\}$. On each $(x_{j_m}, x_{j_{m+1}})$, we pick $\eta_m \in (x_{j_m}, x_{j_m+\sigma_m})$ and we write

$$E_{m,k}(x) = \exp\left\{ -\int_{\eta_m}^{x} \frac{p + ik}{a}(s)ds \right\},$$

and

$$\widehat{u}(x, k) = E_{m,k}(x) \left( C_m + \int_{x_{j_m}}^{x} \frac{\widehat{f}(y, k)}{a(y)E_{m,k}(y)} dy \right),$$

with

$$C_m = \frac{\partial_-^{\ell_m} E_{m-1,k}(x_{j_m})}{\partial_+^{\ell_m} E_{m,k}(x_{j_m})} \left( C_{m-1} + \int_{x_{j_{m-1}}}^{x_{j_m}} \frac{\widehat{f}(y, k)}{a(y)E_{m-1,k}(y)} dy \right).$$

We will show that $\widehat{u}(\cdot, k)$ is smooth at each $x_{j_m}$, $m = 1, \ldots, n$.

On a neighborhood of $x_{j_m}$, we write $\Re p(s) = (s - x_{j_m})^{n_{j_m}-1}\rho(s)$ and $a(s) = (s - x_{j_m})^{n_{j_m}}\alpha(s)$, with $\rho(x_{j_m}) \neq 0$ and $\alpha(x_{j_m}) \neq 0$. Since $\Re p^{(n_{j_m}-1)}(x_{j_m}) = (-\ell_m/n_{j_m})a^{(n_{j_m})}(x_{j_m})$, we obtain $(\rho/\alpha)(x_{j_m}) = -\ell_m$. Hence $(\rho/\alpha)(s) = (s - x_{j_m})\beta(s) - \ell_m$.

It follows that $(\Re p/a)(s) = \beta(s) - \ell_m(s - x_{j_m})^{-1}$, on a neighborhood of $x_{j_m}$, in which $\beta$ is smooth.

For $\epsilon > 0$ small enough and $x \in (x_{j_m}, x_{j_m} + \epsilon)$, we write

$$\begin{aligned} E_{m,k}(x) =& \phi_{m,k}(x) \exp\left\{ \int_{x_{j_m}+\epsilon}^{x} \frac{\ell_m}{s - x_{j_m}} ds \right\} \\ =& \phi_{m,k}(x)(x - x_{j_m})^{\ell_m} \epsilon^{-\ell_m}, \end{aligned}$$

where

$$\phi_{m,k}(x) = \exp\left\{ -\int_{\eta_m}^{x_{j_m}+\epsilon} \frac{p + ik}{a}(s)ds \right\} \exp\left\{ -\int_{x_{j_m}+\epsilon}^{x} \beta(s) + i\frac{\Im p(s) + k}{a(s)} ds \right\}$$

is smooth on a neighborhood of $x_{j_m}$.

We have $\partial_+^{\ell} E_{m,k}(x_{j_m}) = 0$ if $\ell < \ell_m$, and

$$\partial_+^{\ell} E_{m,k}(x_{j_m}) = \frac{\ell!}{(\ell - \ell_m)!} \partial^{\ell-\ell_m} \phi_{m,k}(x_{j_m}) \epsilon^{-\ell_m}, \text{ if } \ell \geq \ell_m.$$

In particular

$$\frac{\partial_+^{\ell} E_{m,k}(x_{j_m})}{\partial_+^{\ell_m} E_{m,k}(x_{j_m})} = \binom{\ell}{\ell_m} \frac{\partial^{\ell-\ell_m} \phi_{m,k}(x_{j_m})}{\phi_{m,k}(x_{j_m})}, \text{ for all } \ell \geq \ell_m.$$

Setting $\gamma(s) = -\beta(s) - i\frac{\Im p(s)+k}{a(s)}$, the above identity shows that there exists a polynomial $q$ in $(\ell - \ell_m)$-variables which satisfies

$$\frac{\partial_+^\ell E_{m,k}(x_{j_m})}{\partial_+^{\ell_m} E_{m,k}(x_{j_m})} = \binom{\ell}{\ell_m} q(\gamma(x_{j_m}), \gamma'(x_{j_m}), \dots, \gamma^{(\ell-\ell_m-1)}(x_{j_m})).$$

With a similar procedure, we also obtain $\partial_-^\ell E_{m-1,k}(x_{j_m}) = 0$ if $\ell < \ell_m$, and

$$\frac{\partial_-^\ell E_{m-1,k}(x_{j_m})}{\partial_-^{\ell_m} E_{m-1,k}(x_{j_m})} = \binom{\ell}{\ell_m} q(\gamma(x_{j_m}), \gamma'(x_{j_m}), \dots, \gamma^{(\ell-\ell_m-1)}(x_{j_m})),$$

if $\ell \geq \ell_m$.

It follows that

$$\frac{\partial_+^\ell E_{m,k}(x_{j_m})}{\partial_+^{\ell_m} E_{m,k}(x_{j_m})} = \frac{\partial_-^\ell E_{m-1,k}(x_{j_m})}{\partial_-^{\ell_m} E_{m-1,k}(x_{j_m})}, \tag{A.1}$$

for all $\ell \geq \ell_m$.

Since $\widehat{f}(\cdot, k)$ is flat at $x_{j_m}$, by the expression of $\widehat{u}(\cdot, k)$ on $(x_{j_{m-1}}, x_{j_m})$ we obtain $\partial_-^\ell \widehat{u}(x_{j_m}, k) = 0$, if $\ell < \ell_m$, and

$$\partial_-^\ell \widehat{u}(x_{j_m}, k) = \partial_-^\ell E_{m-1,k}(x_{j_m}) \left( C_{m-1} + \int_{x_{j_{m-1}}}^{x_{j_m}} \frac{\widehat{f}(y,k)}{a(y) E_{m-1,k}(y)} dy \right),$$

if $\ell \geq \ell_m$.

Similarly, $\partial_+^\ell \widehat{u}(x_{j_m}, k) = 0$ if $\ell < \ell_m$, and $\partial_+^\ell \widehat{u}(x_{j_m}, k) = \partial_+^\ell E_{m,k}(x_{j_m}) C_m$ if $\ell \geq \ell_m$.

Therefore, the definition of $C_m$ and identity (A.1) yield that $\widehat{u}(\cdot, k)$ is smooth at $x_{j_m}$.

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