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Improvement on the polynomial stability for a Timoshenko system with type III thermoelasticity

Marcio A. Jorge Silva [*,1], Sandro B. Pinheiro [2]

Department of Mathematics, State University of Londrina, 86057-970 Londrina, PR, Brazil

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A B S T R A C T

In the present work our main goal is to improve the polynomial decay obtained recently by Santos and Almeida (2017) for a Timoshenko system with type III thermoelasticity. More specifically, in the generic case of different wave speeds of propagation, it is proved by the authors that problem is polynomially stable with decay rate $t^{-1/4}$ for the Dirichlet boundary condition. Here, our objective is to consider the same problem and prove, still in the general situation of different wave speeds, that the decay rate for the Dirichlet boundary condition is $t^{-1/2}$, which consists in a faster decay than the previous one.

1. Introduction

In this work we are going to deal with the following Timoshenko model with thermoelasticity of type III coupled on the shear force:

$$\rho_1\varphi_{tt} - k(\varphi_x+\psi)_x + \sigma\,\theta_{tx} = 0 \quad \text{in} \quad (0,l)\times\mathbb{R}^+, \tag{1.1}$$

$$\rho_2\psi_{tt} - b\,\psi_{xx} + k(\varphi_x+\psi) - \sigma\,\theta_t = 0 \quad \text{in} \quad (0,l)\times\mathbb{R}^+, \tag{1.2}$$

$$\rho_3\theta_{tt} - \delta\theta_{xx} - \gamma\theta_{xxt} + \sigma(\varphi_{xt}+\psi_t) = 0 \quad \text{in} \quad (0,l)\times\mathbb{R}^+, \tag{1.3}$$

subject to initial conditions

$$\begin{gathered}\varphi(\cdot,0)=\varphi_0(\cdot),\quad \varphi_t(\cdot,0)=\varphi_1(\cdot),\quad \psi(\cdot,0)=\psi_0(\cdot),\quad \psi_t(\cdot,0)=\psi_1(\cdot),\\ \theta(\cdot,0)=\theta_0(\cdot),\quad \theta_t(\cdot,0)=\theta_1(\cdot)\quad \text{in}\quad (0,l),\end{gathered} \tag{1.4}$$

and either boundary conditions the full Dirichlet one

$$\varphi(0,t)=\varphi(l,t)=\psi(0,t)=\psi(l,t)=\theta(0,t)=\theta(l,t)=0,\quad t\geq 0, \tag{1.5a}$$

* Corresponding author.
E-mail address: marcioajs@uel.br (M.A. Jorge Silva).

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or the mixed Neumann–Dirichlet one

$$\varphi_x(0,t) = \varphi_x(l,t) = \psi(0,t) = \psi(l,t) = \theta(0,t) = \theta(l,t) = 0, \quad t \geq 0, \tag{1.5b}$$

where $\rho_1, \rho_2, \rho_3, k, b, \delta, \gamma$, σ are positive coefficients, whose physical meanings are very well understood and come from the material that composes a beam with length $l > 0$, and the unknown functions φ, ψ and θ are related to transversal displacement, rotation angle and temperature, respectively.

The above model (1.1)–(1.3) is fully derived by Santos and Almeida Júnior [1] by using the classical governing motions for Timoshenko beams [2] and constitutive thermal law coupled on the shear force where the heat flux conduction has its origins in the Green and Naghdi theories, here called as "type III thermoelasticity", see for instance [3,4]. In the occasion, in [1] the authors considered problem (1.1)–(1.3) with initial–boundary conditions (1.4)–(1.5a) and, instead of (1.5b), they took into account the following mixed Dirichlet–Neumann boundary condition

$$\varphi(0,t) = \varphi(l,t) = \psi_x(0,t) = \psi_x(l,t) = \theta_x(0,t) = \theta_x(l,t) = 0, \quad t \geq 0. \tag{1.6}$$

For more details on the derivation of the model (1.1)–(1.3), we refer to [1, Section 1]. In its remaining sections, the authors proved that the stability of problem (1.1)–(1.4) with boundary conditions (1.5a) or (1.6) depends upon the following parameter referred to the difference of wave speeds of propagation:

$$\chi := \frac{k}{\rho_1} - \frac{b}{\rho_2}. \tag{1.7}$$

More precisely, from [1, Theorem 5.9] one sees that problem (1.1)–(1.4) with boundary condition (1.6) is exponentially stable if and only if $\chi = 0$ in (1.7). For (1.5a), the case $\chi = 0$ is only a necessary assumption for exponential stability. In addition, according to [1, Theorem 6.2] when one considers different speeds of wave propagation, that is, in the case $\chi \neq 0$, then problem (1.1)–(1.4) is only polynomially stable, with decay rates given by:

Case 1. Optimal decay rate $t^{-1/2}$ for (1.6);
Case 2. Slower decay rate $t^{-1/4}$ for (1.5a).

Similar results are also achieved by Fatori et al. [5] for a Timoshenko system with type III thermoelasticity coupled on bending moment. See, for instance, Sections 3 and 4 in [5].

Therefore, motivated by [1,5] and since there is no reason at all to conclude why *Case 1* provides a faster decay than *Case 2* above, our main goal in the present paper is to prove that problem (1.1)–(1.5a) is polynomially stable with decay rate $t^{-1/2}$ (corresponding to the optimal one) in the case $\chi \neq 0$. Moreover, the same result extends to boundary condition (1.5b) and any other boundary conditions for which the system is well posed. Such statements will be clarified in Section 2, where we give our main results, and Section 4. Hence, our achievement in Theorem 2.1 provides the improvement of the polynomial decay rate for (1.1)–(1.5a) when compared with [1, Theorem 6.2] and complements the results of [5] in what concerns the same polynomial decay rate for different boundary conditions, by including the fully Dirichlet case (1.5a).

To the proof of Theorem 2.1, which will only be completed in Section 3, the main difference with the proof of Theorem 6.2 in [1] is that there the authors used a one dimensional version of the Trace Theorem (see on page 663 of [1]) to handle boundary point-wise terms in the case of boundary condition (1.5a). This procedure leads them to a "poor" estimate in the case of such a boundary condition, see e.g. [1, Lemma 5.4], and has been done by using the same techniques as used [5] to handle boundary point-wise terms. Here, differently from [1,5], our proofs are based on local estimates through cut-off functions combined with a recent observability inequality introduced in [6,7] for the resolvent equation related to non-homogeneous Timoshenko systems. Therefore, such a technique allowed us to achieve the same estimate no matter which boundary condition we are taking into account. As a consequence, an improved result on polynomial stability is achieved in the case $\chi \neq 0$. The case $\chi = 0$ is also studied for the sake of completeness. All results and their proofs are given in Sections 2 and 3. We believe the same methodology could be properly extended to other Timoshenko systems with thermal law coupled on bending moment, for example, those considered with type III thermoelasticity [5,8,9] as well as those with distinguished thermal laws [10–13].

2. Semigroup approach and main results

Let us initially consider the phase spaces

$$\mathcal{H}_1 = H_0^1(0,l) \times L^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \quad \text{for} \quad (1.5a),$$

and

$$\mathcal{H}_2 = H_*^1(0,l) \times L_*^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \quad \text{for} \quad (1.5b),$$

where $H_*^1(0,l) = H^1(0,l) \cap L_*^2(0,l)$ and $L_*^2(0,l) = \left\{ u \in L^2(0,l);\ \frac{1}{l}\int_0^l u(x)\,dx = 0 \right\}$. It is well-known that $\mathcal{H}_j$, for each $j = 1, 2$, is a Hilbert space endowed with norm

$$\|U\|_{\mathcal{H}_j}^2 = \int_0^l \Big[\rho_1 |\Phi|^2 + \rho_2 |\Psi|^2 + \rho_3 |\Theta_x|^2 + b|\psi_x|^2 + k|\varphi_x + \psi|^2 + \delta|\theta_x|^2 \Big] dx,$$

for $U = (\varphi, \Phi, \psi, \Psi, \theta, \Theta)^T \in \mathcal{H}_j$, and respective scalar product $(\cdot,\cdot)_{\mathcal{H}_j}$.

As in [1, Section 3] we convert system (1.1)–(1.5) into the following abstract problem

$$\begin{cases} U_t = \mathcal{A}_j U, \quad t > 0, \\ U(0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_1)^T := U_0, \end{cases} \tag{2.1}$$

where $\mathcal{A}_j : D(\mathcal{A}_j) \subset \mathcal{H}_j \to \mathcal{H}_j$ is defined by

$$\mathcal{A}_j U = \begin{pmatrix} \Phi \\ \dfrac{k}{\rho_1}(\varphi_x + \psi)_x - \dfrac{\sigma}{\rho_1}\,\Theta_x \\ \Psi \\ \dfrac{b}{\rho_2}\,\psi_{xx} - \dfrac{k}{\rho_2}(\varphi_x + \psi) + \dfrac{\sigma}{\rho_2}\,\Theta \\ \Theta \\ \dfrac{1}{\rho_3}(\delta\theta_{xx} + \gamma\Theta_{xx}) - \dfrac{\sigma}{\rho_3}(\Phi_x + \Psi) \end{pmatrix}, \tag{2.2}$$

for any $U = (\varphi, \Phi, \psi, \Psi, \theta, \Theta)^T \in D(\mathcal{A}_j)$, with domain

$$D(\mathcal{A}_1) = \left\{ U \in \mathcal{H}_1 \mid \Phi, \Psi, \Theta \in H_0^1(0,l),\ \varphi, \psi, \delta\theta + \gamma\Theta \in H^2(0,l) \right\} \quad \text{for } (1.5a),$$

and

$$D(\mathcal{A}_2) = \left\{ U \in \mathcal{H}_2 \mid \Phi \in H_*^1(0,l),\ \Psi, \Theta \in H_0^1(0,l),\ \varphi, \psi, \delta\theta + \gamma\Theta \in H^2(0,l) \right\} \quad \text{for } (1.5b).$$

As stated in [1, Theorem 3.1], $\mathcal{A}_j$ is the infinitesimal generator of a C_0-semigroup of contractions $T(t) = e^{\mathcal{A}_j t}$ on $\mathcal{H}_j$. Thus, problem (2.1) has a unique solution according to Pazy [14].

Next we present our main result on polynomial stability.

Theorem 2.1. *Under the above notations and assuming $\chi \neq 0$ in (1.7), then there exists a constant $C_n > 0$ independent of $U_0 \in D({\mathcal{A}_j}^n)$, $n \geq 1$ integer, such that the semigroup solution $U(t) = e^{\mathcal{A}_j t} U_0$, for each $j = 1, 2$, satisfies*

$$\|U(t)\|_{\mathcal{H}_j} \leq \frac{C_n}{t^{n/2}} \|U_0\|_{D({\mathcal{A}_j}^n)}, \quad t \to +\infty. \tag{2.3}$$

Just to complement the result given in [1, Theorem 5.9], we state a similar result on exponential stability for (1.1)–(1.5) under the equal wave speeds assumption.

Theorem 2.2. *Under the above notations and assuming* $\chi = 0$ *in* (1.7)*, then there exist constants* $C, \gamma > 0$ *independent of* $U_0 \in \mathcal{H}_j$ *such that the semigroup solution* $U(t) = e^{\mathcal{A}_j t} U_0$ *decays as*

$$\|U(t)\|_{\mathcal{H}_j} \leq Ce^{-\gamma t} \|U_0\|_{\mathcal{H}_j}, \quad t > 0. \tag{2.4}$$

The proofs of Theorems 2.1 and 2.2 will be completed at the end of the next section.

3. Proofs

Let us start by considering the resolvent equation

$$i\beta U - \mathcal{A}_j U = F, \quad j = 1, 2, \tag{3.1}$$

with $U = (\varphi, \Phi, \psi, \Psi, \theta, \Theta)^T$, $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T$ and $\mathcal{A}_j$ defined in (2.2).

Lemma 3.1. *Under the above notations, we have* $i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$*, where* $\rho(\mathcal{A}_j)$ *is resolvent set of* $\mathcal{A}_j$*,* $j = 1, 2$*.*

Proof. The proof is similar to that given in [1, Theorem 5.1]. See also [15, Lemma 4.5]. □

Lemma 3.2. *Under the above notations, there exists a constant* $C > 0$ *such that*

$$\|\Theta_x\|_{L^2}^2 \leq C \|U\|_{\mathcal{H}_j} \|F\|_{\mathcal{H}_j}. \tag{3.2}$$

Proof. First we recall that $\mathcal{A}_j$ is dissipative on $\mathcal{H}_j$ with

$$\text{Re } (\mathcal{A}U, U)_{\mathcal{H}_j} = -\gamma \int_0^l |\Theta_x|^2 \, dx \leq 0, \quad \forall \, U \in D(\mathcal{A}_j), \; j = 1, 2. \tag{3.3}$$

Then, from (3.1) and (3.3) we obtain (3.2) readily for some $C > 0$. □

Lemma 3.3. *Under the above notations, there exists a constant* $C > 0$ *such that*

$$\|\theta_x\|_{L^2}^2, \, \|\delta\theta_x + \gamma\Theta_x\|_{L^2}^2 \leq C \|U\|_{\mathcal{H}_j} \|F\|_{\mathcal{H}_j}. \tag{3.4}$$

Proof. It follows easily from the fifth component of the resolvent equation (3.1) and (3.2). See, for instance, [15, Lemma 4.7]. □

Remark 3.4 (*Cut-off Functions*)**.** Now, it is the precise moment where our arguments are different from [1,5]. For instance, whereas on page 663 of [1] the authors use a Trace Theorem to handle boundary point-wise terms, here we are going to deal with local estimates by using auxiliary cut-off functions motivated by [6,7]. In this way, we do not get different estimates for each boundary term as obtained, for example, in [5, Lemma 3.9] and [1, Lemma 5.4].

Let us consider $l_0 \in (0, l)$ and $\delta > 0$ arbitrary numbers such that $(l_0 - \delta, l_0 + \delta) \subset (0, l)$, and a function $s \in C^2(0, l)$ satisfying

$$\text{supp } s \subset (l_0 - \delta, l_0 + \delta), \quad 0 \leq s(x) \leq 1, \quad x \in (0, l), \tag{3.5}$$

$$s(x) = 1 \quad \text{for} \quad x \in [l_0 - \delta/2, l_0 + \delta/2]. \tag{3.6}$$

Lemma 3.5. *Under the above notations, there exists a constant $C > 0$ such that*

$$\begin{aligned}\int_{l_0-\delta}^{l_0+\delta} s\left(|\varphi_x+\psi|^2+|\Phi|^2\right)dx \leq {} & \frac{C}{|\beta|^{3/2}}\left(\|\delta\theta_x+\gamma\Theta_x\|_{L^2}^{1/2}\|U\|_{\mathcal{H}_j}^{1/2}+\|U\|_{\mathcal{H}_j}^{1/2}\|F\|_{\mathcal{H}_j}^{1/2}\right)\|U\|_{\mathcal{H}_j}\\ & + \frac{C}{|\beta|}\|\delta\theta_x+\gamma\Theta_x\|_{L^2}\|U\|_{\mathcal{H}_j}\\ & + \frac{C}{|\beta|^{4/3}}\|\delta\theta_x+\gamma\Theta_x\|_{L^2}^{2/3}\|U\|_{\mathcal{H}_j}^{4/3}+C\|U\|_{\mathcal{H}_j}\|F\|_{\mathcal{H}_j}+C\|F\|_{\mathcal{H}_j}^2.\end{aligned} \tag{3.7}$$

In addition, given $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$\int_{l_0-\delta/2}^{l_0+\delta/2}\left(|\varphi_x+\psi|^2+|\Phi|^2\right)dx \leq \epsilon\|U\|_{\mathcal{H}_j}^2+C_\epsilon\|F\|_{\mathcal{H}_j}^2. \tag{3.8}$$

Proof. The proof is done by following verbatim the same arguments as in [6, Proposition 3.3]. See also [15, Lemma 4.8] for all computations. □

Now we consider another auxiliary cut-off function $s_1 \in C^2(0,l)$ such that

$$\operatorname{supp} s_1 \subset (l_0-\delta/2, l_0+\delta/2), \quad 0 \leq s_1(x) \leq 1, \quad x \in (0,l), \tag{3.9}$$

$$s_1(x) = 1 \quad \text{for} \quad x \in [l_0-\delta/3, l_0+\delta/3]. \tag{3.10}$$

Since we do not have a dissipative mechanism associated with bending moment, then the next estimates and their consequences will depend on the parameter χ set in (1.7).

Lemma 3.6. *Under the above notations and considering $\epsilon > 0$, we claim:*

(i) If $\chi \neq 0$, then there exists a constant $C_\epsilon > 0$ such that

$$\int_{l_0-\delta/3}^{l_0+\delta/3}\left(|\psi_x|^2+|\Psi|^2\right)dx \;\leq\; \epsilon\|U\|_{\mathcal{H}_j}^2+C_\epsilon|\beta|^4\|F\|_{\mathcal{H}_j}^2. \tag{3.11}$$

(ii) If $\chi = 0$, then there exists a constant $C_\epsilon > 0$ such that

$$\int_{l_0-\delta/3}^{l_0+\delta/3}\left(|\psi_x|^2+|\Psi|^2\right)dx \;\leq\; \epsilon\|U\|_{\mathcal{H}_j}^2+C_\epsilon\|F\|_{\mathcal{H}_j}^2. \tag{3.12}$$

Proof. The computations can be done similarly to [6, Corollary 3.6]. We also refer to [15] (see Lemmas 4.9 and 4.10 and also Corollary 4.11 therein) for all computations. □

Completion of the proof of Theorem 2.1. It is combination of the previous lemmas, along with Lemma 3.6-(i). The approach is similar to [6, Theorem 4.1]. See also on page 110 in [15]. □

Completion of the proof of Theorem 2.2. It is a particular consequence of the previous case, by applying now Lemma 3.6-(ii). It can be done analogously to [6, Theorem 4.3]. See also on page 109 in [15]. □

4. Conclusion

Let us finish by considering some concluding remarks on Theorems 2.1 and 2.2 as follows.

(i) *Improvement.* Theorem 2.1 asserts that, in general, the problem (1.1)–(1.5) is polynomially stable with rates depending on the regularity of the initial data, but independent of the boundary conditions in (1.5). The polynomial decay rate $t^{-1/2}$ achieved in (2.3), for $U_0 \in D(\mathcal{A}_j)$, is independent of the boundary conditions in (1.5). Therefore, for the case of full Dirichlet condition (1.5a), this achievement improves the decay $t^{-1/4}$ obtained in [1, Theorem 6.2].

(ii) *Optimality.* The decay rate $t^{-1/2}$ is optimal for the boundary condition (1.5b) and the proof is similar to that given in [1, Section 6] for the boundary condition (1.6). Consequently, from this and Theorem 2.2, the Timoshenko system (1.1)–(1.4) with boundary condition (1.5b) is exponential stable if and only if $\chi = 0$, which corresponds to [1, Theorem 5.9].

(iii) *Generality.* Our approach on local estimates given in Section 3 along with the observability inequality allows us to conclude the same polynomial decay rate (corresponding to the optimal one $t^{-1/2}$) in the case $\chi \neq 0$ and the same exponential stability in the case $\chi = 0$ for any other different boundary conditions whose problem is well-posed, including (1.6).

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