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ON A BEAM MODEL RELATED TO FLIGHT STRUCTURES WITH NONLOCAL ENERGY DAMPING

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Abstract. This paper deals with new results on existence, uniqueness and stability for a class of nonlinear beams arising in connection with nonlocal dissipative models for flight structures with *energy damping* first proposed by Balakrishnan-Taylor [2]. More precisely, the following n-dimensional model is addressed

$$u_{tt} - \kappa\Delta u + \Delta^2 u - \gamma\left[\int_\Omega \left(|\Delta u|^2 + |u_t|^2\right) dx\right]^q \Delta u_t + f(u) = 0 \ \text{ in } \ \Omega\times\mathbb{R}^+,$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary, the coefficient of extensibility κ is nonnegative, the damping coefficient γ is positive and $q\geq 1$. The nonlinear source $f(u)$ can be seen as an external forcing term of lower order. Our main results feature global existence and uniqueness, polynomial stability and a non-exponential decay prospect.

1. Introduction.

Let $\Omega\subset\mathbb{R}^n$ be a domain with smooth boundary $\Gamma=\partial\Omega$ and $\mathbb{R}^+=(0,\infty)$. In this paper we study the following beam model with nonlocal (Balakrishnan-Taylor) energy damping

$$u_{tt} - \kappa\Delta u + \Delta^2 u - \gamma\left[\int_\Omega \left(|\Delta u|^2 + |u_t|^2\right) dx\right]^q \Delta u_t + f(u) = 0 \ \text{ in } \ \Omega\times\mathbb{R}^+, \quad (1)$$

$$u(x,0)=u_0(x) \quad \text{and} \quad u_t(x,0)=u_1(x), \quad x\in\Omega, \quad (2)$$

where $\kappa\geq 0$ (for simplicity[1]), $\gamma>0$ and $q\geq 1$. The source term $f(u)$ is added as a nonlinear lower order perturbation to deal with a more general problem. We

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[1]As a matter of fact, the extensibility coefficient could be taken as $\kappa\geq-\lambda_1^{1/2}$, where λ_1 is the first eigenvalue of the bi-harmonic operator Δ^2 appearing later in (12). See, for instance, [18].

consider (1)-(2) subject to clamped boundary condition

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+, \tag{3}$$

where ν is the outward normal to Γ. In the next paragraphs, we are going to provide a brief inspiration to consider equation (1) and a comparison with the literature on the subject.

In 1989 Balakrishnan and Taylor [2], inspired by a method of *Energy Approximation* due to Zhang [30], presented some models for vibrating extensible beams with nonlocal nonlinear damping terms that are intrinsically connected with the study of damping phenomena in flight structures. For this reason, we genuinely associate such nonlocal beams models with those related to nonlocal energy damping in flight structures. Accordingly, one knows that the basic one-dimensional (in time) dynamic can be described by

$$x''(t) + w^2 x(t) + \gamma D\big(x(t), x'(t)\big) = 0, \tag{4}$$

where the variable $x(t)$ stands for the displacement of a point x of the flight structure at time t, w and γ are positive constants and $D(\cdot,\cdot)$ is a given function. As proposed in [2], the following class of damping models (also called *energy damping*) can be considered

$$D\big(x(t), x'(t)\big) = \left(\frac{w^2}{2}[x(t)]^2 + \frac{1}{2}[x'(t)]^2\right)^q x'(t), \quad q > 0. \tag{5}$$

As usual, denoting the energy functional by $E(t) = \frac{w^2}{2}[x(t)]^2 + \frac{1}{2}[x'(t)]^2$, then the model (4) – under the featured damping (5) – turns into the following

$$x''(t) + w^2 x(t) + \gamma [E(t)]^q x'(t) = 0, \tag{6}$$

which provides a new class of dissipative models with nonlocal energy damping. Indeed, formal computations on (6) gives

$$E'(t) \;=\; -\gamma [E(t)]^q [x'(t)]^2 \;\leq\; 0.$$

In the above direction, it is proposed in [2, Section 4] several models corresponding to one dimensional beam equations. For instance, we highlight the next prototype

$$u_{tt} - 2\zeta\sqrt{\lambda}u_{xx} + \lambda u_{xxxx} - \gamma\left[\int_{-L}^{L}\big(\lambda|u_{xx}|^2 + |u_t|^2\big)dx\right]^q u_{xxt} = 0, \tag{7}$$

where $u = u(x,t)$ represents the transversal deflection of a beam with length $2L > 0$ in the rest position, $\gamma > 0$ is a damping coefficient, ζ is a constant appearing in the approximation of Krylov-Bogoliubov and $\lambda = \frac{2\zeta w}{\sigma^2}$ with w being the mode frequency and σ^2 the spectral density of a Gaussian external force. We refer to equation (4.2) in [2] for the modeling of (7). See also [1, 5, 12, 15, 24, 29] for related models.

In order to raise the model (7) at a n-dimensional context, say on a bounded domain $\Omega \subset \mathbb{R}^n$, the normalized equation (7) can be read as follows

$$u_{tt} - \Delta u + \Delta^2 u - \left[\int_\Omega \big(|\Delta u|^2 + |u_t|^2\big)\,dx\right]^q \Delta u_t = 0, \tag{8}$$

which is a particular case of (1) with $f \equiv 0$. Due to the character of the nonlocal (and possibly degenerate) energy damping

$$D(u,u_t) = -\left[\int_\Omega \left(|\Delta u|^2 + |u_t|^2\right) dx\right]^q \Delta u_t := -\left[\|\Delta u\|^2 + \|u_t\|^2\right]^q \Delta u_t, \tag{9}$$

the stability of (8) is very little known in the literature. As a matter of fact, to our best knowledge, there is no result on stability for the degenerate beam model (8). Therefore, the main purpose of this paper is twofold: the first one deals with the existence and uniqueness of global solution to (1) (and consequently to (8)) whereas the second one is concerned with its stability when the time t goes to infinity. Due to technical tools, the results will be proved for $q \geq 1$ as clarified in the next sections.

Related models to the Balakrishnan-Taylor one like (7) or else (8) with respect to nonlocal damping terms can be found in the work by Woinowsky and Krieger [28] that derived the extensible beam equation

$$u_{tt} + \frac{EI}{\rho} u_{xxxx} - \left[\frac{H}{\rho} + \frac{EA}{2\rho L}\int_0^L |u_x|^2\, dx\right] u_{xx} = 0, \tag{10}$$

where L, E, I, ρ, H and A are physical constants and $u(x,t)$ represents the displacement. Such model has been widely studied in several mathematical aspects so far. We first quote the pioneer works by Dickey [11] and Ball [3, 4]. Just to name a few more, we refer to [6, 7, 8, 9, 10, 13, 20, 14, 17, 22, 23, 27, 31] where existence, uniqueness, stability and asymptotic behavior of solutions were object of study for problems like (10) under different damping mechanisms. Among several generalizations of (10) addressed in the above papers, we consider the next normalized n-dimensional model

$$u_{tt} + \Delta^2 u - M\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u - N\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u_t = 0, \tag{11}$$

where the nonlocal damping coefficient $N(\cdot)$ depends on the same quantity as the generalized Berger extensibility coefficient $M(\cdot)$, which differs from the energy damping coefficient in (9). The model (11) was studied by the authors in [17] whose inspiration arose from works by Lange and Perla Menzala [19] and Jorge Silva and Narciso [16]. In all these latter works [19, 16, 17], the nonlocal damping damping is bounded from below, which means that there exists a constant $\alpha_0 > 0$ such that $N(s) > \alpha_0 > 0$ for all $s \geq 0$. Thus, the damping in (11) is, in fact, nonlocal but it does not degenerate along the time. This constitutes the main difference between the damping of Kirchhoff type $-N\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u_t$ and the energy damping $D(u,u_t)$ in (9) for the Balakrishnan-Taylor model (8), which leads to a more challenging work in the sense of existence and stability of solution as clarified below. Another difference is that we are now supported by a consistent physical modeling in terms of the nonlocal damping coefficient due to the Balakrishnan and Taylor's paper [2]. Therefore, we can also say that our work extends, in some sense, the previews ones working with extensible models under nonlocal (non-degenerated) dissipations because we will not assume that the energy damping coefficient in (9) is bounded from below by a positive constant. It brings some technical difficulties even in the existence of solutions as detailed in Section 2. Moreover, the model (11) is known to be exponential stable according to [17, Remark 7], whereas such stability property seems to be a hard task to prove for (8) due to the degeneracy of the damping coefficient $\left[\|\Delta u\|^2 + \|u_t\|^2\right]^q$, $q \geq 1$, which clarified in Section 4. However, we prove that problem (8) has a stability of polynomial type in terms of the exponent $q \geq 1$, which is done in Section 3. As far as we know, this constitutes a first stability result for problems with energy damping like (7). For technical reasons, the case $0 < q < 1$ is not approached in the present work. We still observe that, in mathematical aspects, the case $q = 0$ can be considered in (8) and it represents a particular case of (11) when $N(s) = \alpha_0 > 0$ is a constant function. Thus, the system turns into exponentially stable, which is very well-known in the literature and it is not necessary to be treated here.

The remaining paper is organized as follows. In Section 2 we introduce some initial notations, state and prove the existence and uniqueness result to problem (1)-(3). In Section 3 we consider the stability result to the corresponding energy functional, see e.g. $E(t)$ set in (13) below, by proving its polynomial decay rate of type $(1+t)^{-\frac{1}{q}}$, $q \geq 1$, that depends on the size of the initial energy $E(0)$. Finally, in Section 4 we provide a new and interesting estimate to the energy, which indicates that $E(t)$ is not exponentially stable.

2. **Existence and uniqueness.** Let us begin by introducing the notations that will be used throughout the remaining work. We denote by $W_0 = L^2(\Omega)$, $W_1 = H_0^1(\Omega)$, $W_2 = H_0^2(\Omega)$ and for $m = 3, 4$, we consider $W_m = H^m(\Omega) \cap H_0^2(\Omega)$. The notation $(\cdot, \cdot)$ stands for the L^2-inner product and $\|\cdot\|_p$ denotes the L^p-norm, which for $p = 2$ is simplified to $\|\cdot\|$. Thus, $\|\nabla \cdot\|$ and $\|\Delta \cdot\|$ represent the norms in W_1 and W_2, respectively. When there is no possibility of confusion, we shall use the same notation $(\cdot, \cdot)$ to represent the duality pairing between any Banach space W and its dual W'. We also consider the following Hilbert phase spaces

$$\mathcal{H}_i = W_{i+2} \times W_i, \quad i = 0, 1, 2,$$

with standard inner products and norms. To simplify, for $i = 0$, we just denote $\mathcal{H}_0 = \mathcal{H}$ with norm

$$||(u, v)||^2_{\mathcal{H}} = \|\Delta u\|^2 + \|v\|^2, \quad (u, v) \in \mathcal{H}.$$

Denoting by $\lambda_1 > 0$ the first eigenvalue of the bi-harmonic operator Δ^2 with boundary condition (3), then it hold the inequalities

$$\lambda_1 \|u\|^2 \leq \|\Delta u\|^2, \quad \lambda_1^{1/2} \|\nabla u\|^2 \leq \|\Delta u\|^2, \quad \forall\, u \in W_2. \tag{12}$$

Under the above notations and setting $F(z) = \int_0^z f(\tau)d\tau$, the energy functional $E(t) = E(u(t), u_t(t))$ associated with problem (1)-(3) is given by

$$E(t) = \frac{1}{2}\left[\|u_t(t)\|^2 + \|\Delta u(t)\|^2 + \kappa\|\nabla u(t)\|^2\right] + \int_\Omega F(u(t))dx, \quad t \geq 0. \tag{13}$$

The existence and uniqueness of regular solution to problem (1)-(3) is presented below as well as an identity showing that the energy $E(t)$ is a non-increasing functional.

Theorem 2.1. *Let us assume that the coefficients satisfy $\kappa \geq 0$, $\gamma > 0$ and $q \geq 1$. In addition, we suppose that $f \in C^1(\mathbb{R})$ is a function with $f(0) = 0$ and there exist constants $C_{f'} > 0$ and $\theta \in [0, \lambda_1)$ such that*

$$|f'(u)| \quad \leq C_{f'}(1 + |u|^\rho), \quad u \in \mathbb{R}, \tag{14}$$

$$-\frac{\theta}{2}|u|^2 \leq F(u) \leq f(u)u + \frac{\theta}{2}|u|^2, \quad u \in \mathbb{R}. \tag{15}$$

where the growth exponent ρ satisfies

$$\rho \geq 0 \;\; \textit{if} \;\; 1 \leq n \leq 4 \quad \textit{or} \quad 0 \leq \rho \leq \frac{4}{n-4} \;\; \textit{if} \;\; n \geq 5. \tag{16}$$

Then, we have:

(i) *If* $(u_0, u_1) \in \mathcal{H}_2$*, then problem* (1)-(3) *has a regular (strong) solution* u *satisfying*

$$(u, u_t) \in L^\infty(0, T; \mathcal{H}_2) \quad \text{and} \quad u_{tt} \in L^\infty(0, T; W_0), \quad \forall\, T > 0. \tag{17}$$

(ii) *If* $U^1(t) = (u^1(t), u_t^1(t))$ *and* $U^2(t) = (u^2(t), u_t^2(t))$ *are regular solutions of* (1)-(3) *corresponding to* $U_0^1 = (u_0^1, u_1^1)$*,* $U_0^2 = (u_0^2, u_1^2)$*, respectively, then there exists a positive constant* $C = C(||U_0||_{\mathcal{H}_2}, ||U_1||_{\mathcal{H}_2}) > 0$ *such that*

$$||U^1(t) - U^2(t)||_{\mathcal{H}} \leq C||U_0^1 - U_0^2||_{\mathcal{H}}, \quad t \in [0, T]. \tag{18}$$

In particular, (18) *states that for each initial data there exists a unique regular solution.*

(iii) *The energy* $E(t)$ *given in* (13)*, defined over a regular solution, is non-increasing. More precisely, it satisfies identity*

$$E(t) + \gamma \int_s^t ||(u(\tau), u_t(\tau))||_{\mathcal{H}}^{2q} \|\nabla u_t(\tau)\|^2 \, d\tau = E(s), \quad t > s \geq 0. \tag{19}$$

Remark 1. Assumption (14) along with the Mean Value Theorem imply that there exists a constant $C_f > 0$ such that

$$|f(u)| \leq C_f(1 + |u|^\rho)|u|, \quad u \in \mathbb{R}. \tag{20}$$

The proof of Theorem 2.1 relies on the Faedo-Galerkin method, where we use compactness arguments as provided by Lions' book [21]. It will be given in the next subsections.

2.1. **Proof of Theorem 2.1**$-(i)$**.** Let $(\omega_j)_{j\in\mathbb{N}}$ be the complete orthonormal set of W_4 given by the eigenfunctions of Δ^2 with boundary condition (3) and consider

$$V_m = \text{Span}\{\omega_1, \omega_2, \dots, \omega_m\}$$

the subspace of W_4 generated by the first m elements of $(\omega_j)_{j\in\mathbb{N}}$. For each $m \in \mathbb{N}$, we can construct a function u^m given by

$$u^m(t) = \sum_{j=1}^m y_{jm}(t)\omega_j \in \text{Span}\{\omega_1, \dots, \omega_m\}, \quad t \in [0, T_m),$$

where (y_{jm}) is a local solution on $[0, T_m) \subset [0, T)$ of the following system of ODEs:

$$\begin{cases} (u_{tt}^m(t), \omega_j) + (\Delta u^m(t), \Delta\omega_j) - \kappa(\Delta u^m(t), \omega_j) \\ \quad -\gamma||(u^m(t), u_t^m(t))||_{\mathcal{H}}^{2q}(\Delta u_t^m(t), \omega_j) + (f(u^m(t)), \omega_j) = 0, \\ \quad u^m(0) = u_0^m, \quad u_t^m(0) = u_1^m, \quad j = 1, \dots, m. \end{cases} \tag{21}$$

In what follows, our first a priori estimate (see (26) below) allows to extend the local solution to the whole interval $[0, T)$. The following estimates of this section will be used to prove the existence of strong solution for (1)-(3).

In order to simplify the text, we are going to use the same parameter C to denote different positive constants that will appear in the computations, but we will also specify its dependence on time and initial data whenever necessary.

A Priori Estimate I. We first consider the approximate system (21) with

$$(u_0^m, u_1^m) \to (u_0, u_1) \quad \text{strongly in } \mathcal{H} = W_2 \times W_0. \tag{22}$$

Replacing ω_j by $u_t^m(t)$ in (21), then a straightforward computation yields

$$\frac{d}{dt}E^m(t) + \gamma||(u^m(t), u_t^m(t))||_{\mathcal{H}}^{2q}\|\nabla u_t^m(t)\|^2 = 0, \quad t \in [0, T_m), \tag{23}$$

where $E^m(t)$ is the energy functional (13) for Galerkin solutions u^m. Integrating (23) from 0 to $t \leq T_m$, we have

$$E^m(t) + \gamma \int_0^t ||(u^m(s), u_t^m(s))||_{\mathcal{H}}^{2q} \|\nabla u_t^m(s)\|^2 \, ds = E^m(0). \tag{24}$$

On the other hand, from (12) and (15), we get

$$\int_\Omega F(u^m) dx \geq -\frac{\theta}{2} \|u^m(t)\|^2 \geq -\frac{\theta}{2\lambda_1} \|\Delta u^m(t)\|^2.$$

Denoting $\omega := 1 - \frac{\theta}{\lambda_1} > 0$ and using the definition of the energy in (13), we obtain

$$E^m(t) \geq \frac{\omega}{2} \|\Delta u^m(t)\|^2 + \frac{1}{2} \|u_t^m(t)\|^2 \geq \frac{\omega}{2} ||(u^m(t), u_t^m(t))||_{\mathcal{H}}^2. \tag{25}$$

Combining (24) and (25), and using (22), we infer

$$||(u^m(t), u_t^m(t))||_{\mathcal{H}}^2 + \int_0^t ||(u^m(s), u_t^m(s))||_{\mathcal{H}}^{2q} \|\nabla u_t^m(s)\|^2 \, ds \leq \frac{E^m(0)}{\min\{\frac{\omega}{2}, \gamma\}} \leq C, \tag{26}$$

for $t \in [0, T_m)$, where $C := C(||(u_0, u_1)||_{\mathcal{H}}) > 0$ is a constant depending on weak initial data. Moreover, estimate (26) allows us to extend the local solution of the approximate problem to whole interval $[0, T)$, for any given $T > 0$, and using the same procedure to obtain (23)-(26) one sees that (26) holds true for all $t \in [0, T)$. Therefore, we conclude

$$(u^m, u_t^m) \quad \text{is bounded in} \quad L^\infty(0, T; \mathcal{H}), \tag{27}$$

with additional boundedness

$$||(u^m, u_t^m)||_{\mathcal{H}}^q |\nabla u_t^m| \quad \text{is bounded in} \quad L^2(0, T; L^2(\Omega)). \tag{28}$$

A Priori Estimate II. Now we consider the approximate problem (21) with

$$(u_0^m, u_1^m) \;\to\; (u_0, u_1) \quad \text{strongly in} \;\; \mathcal{H}_2 = W_4 \times W_2. \tag{29}$$

Deriving the approximate equation in (21) with respect to t and taking $w_j = u_{tt}^m(t)$ in the resulting expression, it results

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left[\|u_{tt}^m(t)\|^2 + \|\Delta u_t^m(t)\|^2 + \kappa \|\nabla u_t^m(t)\|^2 \right] \\ &+ \gamma ||(u^m(t), u_t^m(t))||_{\mathcal{H}}^{2q} \|\nabla u_{tt}^m(t)\|^2 = I_1 + I_2, \end{aligned} \tag{30}$$

for all $0 \leq t \leq T$, where we denote

$$\begin{aligned} I_1 &= \gamma \frac{d}{dt} ||(u^m(t), u_t^m(t))||_{\mathcal{H}}^{2q} \int_\Omega \Delta u_t^m(t) u_{tt}^m(t) dx, \\ I_2 &= -\int_\Omega f'(u^m(t))\, u_t^m(t)\, u_{tt}^m(t)\, dx. \end{aligned}$$

Let us estimate the terms on the right hand side of identity (30). Using the Hölder and Young's inequalities, we get

$$|I_1| \leq \frac{\gamma}{2} \left| \frac{d}{dt} ||(u^m(s), u_t^m(s))||_{\mathcal{H}}^{2q} \right| \left[\|u_{tt}^m(t)\|^2 + \|\Delta u_t^m(t)\|^2 \right]. \tag{31}$$

Applying (14), Hölder's inequality with $\frac{\rho}{2(\rho+1)}+\frac{1}{2(\rho+1)}+\frac{1}{2}=1$, the embedding $W_2 \hookrightarrow L^{2(\rho+1)}(\Omega)$, estimate (26) and Young's inequality, we have

$$\begin{aligned}
|I_2| &\leq C_{f'}\int_\Omega [1+|u^m(t)|^\rho]|u_t^m(t)||u_{tt}^m(t)|dx \\
&\leq 2^{\frac{3\rho+2}{2(\rho+1)}}C_{f'}\left[|\Omega|^{\frac{\rho}{2(\rho+1)}}+\|u^m(t)\|_{2(\rho+1)}^\rho\right]\|u_t^m(t)\|_{2(\rho+1)}\|u_{tt}^m(t)\| \\
&\leq 2^{\frac{3\rho+2}{2(\rho+1)}}C_{f'}C_{|\Omega|}\left[|\Omega|^{\frac{\rho}{2(\rho+1)}}+C_{|\Omega|}^\rho\|\Delta u^m(t)\|_2^\rho\right]\|\Delta u_t^m(t)\|\|u_{tt}^m(t)\| \qquad (32)\\
&\leq 2^{\frac{\rho}{2(\rho+1)}}C_{f'}C_{|\Omega|}\left[|\Omega|^{\frac{\rho}{2(\rho+1)}}+C_{|\Omega|}^\rho\sqrt{\frac{2}{\omega}}C^{\frac{\rho}{2}}\right]\left[\,\|u_{tt}^m(t)\|^2+\|\Delta u_t^m(t)\|^2\,\right],
\end{aligned}$$

where $C_{|\Omega|}>0$ is a constant given by the embedding inequality $\|v\|_{2(\rho+1)}\leq C_{|\Omega|}\|\Delta v\|$.

Thus, inserting the estimates (31)-(32) in (30), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\|u_{tt}^m(t)\|^2+\|\Delta u_t^m(t)\|^2+\kappa\|\nabla u_t^m(t)\|^2\right]+2\gamma||(u^m(t),u_t^m(t))||_{\mathcal{H}}^{2q}\|\nabla u_{tt}^m(t)\|^2 \\
&\qquad\leq \left[C+\gamma\left|\frac{d}{dt}||(u^m(t),u_t^m(t))||_{\mathcal{H}}^{2q}\right|\right]\left[\,\|u_{tt}^m(t)\|^2+\|\Delta u_t^m(t)\|^2\,\right], \qquad (33)
\end{aligned}$$

for all $0\leq t\leq T$ and some constant $C=C(\rho,\gamma,|\Omega|,C_{f'})>0$.

Now we define the functional

$$\chi(t):=C+\gamma\left|\frac{d}{dt}||(u^m(t),u_t^m(t))||_{\mathcal{H}}^{2q}\right|.$$

Then, $\chi\in L^1(0,T)$ and for every $t\geq 0$,

$$\begin{aligned}
\int_0^t\chi(s)ds &= Ct+\gamma\int_0^t\left|\frac{d}{dt}||(u^m(s),u_t^m(s))||_{\mathcal{H}}^{2q}\right|ds \\
&\leq Ct+\gamma\left[\,||(u^m(t),u_t^m(t)||_{\mathcal{H}}^{2q}+||(u^m(0),u_t^m(0))||_{\mathcal{H}}^{2q}\,\right]\leq C(t+1),
\end{aligned}$$

for some constant $C:=C(||(u_0,u_1)||_{\mathcal{H}})>0$, where we have used the global estimate (26). Thus, integrating (33) on $(0,t)$ and using Gronwall's inequality, we arrive at

$$\begin{aligned}
&\|u_{tt}^m(t)\|^2+\|\Delta u_t^m(t)\|^2+\kappa\|\nabla u_t^m(t)\|^2 \\
&+2\gamma e^C\int_0^t e^{C(t-s)}||(u^m,u_t^m)||_{\mathcal{H}}^{2q}\|\nabla u_{tt}^m\|^2ds \\
&\quad\leq\ e^{C(1+t)}\left[\,\|u_{tt}^m(0)\|^2+\|\Delta u_1^m\|^2+\kappa\|\nabla u_1^m\|^2\,\right], \qquad (34)
\end{aligned}$$

for all $0\leq t\leq T$. Taking $t=0$ in (21) and substituting $w_j=u_{tt}(0)$ yields

$$\|u_{tt}^m(0)\|\leq\|\Delta^2u_0^m\|+\kappa\|\Delta u_0^m\|+\gamma||(u_0^m,u_1^m)||_{\mathcal{H}}^{2q}\|\Delta u_1^m\|+\|f(u_0^m)\|\leq C,$$

$C=C(\|\Delta^2u_0\|,\|\Delta u_1\|)>0$ is a constant depending on regular initial data, which comes from (29), but independent of m . Thus, from (34) and using again (29), we have

$$\begin{aligned}
&\|u_{tt}^m(t)\|^2+\|\Delta u_t^m(t)\|^2+\kappa\|\nabla u_t^m(t)\|^2 \\
&\quad+\int_0^t||(u^m(s),u_t^m(s))||_{\mathcal{H}}^{2q}\|\nabla u_{tt}^m(s)\|^2ds\leq C, \qquad (35)
\end{aligned}$$

for all $t \in [0,T]$, $m \in \mathbb{N}$, and some $C = C(||(u_0,u_1)||_{\mathcal{H}_2}) > 0$. Thus, (35) implies that

$$(u_t^m) \quad \text{is bounded in} \quad L^\infty(0,T;W_2), \tag{36}$$
$$(u_{tt}^m) \quad \text{is bounded in} \quad L^\infty(0,T;W_0), \tag{37}$$

with additional boundedness

$$||(u^m,u_t^m)||_{\mathcal{H}}^q|\nabla u_{tt}^m| \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)). \tag{38}$$

Moreover, taking $\omega_j = \Delta^2 u^m$ in (21) and integrating over Ω, there exists also a constant $C = C(||(u_0,u_1)||_{\mathcal{H}_2}) > 0$, independent of t, such that

$$\|\Delta^2 u^m(t)\| \leq [\,\|u_{tt}^m(t)\| + \kappa\|\Delta u^m(t)\| + \gamma||(u^m,u_t^m)||_{\mathcal{H}}^{2q}\|\Delta u_t^m(t)\| + \|f(u^m)\|\,] < C, \tag{39}$$

for all $t \in [0,T]$ and $m \in \mathbb{N}$, which results in the next boundedness

$$(u^m) \quad \text{is bounded in} \quad L^\infty(0,T;W_4). \tag{40}$$

Also, using assumption (16) and estimates (20) and (27), we conclude

$$f(u^m) \quad \text{is bounded in} \quad L^\infty(0,T;W_0). \tag{41}$$

With these estimates, we have gathered all tools to pass the limit in the approximate problem (21) as follows.

Passage to the limit and existence of strong solution. From (36), (37) and (40), passing to a subsequence if necessary, we get the limits

$$u^m \rightharpoonup u \quad \text{weak star in} \quad L^\infty(0,T;W_4), \tag{42}$$
$$u_t^m \rightharpoonup u_t \quad \text{weak star in} \quad L^\infty(0,T;W_2), \tag{43}$$
$$u_{tt}^m \rightharpoonup u_{tt} \quad \text{weak star in} \quad L^\infty(0,T;W_0). \tag{44}$$

Since the embeddings $W_4 \hookrightarrow W_2 \hookrightarrow W_0$ are compact, then from Aubin-Lions Lemma, cf. [21], the convergences (42)-(44) imply that there exists a subsequence, still denoted by u^n, such that

$$u^m \to u \quad \text{strongly in} \quad L^2(0,T;W_2), \tag{45}$$
$$u_t^m \to u_t \quad \text{strongly in} \quad L^2(0,T;W_0), \tag{46}$$

and also (using that f is continuous)

$$u^m \to u \quad \text{a. e. in} \quad (0,T)\times\Omega. \tag{47}$$
$$f(u^m) \to f(u) \quad \text{a. e. in} \quad (0,T)\times\Omega. \tag{48}$$

In addition, from (41), (48) and Lions' Lemma (see again [21]) we infer

$$f(u^m) \rightharpoonup f(u) \;\; \text{weak in} \;\; L^2(0,T;W_0). \tag{49}$$

Now, at light of the limits (42)-(49), we claim that the approximated problem (21) can be passed to the limit when $n \to \infty$ to prove that u satisfies the equation

$$u_{tt} + \Delta^2 u - \kappa\Delta u - \gamma[\|\Delta u\|^2 + \|u_t\|^2]^q\Delta u_t + f(u) = 0 \quad \text{in} \quad L^\infty(0,T;W_0). \tag{50}$$

Indeed, such assertion for the linear terms and the nonlinear source $f(u)$ are standard, being already concluded in previous papers by authors, see for instance [17].

Next, we are going to prove this statement for the nonlinear term involving the damping. We observe

$$
\begin{aligned}
&\left|\int_0^T ||(u^m(t),u_t^m(t))||_{\mathcal{H}}^{2q}\,(\Delta u_t^m(t),w)\,\theta(t)\,dt\right.\\
&\left.-\int_0^T ||(u(t),u_t(t))||_{\mathcal{H}}^{2q}\,(\Delta u_t(t),w)\,\theta(t)\,dt\right|\\
&\quad\le\left|\int_0^T ||(u^m(t),u_t^m(t))||_{\mathcal{H}}^{2q}\,(\Delta u_t^m(t)-\Delta u_t(t),w)\,\theta(t)\,dt\right| \qquad (51)\\
&\quad+\left|\int_0^T \Big[\,||(u^m(t),u_t^m(t))||_{\mathcal{H}}^{2q}-||(u(t),u_t(t))||_{\mathcal{H}}^{2q}\Big]\,(\Delta u_t(t),w)\,\theta(t)\,dt\right|,
\end{aligned}
$$

for all $w\in W_1$ and $\theta\in C_0^\infty(0,T)$. In what follows, we are going to show that each term on the right hand side of (51) converges to zero. To the first one, it is enough to observe (27) and the convergence (43). To the second one, we use the Mean Value Theorem and (27) to obtain

$$
\begin{aligned}
&\left|\int_0^T \Big[\,||(u^m(t),u_t^m(t))||_{\mathcal{H}}^{2q}-||(u(t),u_t(t))||_{\mathcal{H}}^{2q}\Big]\,(\Delta u_t(t),w)\,\theta(t)\,dt\right|\\
&\le q\int_0^T \big[\,||(u^m,u_t^m)||_{\mathcal{H}}^{2}+||(u,u_t)||_{\mathcal{H}}^{2}\,\big]^{q-1}\big[\|\Delta u^m\|^2+\|u_t^m\|^2\\
&\quad-\|\Delta u\|^2-\|u_t^m\|^2\big]\,dt \qquad (52)\\
&\le C\int_0^T \Big(\|\Delta u^m(t)-\Delta u(t)\|+\|u_t^m(t)-u_t(t)\|\Big)\,dt,
\end{aligned}
$$

for all $w\in W_2$ and $\theta\in C_0^\infty(0,T)$. This is exact moment where we have used the technical assumption $q\ge 1$ to control proper terms. Above, we have omitted the parameter t in some places to simplify the notations. Thus, (45), (46) and (52) give us the convergence of the second term of (51).

Finally, multiplying (21) by θ, integrating on $(0,T)$ and passing the resulting expression to the limit, and observing that (ω_j) is an orthonormal basis for W_0, we get

$$
\begin{aligned}
&\int_0^T (u_{tt}(t),\omega)\theta(t)\,dt+\int_0^T (\Delta u(t),\Delta\omega)\theta(t)\,dt-\kappa\int_0^T (\Delta u(t),\omega)\theta(t)\,dt\\
&\quad-\gamma\int_0^T ||(u(t),u_t(t))||_{\mathcal{H}}^{2q}(\Delta u_t(t),\omega)\theta(t)\,dt+\int_0^T (f(u(t)),\omega)\theta(t)\,dt=0,
\end{aligned}
$$

for all $\omega\in W_0$ and $\theta\in C_0^\infty(0,T)$. From this and the regularity of the functions in (42)-(44), we conclude that (50) holds true. The initial conditions in (2) are obtained by using (29) and (50), being proved in a standardly way.

Therefore, problem (1)-(3) has a strong solution u in the class (17), which proves Theorem 2.1$-(i)$. □

2.2. **Proof of Theorem 2.1–(ii).** Setting $w = u^1 - u^2$ and $\tilde{f}(w) = f(u^1) - f(u^2)$, the difference $U^1 - U^2 = (w, w_t)$ is a regular solution of the following problem

$$\begin{cases} w_{tt} - \kappa\Delta w + \Delta^2 w - \dfrac{\gamma}{2}\Pi_1 \Delta w_t - \dfrac{\gamma}{2}\Pi_2 \left[\Delta u_t^1 + \Delta u_t^2\right] + \tilde{f}(w) = 0, \\ w(0) = u_0^1 - u_0^2 := w_0, \quad w_t(0) = u_1^1 - u_1^2 := w_1, \end{cases} \tag{53}$$

where we denote

$$\Pi_1(t) = ||U^1(t)||_{\mathcal{H}}^{2q} + ||U^2(t)||_{\mathcal{H}}^{2q} \quad \text{and} \quad \Pi_2(t) = ||U^1(t)||_{\mathcal{H}}^{2q} - ||U^2(t)||_{\mathcal{H}}^{2q}.$$

Multiplying the equation in (53) by w_t and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\, \|w_t(t)\|^2 + \kappa\|\nabla w(t)\|^2 + \|\Delta w(t)\|^2 \,\right] + \frac{\gamma}{2}\Pi_1(t)\|\nabla w_t(t)\|^2 = \mathcal{J}_1 + \mathcal{J}_2. \tag{54}$$

where

$$\begin{aligned} \mathcal{J}_1 &= -\int_\Omega \tilde{f}(w(t)) w_t(t) dx, \\ \mathcal{J}_2 &= -\frac{\gamma}{2}\Pi_2(t)\int_\Omega \left[\nabla u_t^1(t) + \nabla u_t^2(t)\right] \nabla w_t(t) dx. \end{aligned}$$

Hereafter, we still denote by C several constants depending on initial data. From condition (14) and the Mean Value Theorem, the generalized Hölder's inequality with $\frac{\rho}{2(\rho+1)} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1$ and the embedding $W_2 \hookrightarrow L^{2(\rho+1)}(\Omega)$, we can estimate the term $\mathcal{J}_1$ as

$$\begin{aligned} |\mathcal{J}_1| &\le C_{f'}\left[|\Omega|^{\frac{\rho}{2(\rho+1)}} + \|u^1(t)\|_{2(\rho+1)}^\rho + \|u^2(t)\|_{2(\rho+1)}^\rho\right] \|w(t)\|_{2(\rho+1)}\|w_t(t)\| \\ &\le C\|w(t)\|_{2(\rho+1)}\|w_t(t)\| \\ &\le C\left(\|\Delta w(t)\|^2 + \|w_t(t)\|^2\right), \end{aligned} \tag{55}$$

for some $C = C(||U_0||_{\mathcal{H}}, ||U_1||_{\mathcal{H}}) > 0$. The term $\mathcal{J}_2$ can be firstly estimated as

$$|\mathcal{J}_2| \le \frac{\gamma}{2}|\Pi_2(t)| \left[\|\nabla u_t^1(t)\| + \|\nabla u_t^2(t)\|\right] \|\nabla w_t(t)\| \le C|\Pi_2(t)|\|\nabla w_t(t)\|,$$

for some $C = C(||U_0||_{\mathcal{H}_2}, ||U_1||_{\mathcal{H}_2}) > 0$. In addition, since $g : \mathbb{R} \to \mathbb{R}$ given by $g(s) = |s|^q$ satisfies $g'(s) = q|s|^{q-1}\frac{s}{|s|}$, $q \ge 1$, then from the Mean Value Theorem, we obtain

$$\begin{aligned} |\Pi_2(t)| &\le q\left|\, ||U^1(t)||_{\mathcal{H}}^2 + ||U^2(t)||_{\mathcal{H}}^2 \,\right|^{q-1}\left[||U^1(t)||_{\mathcal{H}}^2 - ||U^2(t)||_{\mathcal{H}}^2\right] \\ &\le q\left|\, ||U^1(t)||_{\mathcal{H}}^2 + ||U^2(t)||_{\mathcal{H}}^2 \,\right|^{q-1}\left[||U^1(t)||_{\mathcal{H}}\right] ||(w(t), w_t(t))||_{\mathcal{H}} \\ &+ q\left|\, ||U^1(t)||_{\mathcal{H}}^2 + ||U^2(t)||_{\mathcal{H}}^2 \,\right|^{q-1}\left[||U^2(t)||_{\mathcal{H}}\right] ||(w(t), w_t(t))||_{\mathcal{H}} \\ &\le C\left[||U^1(t)||_{\mathcal{H}} + ||U^2(t)||_{\mathcal{H}}\right]^q ||(w(t), w_t(t))||_{\mathcal{H}}, \end{aligned}$$

for some $C = C(||U_0||_{\mathcal{H}}, ||U_1||_{\mathcal{H}}) > 0$, where we have used again the assumption $q \ge 1$. Thus, using Young's inequality with $ab \le \frac{2}{\gamma}a^2 + \frac{\gamma}{8}b^2$, we have

$$\begin{aligned} |\mathcal{J}_2| &\le C||(w(t), w_t(t))||_{\mathcal{H}}\left[\, ||U^1(t)||_{\mathcal{H}}^q + ||U^2(t)||_{\mathcal{H}}^q \,\right] \|\nabla w_t(t)\| \\ &\le \frac{2C^2}{\gamma}||(w(t), w_t(t))||_{\mathcal{H}}^2 + \frac{\gamma}{8}\left[||U^1(t)||_{\mathcal{H}}^q + ||U^2(t)||_{\mathcal{H}}^q\right]^2 \|\nabla w_t(t)\|^2 \\ &\le C||(w(t), w_t(t))||_{\mathcal{H}}^2 + \frac{\gamma}{4}\Pi_1(t)\|\nabla w_t(t)\|^2, \end{aligned} \tag{56}$$

for some $C = C(||U_0||_{\mathcal{H}_2}, ||U_1||_{\mathcal{H}_2}) > 0$. Replacing (55) and (56) in (54), we arrive at

$$\frac{1}{2}\frac{d}{dt}\left[\, \|w_t(t)\|^2 + \kappa\|\nabla w(t)\|^2 + \|\Delta w(t)\|^2 \,\right] + \frac{\gamma}{4}\Pi_1(t)\|\nabla w_t(t)\|^2 \\ \leq\ C\left[\, \|w_t(t)\|^2 + \kappa\|\nabla w(t)\|^2 + \|\Delta w(t)\|^2 \,\right], \tag{57}$$

for all $t \in [0,T]$ and some $C = C(||U_0||_{\mathcal{H}_2}, ||U_1||_{\mathcal{H}_2}) > 0$. Hence, using that

$$||U_1(t)-U_2(t)||^2_{\mathcal{H}} \leq \|w_t(t)\|^2+\kappa\|\nabla w(t)\|^2+\|\Delta w(t)\|^2 \leq (1+\frac{\kappa}{\lambda_1^{1/2}})||U_1(t)-U_2(t)||^2_{\mathcal{H}},$$

integrating (57) on $[0,t]$, applying Gronwall's inequality, we conclude

$$||U^1(t) - U^2(t)||^2_{\mathcal{H}} \leq C||U^1_0 - U^2_0||^2_{\mathcal{H}},$$

for some constant $C(||U_0||_{\mathcal{H}_2}, ||U_1||_{\mathcal{H}_2})$, which implies (18) as desired.

In particular, for $U^1_0 = U^2_0$, we have uniqueness. □

2.3. **Proof of Theorem 2.1**$-(iii)$**.** The proof of (19) is similar to (24). Indeed, it can be obtained by multiplying equation (1) by u_t and integrating on $\Omega\times(s,t)$, $t > s \geq 0$. □

From Theorem 2.1$-(iii)$, we conclude that the energy is non-increasing over the regular solution of (1)-(3). In the next section we are going to study its stability.

Remark 2. The main difficulty to obtain the existence and uniqueness of weak solution comes from the fact that the damping coefficient is not bounded from below. In other words, it seems to be a hard task to extract from (28) that

$$\|\nabla u^m_t\| \quad \text{is bounded in} \quad L^2(\Omega\times[0,T]).$$

This would be a crucial boundedness in the related *"A Priori Estimate I"* to the passage of limit as well as the continuous dependence of initial data in $\mathcal{H}$. Therefore, the weak solution is not gotten so standardly as in the previous works by authors [16, 17, 18].

3. **Polynomial stability.**

In this section we analyze the stability of the energy $E(t)$ defined in (13). More precisely, we are going to present below a (non-uniform) decay of polynomial type depending on the exponent q of the damping coefficient and the size of the initial energy. Our arguments are based on the Nakao's Lemma that can be found, for instance, in [25, 26].

Theorem 3.1. *Let us suppose that assumptions of Theorem 2.1 hold. Then, the energy $E(t)$ defined in (13) satisfies*

$$E(t) \leq \left[\, \frac{q}{\mu}(t-1)^+ + (E(0))^{-q}\,\right]^{-\frac{1}{q}}, \quad t>0, \tag{58}$$

where $s^+ = \frac{s+|s|}{2}$ and $\mu = \mu(E(0)) > 0$ a constant depending on $E(0)$.

Proof. Multiplying (1) by u_t and integrating over Ω, we obtain

$$\frac{d}{dt}E(t) + \gamma\|(u(t),u_t(t)\|^{2q}_{\mathcal{H}}\|\nabla u_t(t)\|^2 = 0, \quad t>0. \tag{59}$$

Now, from Poincaré's inequality $\|u_t\|^2 \leq C_{|\Omega|}\|\nabla u_t\|^2$, we have

$$\gamma\|(u(t),u_t(t)\|^{2q}_{\mathcal{H}}\|\nabla u_t(t)\|^2 \geq \gamma\|u_t(t)\|^{2q}_2\|\nabla u_t(t)\|^2 \geq \frac{\gamma}{C_{|\Omega|}}\|u_t(t)\|^{2(q+1)}_2,$$

and replacing it in (59), we get

$$\frac{d}{dt}E(t) + \frac{\gamma}{C_{|\Omega|}}\|u_t(t)\|_2^{2(q+1)} \leq 0, \quad t > 0. \tag{60}$$

In addition, integrating (60) from t to $t+1$, we obtain

$$\frac{\gamma}{C_{|\Omega|}}\int_t^{t+1}\|u_t(s)\|_2^{2(q+1)}\,ds \leq E(t) - E(t+1) := [D(t)]^2. \tag{61}$$

Using Hölder inequality with $\frac{q}{q+1} + \frac{1}{q+1} = 1$ and (61), we deduce

$$\begin{aligned}\int_t^{t+1}\|u_t(s)\|^2 ds &\leq \left(\int_t^{t+1} 1^{\frac{q+1}{q}}\,ds\right)^{\frac{q}{q+1}}\left(\int_t^{t+1}\|u_t(s)\|_2^{2(q+1)}ds\right)^{\frac{1}{q+1}} \\ &\leq \left(\frac{C_{|\Omega|}}{\gamma}\right)^{\frac{1}{q+1}}[D(t)]^{\frac{2}{q+1}}. \end{aligned}\tag{62}$$

From (62) and the Mean Value Theorem for integrals, we infer that there exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_t(t_i)\|^2 \leq 4\int_t^{t+1}\|u_t(s)\|^2 ds \leq 4\left(\frac{C_{|\Omega|}}{\gamma}\right)^{\frac{1}{q+1}}[D(t)]^{\frac{2}{q+1}}. \tag{63}$$

On the other hand, multiplying equation (1) by u and integrating over $[t_1, t_2]$ $\times\Omega$, we have

$$\begin{aligned}&\int_{t_1}^{t_2}\left[\|\Delta u(s)\|^2 + \kappa\|\nabla u(s)\|^2 + \int_\Omega f(u(s))u(s)dx\right]ds \\ &\quad = \int_{t_1}^{t_2}\|u_t(s)\|^2 ds + \sum_{i=1}^{2}\mathcal{I}_i,\end{aligned}\tag{64}$$

where we set

$$\mathcal{I}_1 = [(u(t_1), u_t(t_1)) - (u(t_2), u_t(t_2))]$$

and

$$\mathcal{I}_2 = \gamma\int_{t_1}^{t_2}\|(u(s), u_t(s))\|_{\mathcal{H}}^{2q}\int_\Omega \Delta u_t(s)u(s)dx\,ds.$$

From conditions (12) and (15), we have

$$\int_{t_1}^{t_2}\int_\Omega f(u(s))u(s)dx\,ds \geq \int_{t_1}^{t_2}\int_\Omega F(u(s))dxds - \frac{\theta}{2\lambda_1}\int_{t_1}^{t_2}\|\Delta u(s)\|^2 ds. \tag{65}$$

Replacing (65) in (64), adding $\frac{1}{2}\int_{t_1}^{t_2}\|u_t(s)\|^2 ds$ in both sides of the resulting expression and using that $\frac{1}{2}\omega = \frac{1}{2}(1 - \frac{\theta}{\lambda_1}) > 0$, we get

$$\frac{1}{2}\int_{t_1}^{t_2}E(s)\,ds \leq \frac{3}{2}\int_{t_1}^{t_2}\|u_t(s)\|^2 ds + \sum_{i=1}^{2}\mathcal{I}_i. \tag{66}$$

Now let us estimate the terms $\mathcal{I}_1$ and $\mathcal{I}_2$ in the right hand side of (66). In both cases, we use Young's inequality with $\varepsilon > 0$. Indeed, first we note that similarly to

(25) one has $\|\Delta u(t)\| \leq \frac{2}{\omega^{1/2}} E(t)^{\frac{1}{2}}$. Thus, using (63), we obtain

$$\begin{aligned}
\mathcal{I}_1 &\leq \|u(t_1)\|\|u_t(t_1)\| + \|u(t_2)\|\|u_t(t_2)\| \\
&\leq \frac{8}{\omega^{1/2}\lambda_1^{1/2}} \left(\frac{C_{|\Omega|}}{\gamma}\right)^{\frac{1}{2(q+1)}} [D(t)]^{\frac{1}{q+1}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{2}}(s) \\
&\leq \frac{16}{\varepsilon\omega\lambda_1} \left(\frac{C_{|\Omega|}}{\gamma}\right)^{\frac{1}{q+1}} [D(t)]^{\frac{2}{q+1}} + \varepsilon \sup_{t_1 \leq s \leq t_2} E(s). \qquad (67)
\end{aligned}$$

Now, applying (19) and again (25) it follows that

$$\|(u(t), u_t(t))\|_{\mathcal{H}}^{2q} \leq C, \quad t > 0,$$

where $C = C(E(0)) > 0$ is a constant depending on initial energy $E(0)$. From this and (62), we have

$$\begin{aligned}
\mathcal{I}_2 &\leq \gamma C \int_{t_1}^{t_2} \|u_t(s)\|\|\Delta u(s)\|\, ds \\
&\leq \gamma C \left(\frac{C_{|\Omega|}}{\gamma}\right)^{\frac{1}{2(q+1)}} [D(t)]^{\frac{1}{q+1}} \sup_{t_1 \leq s \leq t_2} \|\Delta u(s)\| \qquad (68) \\
&\leq \frac{2\gamma C}{\omega^{\frac{1}{2}}} \left(\frac{C_{|\Omega|}}{\gamma}\right)^{\frac{1}{2(q+1)}} [D(t)]^{\frac{1}{q+1}} \sup_{t_1 \leq s \leq t_2} E^{\frac{1}{2}}(s) \\
&\leq \frac{\gamma^2 C^2}{\varepsilon\omega} \left(\frac{C_{|\Omega|}}{\gamma}\right)^{\frac{1}{q+1}} [D(t)]^{\frac{2}{q+1}} + \varepsilon \sup_{t_1 \leq s \leq t_2} E(s),
\end{aligned}$$

for some constant $C = C(E(0)) > 0$. Replacing (62), (67) and (68) in (66), we obtain

$$\int_{t_1}^{t_2} E(s)ds \leq \mu_0 [D(t)]^{\frac{2}{q+1}} + 4\,\varepsilon \sup_{t_1 \leq s \leq t_2} E(s), \qquad (69)$$

where $\mu_0 = \mu_0(E(0)) > 0$ is given by

$$\mu_0 = \left[\frac{32}{\varepsilon\omega\lambda_1} + \frac{2\gamma^2 C^2}{\varepsilon\omega} + 3\right] \left(\frac{C_{|\Omega|}}{\gamma}\right)^{\frac{1}{q+1}}.$$

On the other hand, since $E(t)$ is non-increasing (see (19)) and applying again the Mean Value Theorem for integrals, there exists $\zeta \in [t_1, t_2] \subset [t, t+1]$ such that

$$\int_{t_1}^{t_2} E(s)\, ds = E(\zeta)(t_2 - t_1) \geq \frac{1}{2} E(t+1). \qquad (70)$$

Now, using again that $E(t)$ is non-increasing, recalling the definition of $D(t)$ in (61), and applying (70) and (69), we infer

$$\begin{aligned}
\sup_{t \leq s \leq t+1} E(s) &= E(t) \\
&= [D(t)]^2 + E(t+1) \\
&\leq [D(t)]^2 + 2\int_{t_1}^{t_2} E(s)ds \\
&\leq [D(t)]^2 + 2\mu_0 [D(t)]^{\frac{2}{q+1}} + 8\,\varepsilon \sup_{t \leq s \leq t+1} E(s).
\end{aligned}$$

Thus, choosing $\varepsilon = \frac{1}{16}$, one has

$$\sup_{t\leq s\leq t+1} E(s) \leq 2[D(t)]^2 + 4\mu_0[D(t)]^{\frac{2}{q+1}}, \quad t > 0,$$

and since $0 < \frac{2}{q+1} \leq 1$, one gets

$$\sup_{t\leq s\leq t+1} E(s) \leq [D(t)]^{\frac{2}{q+1}} \left[4\mu_0 + 2[D(t)]^{\frac{2q}{q+1}}\right], \quad t > 0. \tag{71}$$

Using (19), regarding (61) and the expression for μ_0, there exists a positive constant $\mu = \mu_1(E(0)) > 0$ depending on initial energy, but independent of $t > 0$, such that

$$\left[4\mu_0 + 2[D(t)]^{\frac{2q}{q+1}}\right]^{q+1} < \mu, \quad t > 0,$$

and from (71) we conclude

$$\sup_{t\leq s\leq t+1} [E(s)]^{q+1} \leq \mu[E(t) - E(t+1)], \quad t > 0. \tag{72}$$

Therefore, by using the Nakao's Lemma, cf. [25, 26], the estimate (58) is achieved from (72). This ends the proof of Theorem 3.1. □

4. **A peculiar estimate.** In this section we prove a particular estimate to the energy $E(t)$ defined in (13), which indicates that $E(t)$ does not have exponential decay rates. This will allow us to deduce that problem (1)-(3) does not have exponential decay patterns as well. On the other hand, such estimate does not prevent polynomial decay rate like $(1+t)^{-\frac{1}{q}}$, $q \geq 1$, as provided in (58) by Theorem 3.1. All these statements will be clarified at the end of the section.

In what follows, in order to simplify the notations, we are going to deal in the particular case where $f \equiv 0$ in (1)-(3). Thus, the corresponding energy becomes to

$$E(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{\kappa}{2}\|\nabla u(t)\|^2, \quad t \geq 0, \tag{73}$$

with Theorems 2.1 and 3.1 remaining unchanged. In particular, for $s = 0$, (19) reads as

$$E(t) + \gamma\int_0^t ||(u(\tau), u_t(\tau))||_{\mathcal{H}}^{2q}\|\nabla u_t(\tau)\|^2\, d\tau = E(0), \quad t > 0. \tag{74}$$

It is worth noting that (74) is obtained by integrating the following energy relation

$$\frac{d}{dt}E(t) = -\gamma||(u(t), u_t(t))||_{\mathcal{H}}^{2q}\|\nabla u_t(t)\|^2, \quad t > 0, \tag{75}$$

which in turn can be achieved similarly to (23), namely, multiplying (1) by u_t and integrating on Ω. In addition, from (73)-(74) and regarding (12), it is easy to get the following estimates:

$$\|u(t)\|^{2q} \leq \left[\frac{2}{\lambda_1}E(t)\right]^q \leq \left[\frac{2}{\lambda_1}E(0)\right]^q, \quad t > 0, \tag{76}$$

$$\|u(t)\|^{2q} \leq \frac{1}{\lambda_1^q}||(u(t), u_t(t))||_{\mathcal{H}}^{2q}, \quad t > 0, \tag{77}$$

and

$$\|(u(t), u_t(t))\|_{\mathcal{H}}^{2q} \leq [2E(0)]^q, \quad t > 0. \tag{78}$$

Under the above notations, our main result in this section reads as follows:

Theorem 4.1. *Under the assumptions of Theorem 2.1 with $f = 0$ and the notations above with finite initial energy $0 < E(0) < \infty$, then the energy $E(t)$ given in (73) satisfies*

$$E(t) \leq 3E(0)e^{-\delta \int_0^t \|u(s)\|^{2q} ds}, \quad t > 0, \tag{79}$$

where $\delta = \delta(\frac{1}{E(0)}) > 0$ a constant inversely proportional to $E(0)$.

Proof. Let us start by setting the following functional

$$\Phi(t) \;=\; \|u(t)\|^{2q}(u(t), u_t(t)), \quad t \geq 0, \tag{80}$$

and the perturbed energy

$$E_\varepsilon(t) \;=\; E(t) + \varepsilon\Phi(t), \quad t \geq 0, \tag{81}$$

where $\varepsilon > 0$ will be chosen later. The proof of (79) is done in the next steps.

Step 1. Choosing

$$0 < \varepsilon \leq \frac{\lambda_1^{q+\frac{1}{2}}}{2[2E(0)]^q}, \tag{82}$$

then

$$\frac{1}{2}E(t) \leq E_\varepsilon(t) \leq \frac{3}{2}E(t), \quad t \geq 0. \tag{83}$$

Indeed, using (76) and Young's inequality, a straightforward computation gives us

$$|E_\varepsilon(t) - E(t)| \leq \frac{\varepsilon}{\lambda_1^{q+\frac{1}{2}}}[2E(0)]^q E(t), \quad t \geq 0,$$

from where (83) follows by taking (82) into account.

Step 2. There exists a constant $C_0 = C_0(E(0)) > 0$ such that

$$\begin{aligned} \frac{d}{dt}\Phi(t) \;=\; &-\frac{1}{2}\|u(t)\|^{2q}\|\Delta u(t)\|^2 - \kappa\|u(t)\|^{2q}\|\nabla u(t)\|^2 \\ &+ C_0||(u(t), u_t(t))||_{\mathcal{H}}^{2q}\|\nabla u_t(t)\|^2, \quad t > 0. \end{aligned} \tag{84}$$

Indeed, deriving $\Phi(t)$ in (80) with respect to t, using equation (1) and integrating by parts, we have

$$\begin{aligned} \frac{d}{dt}\Phi(t) \;=\; &2q\|u(t)\|^{2q-2}(u(t), u_t(t))^2 + \|u(t)\|^{2q}\|u_t(t)\|^2 - \|u(t)\|^{2q}\|\Delta u(t)\|^2 \\ &- \kappa\|u(t)\|^{2q}\|\nabla u(t)\|^2 + \gamma\|u(t)\|^{2q}||(u(t), u_t(t))||_{\mathcal{H}}^{2q}(\Delta u(t), u_t(t)). \end{aligned}$$

Since $q \geq 1$ and using Young's inequality with $\eta > 0$, then

$$\begin{aligned} \frac{d}{dt}\Phi(t) \;\leq\; &3q\|u(t)\|^{2q}\|u_t(t)\|^2 - \|u(t)\|^{2q}\|\Delta u(t)\|^2 - \kappa\|u(t)\|^{2q}\|\nabla u(t)\|^2 \\ &+ \frac{\gamma}{2}\eta\|u(t)\|^{2q}||(u(t), u_t(t))||_{\mathcal{H}}^{2q}\|\Delta u(t)\|^2 \\ &+ \frac{\gamma}{2\eta}\|u(t)\|^{2q}||(u(t), u_t(t))||_{\mathcal{H}}^{2q}\|u_t(t)\|^2. \end{aligned} \tag{85}$$

Now, replacing the initial estimates (76)-(78) in (85) we arrive at

$$\begin{aligned} \frac{d}{dt}\Phi(t) \;\leq\; &\left(\frac{3q}{\lambda_1^q} + \frac{\gamma}{2\eta}\left[\frac{2}{\lambda_1}E(0)\right]^q\right)||(u(t), u_t(t))||_{\mathcal{H}}^{2q}\|u_t(t)\|^2 \\ &- \left(1 - \frac{\gamma}{2}\eta[2E(0)]^q\right)\|u(t)\|^{2q}\|\Delta u(t)\|^2 - \kappa\|u(t)\|^{2q}\|\nabla u(t)\|^2. \end{aligned} \tag{86}$$

Choosing $\eta = 1/\gamma[2E(0)]^q > 0$, applying Poincaré's inequality with

$$\|u\|^2 \leq c_{|\Omega|}\|\nabla u\|^2$$

and denoting $C_0 = \frac{c_{|\Omega|}}{2\lambda_1^q}\left(6q + \gamma^2[2E(0)]^{2q}\right) > 0$, we obtain from (86) that (84) holds true.

Step 3. Choosing

$$0 < \varepsilon \leq \frac{\gamma\lambda_1^q}{c_{|\Omega|}\left(6q + \gamma^2[2E(0)]^{2q}\right)} \quad \text{and} \quad \varepsilon_0 = \min\{\varepsilon, \gamma\lambda_1^q/c_{|\Omega|}\}, \tag{87}$$

then

$$\frac{d}{dt}E_\varepsilon(t) \leq -\varepsilon_0\|u(t)\|^{2q}E(t), \quad t > 0. \tag{88}$$

In fact, taking the derivative of $E_\varepsilon(t)$ in (81) with respect to variable t and replacing (75) and (84) in the resulting expression, we get

$$\begin{aligned}\frac{d}{dt}E_\varepsilon(t) \leq & -\frac{\varepsilon}{2}\|u(t)\|^{2q}\|\Delta u(t)\|^2 - \varepsilon\kappa\|u(t)\|^{2q}\|\nabla u(t)\|^2 \\ & - (1-\varepsilon C_0)||(u(t),u_t(t))||_{\mathcal{H}}^{2q}\|\nabla u_t(t)\|^2, \quad t > 0.\end{aligned}$$

Thus, picking up $\varepsilon > 0$ like in (87), using Poincaré's inequality and (77), we infer

$$\frac{d}{dt}E_\varepsilon(t) \leq \|u(t)\|^{2q}\left\{-\frac{\gamma\lambda_1^q}{2c_{|\Omega|}}\|u_t(t)\|^2 - \frac{\varepsilon}{2}\|\Delta u(t)\|^2 - \varepsilon\frac{\kappa}{2}\|\nabla u(t)\|^2\right\}, \quad t > 0,$$

from where it follows (88) after taking $\varepsilon_0 > 0$ as in (87).

Step 4. Conclusion. Keeping in mind the helpful estimates (83) and (88) we are able to conclude the proof of (79). In fact, taking $\varepsilon > 0$ the minimum of the expressions (82) and (87), noting that $\varepsilon \sim \frac{1}{E(0)}$, and then $\varepsilon_0 = \min\{\varepsilon, \gamma\lambda_1^q/c_{|\Omega|}\} \sim \frac{1}{E(0)}$, we obtain from (83) and (88) the following ODE inequality

$$\frac{d}{dt}E_\varepsilon(t) \leq -\frac{2}{3}\varepsilon_0\|u(t)\|^{2q}E_\varepsilon(t), \quad t > 0,$$

and, therefore, a standard computation leads to (79) with $\delta = \frac{2}{3}\varepsilon_0 \sim \frac{1}{E(0)}$. □

Remark 3. From Theorem 4.1 one sees that the energy $E(t)$ is under the following functional $S(t) = S(u(t))$ given by

$$S(t) = 3E(0)e^{-\delta\int_0^t\|u(s)\|^{2q}ds}, \quad t > 0. \tag{89}$$

In what follows, from (79) and (89), we are going to conclude that the energy $E(t)$ does not enjoy exponential decay rates $e^{-\alpha t}$ for $\alpha > 0$ nor (faster) polynomial stability like $\frac{1}{(1+t)^\alpha}$ for $\alpha > 1$. Therefore, it suggests that the (slower) polynomial decay rate $\frac{1}{(1+t)^{\frac{1}{q}}}$, $q \geq 1$, proved in Theorem 3.1 seems to be more admissible for the energy $E(t)$ corresponding to problem (1)-(3) with $f = 0$, and probably the same occurs to the general case with non-null source term $f(u)$.

In the next final considerations, our statements shall be made for non-null finite initial energy $E(0) \leq R$, for every $R > 0$. In both cases, we argue by contradiction.

Non-exponential stability. Let us suppose that $S(t)$ in (89) is of exponential type

$$S(t) = 3E(0)e^{-\alpha t}, \quad \alpha > 0.$$

This implies that $\delta\int_0^t\|u(s)\|^{2q}ds = \alpha t$, $t > 0$, and, consequently, from (76) and finite initial energy we get

$$\int_0^\infty [E(t)]^q dt \geq \left(\frac{\lambda_1}{2}\right)^q\int_0^\infty \|u(t)\|^{2q}dt = +\infty. \tag{90}$$

On the other hand, from (79) we also have

$$[E(t)]^q \leq [3E(0)]^q e^{-\alpha q\, t}, \quad t > 0.$$

This yields

$$\int_0^\infty [E(t)]^q dt \leq \frac{[3E(0)]^q}{\alpha q} < +\infty,$$

which contradicts (90). Therefore, we conclude that (89) is never of exponential type, which predicts the non-exponential stability for $E(t)$.

No faster polynomial stability. Let us admit now that $S(t)$ in (89) has a polynomial decay rate like

$$S(t) = \frac{3E(0)}{(1+t)^\alpha}, \quad \alpha > 1. \tag{91}$$

In this case, one gets $\delta \int_0^t \|u(s)\|^{2q} ds = \ln\left[(1+t)^\alpha\right],\ t > 0$, and also from (76) we obtain

$$\int_0^\infty [E(t)]^q dt \geq \left(\frac{\lambda_1}{2}\right)^q \int_0^\infty \|u(t)\|^{2q} dt = +\infty. \tag{92}$$

On the other hand, from (79) it follows that

$$[E(t)]^q \leq \frac{[3E(0)]^q}{(1+t)^{\alpha q}}, \quad t > 0,$$

and since $\alpha q > 1$, one concludes

$$\int_0^\infty [E(t)]^q dt \leq \frac{[3E(0)]^q}{\alpha q - 1} < +\infty,$$

which is a contradiction with (92). Hence, the expression (89) can not represent a better polynomial decay rate for $E(t)$. Last, but not least, it is worth mentioning that the above procedure does not prevent polynomial decay like (91) with $\alpha = \frac{1}{q},\ q \geq 1$. Indeed, in such a case the desired contradiction does not happen since $\alpha q = 1$.

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REFERENCES

[1] A. V. Balakrishnan, A theory of nonlinear damping in flexible structures, *Stabilization of flexible structures*, (1988), 1–12.

[2] A. V. Balakrishnan and L. W. Taylor, *Distributed Parameter Nonlinear Damping Models for Flight Structures*, Proceedings Damping 89, Flight Dynamics Lab and Air Force Wright Aeronautical Labs, WPAFB, 1989.

[3] J. M. Ball, Initial-boundary value problems for an extensible beam, *J. Math. Anal. Appl.*, **42** (1973), 61–90.

[4] J. M. Ball, Stability theory for an extensible beam, *J. Differential Equations*, **14** (1973), 399–418.

[5] R. W. Bass and D. Zes, Spillover, Nonlinearity, and flexible structures, *The Fourth NASA Workshop on Computational Control of Flexible Aerospace Systems, NASA Conference Publication 10065 ed. L.W.Taylor*, (1991), 1–14.

[6] A. C. Biazutti and H. R. Crippa, Global attractor and inertial set for the beam equation, *Applicable Analysis*, **55** (1994), 61–78.

[7] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, Global existence and asymptotic stability for the nonlinear and generalized damped extensible plate equation, *Commun. Contemp. Math.*, **6** (2004), 705–731.

[8] I. Chueshov and I. Lasiecka, Long-time behavior of second order evolution equations with nonlinear damping, *Mem. Amer. Math.,* **195** (2008), viii+183 pp.

[9] H. R. Clark, M. A. Rincon and R. D. Rodrigues, Beam equation with weak-internal damping in domain with moving boundary, *Applied Numerical Mathematics*, **47** (2003), 139–157.

[10] H. R. Clark, Elastic membrane equation in bounded and unbounded domains, *EJQTDE*, **11** (2002), 1–21.

[11] R. W. Dickey, Free vibrations and dynamic buckling of the extensible beam, *J. Math. Anal. Appl.*, **29** (1970), 443–454.

[12] E. H. Dowell, *Aeroelasticity of Plates and Shells*, Groninger, NL, Noordhoff Int. Publishing Co., 1975.

[13] A. Eden and A. J. Milani, Exponential attractor for extensible beam equations, *Nonlinearity*, **6** (1993), 457–479.

[14] C. Giorgi, M. G. Naso, V. Pata and M. Potomkin, Global attractors for the extensible thermoelastic beam system, *J. Differential Equations*, **246** (2009), 3496–3517.

[15] T. J. Hughes and J. E. Marsden, *Mathematical Foundation of Elasticity*, Dover Publications, Inc., New York, 1994.

[16] M. A. Jorge Silva and V. Narciso, Long-time behavior for a plate equation with nonlocal weak damping, *Differential Integral Equations*, **27** (2014), 931–948.

[17] M. A. Jorge Silva and V. Narciso, Attractors and their properties for a class of nonlocal extensible beams, *Discrete Contin. Dyn. Syst.*, **35** (2015), 985–1008.

[18] M. A. Jorge Silva and V. Narciso, Long-time dynamics for a class of extensible beams with nonlocal nonlinear damping, *Evol. Equ. Control Theory*, **6** (2017), 437–470.

[19] H. Lange and G. Perla Menzala, Rates of decay of a nonlocal beam equation, *Differential Integral Equations*, **10** (1997), 1075–1092.

[20] J. Limaco, H. R. Clark and A. J. Feitosa, Beam evolution equation with variable coeficients, *Math. Meth. Appl. Sci.*, **28** (2005), 457–478.

[21] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[22] T. F. Ma and V. Narciso, Global attractor for a model of extensible beam with nonlinear damping and source terms, *Nonlinear Anal.*, **73** (2010), 3402–3412.

[23] T. F. Ma, V. Narciso and M. L. Pelicer, Long-time behavior of a model of extensible beams with nonlinear boundary dissipations, *J. Math. Anal. Appl.*, **396** (2012), 694–703.

[24] C. Mu and J. Ma, On a system of nonlinear wave equations with Balakrishnan-Taylor damping, *Z. Angew. Math. Phys.*, **65** (2014), 91–113.

[25] M. Nakao, Convergence of solutions of the wave equation with a nonlinear dissipative term to the steady state, *Mem. Fac. Sci. Kyushu Univ. Ser. A*, **30** (1976), 257–265.

[26] M. Nakao, A difference inequality and its application to nonlinear evolution equations, *J. Math. Soc. Japan*, **30** (1978), 747–762.

[27] S. K. Patcheu, On a global solution and asymptotic behaviour for the generalized damped extensible beam equation, *J. Differential Equations*, **135** (1997), 299–314.

[28] S. Woinowsky-Krieger, The effect of axial force on the vibration of hinged bars, *Journal of Applied Mechanics*, **17** (1950), 35–36.

[29] Y. You, Inertial manifolds and stabilization of nonlinear beam equations with Balakrishnan-Taylor damping, *Abstr. Appl. Anal.*, **1** (1996), 83–102.

[30] W. Zhang, Nonlinear damping model: Response to random excitation, *5th Annual NASA Spacecraft Control Laboratory Experiment (SCOLE) Workshop*, (1988), 27–38.

[31] Y. Zhijian, On an extensible beam equation with nonlinear damping and source terms, *J. Differential Equations*, **254** (2013), 3903–3927.

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