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ON MODELING AND UNIFORM STABILITY OF A PARTIALLY DISSIPATIVE VISCOELASTIC TIMOSHENKO SYSTEM*

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Dedicated to Professor To Fu Ma on the occasion of his 60th birthday

Abstract. In this paper we first explore the deduction of the mathematical model for some viscoelastic Timoshenko systems. As a consequence, a new partially dissipative viscoelastic Timoshenko system arises with damping mechanism acting only on the shear force. Then, we prove uniform decay rates for this new system with the help of a modern observability inequality, where the assumption of equal speeds of wave propagation is regarded as a sufficient condition. Moreover, we prove that equal wave speeds is also a necessary condition to establish uniform decay rates.

Key words. Timoshenko systems, viscoelasticity, stability, decay rates, equal wave speeds

AMS subject classifications. 35B35, 35B40, 35L53, 35Q74, 74D05, 74K10

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1. Introduction.

This paper addresses the model deduction, uniform, and nonuniform stability results to the following Timoshenko system with a viscoelastic dissipation mechanism coupled on the shear force:

(1.1)
$$\begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x+\psi)_x + \kappa\displaystyle\int_0^t g(t-s)(\phi_x+\psi)_x(s)ds = 0 \quad \text{in} \quad (0,L)\times\mathbb{R}^+, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\phi_x+\psi) - \kappa\displaystyle\int_0^t g(t-s)(\phi_x+\psi)(s)ds = 0 \quad \text{in} \quad (0,L)\times\mathbb{R}^+, \end{cases}$$

where $L>0$ is the length of the beam and $\mathbb{R}^+=(0,\infty)$. Corresponding to the unknown variables ϕ and ψ, we consider the Dirichlet–Neumann boundary condition and initial data to be set later. The physical meaning of the positive constants $\rho_1,\rho_2,\kappa,b>0$ as well as the relaxation function $g>0$, the latter also known as the *memory kernel*, will be precisely introduced in the next section.

To the best of our knowledge, system (1.1) has not been considered in the literature. Its mathematical formulation consists by taking a viscoelastic deformation on the shear force only as presented in section 2, where we use the ideas introduced by Prüss [30, Chapter 9] and Drozdov and Kolmanovskii [12, Chapter 5] on integro-differential (viscoelastic) equations.

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On the other hand, when the viscoelastic law is only applied to the bending moment, the classical viscoelastic Timoshenko system emerges:

$$
(1.2)\quad \begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x + \psi)_x = 0 \ \text{ in } \ (0,L)\times\mathbb{R}^+, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\phi_x+\psi) + b\displaystyle\int_0^t g(t-s)\psi_{xx}(s)\,ds = 0 \ \text{ in } \ (0,L)\times\mathbb{R}^+, \end{cases}
$$

which has been studied by several authors in recent years. Concerning this problem (1.2), under Dirichlet boundary conditions, we would like to mention here the pioneer work by Ammar-Khodja et al. [3], whose main results on the corresponding energy functional are summarized as follows (see [3, Theorems 2.7, 3.5, and 4.1]).

1. Under the assumption of equal wave speeds of propagation $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$, one has
   - if g is of exponential type, then the energy functional also decays exponentially;
   - if g is of polynomial type, then the energy functional also decays polynomially.
2. Under the assumption of different wave speeds of propagation $\frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2}$, one has that
   - even though g is of exponential type, the energy functional does not decay uniformly (for weak initial data).

Since problem (1.2) is partially dissipative, as we can see in case 1, both g and the energy functional related to the solutions decay accordingly and uniformly, provided that $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$. This scenario attracted the attention of many mathematicians and several papers have been published in the literature, where more general uniform decay rates are established. On the other hand, in case 2, just a few papers can be found in the literature on the subject. As a matter of fact, in this scenario, a nonuniform stabilization of the energy has been done by taking into account the regularity of initial data and more regular solutions. For such generalizations and related problems, we refer to [7, 10, 15, 16, 17, 25, 26, 27, 31] and references therein.

According to the above considerations and having in mind that system (1.2) has been exhaustively studied lately, we turn our attention back to the new system (1.1). The main novelties and contributions in the present paper are threefold:

1. To give a precise deduction on the mathematical modeling for system (1.1) by using viscoelastic constitutive laws, which are physically consistent; see section 2.
2. To provide uniform stability results for (1.1), under the mathematical assumption of *equal speeds of wave propagation* (equal wave speeds, for short)

$$
(1.3)\qquad \frac{\kappa}{\rho_1} = \frac{b}{\rho_2},
$$

   and combining a new *observability inequality* with recent abstract results on stability for problems with memory as introduced by Lasiecka et al. [18, 21]; see section 3.
3. To prove that condition (1.3), which puts emphasis on the physical character of the system, is also necessary to obtain uniform decay rates for the energy functional associated with (1.1). To this purpose, we follow the ideas as introduced by Ammar-Khodja et al. [3]; see section 4.

Remark 1.1. Let us give some comments on the equal wave speeds condition (1.3). Due to the physical meaning of the coefficients (see, e.g., (2.13)), it is worth pointing out that (1.3) is physically never satisfied. Indeed, under the notation established in (2.13), the assumption (1.3) turns into $G = E/k$. On the other hand, as highlighted, e.g., in [24, 29] we have from the theory of elasticity that the relation between these two elastic modulus is given by $G = \frac{E}{2(1+\nu)}$, where $\nu \in (0, \frac{1}{2})$ is the Poisson's ratio. It means that the identity $k = 2(1+\nu)$ must hold true, which is physically impossible since $k < 1$. Therefore, the assumption (1.3) and the results in section 3 are only considered from a mathematical point of view. However, in order to address the problem from a physical aspect as well, we still consider the case of different wave speeds $\frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2}$, which led to the results in section 4.

2. Deduction of viscoelastic Timoshenko beams. In this section, in order to derive some viscoelastic Timoshenko beams models, mainly in what concerns (1.1), we combine some classical elastic equations that arise from Timoshenko ideas on beams (see [32, 33]), together with constitutive relations on viscoelasticity for materials containing hereditary (history) properties; see, for instance, Prüss [30] and Drozdov and Kolmanovskii [12]. A classical modeling of linear viscoelastic (wave-like) equations was first provided by Dafermos [8, 9] in the 1970s. Lately, a quite new approach on wave equations with memory was provided by Frabrizio, Giorgi, and Pata in [13], where the authors came up with a new treatment for integro-differential equations with general kernels, providing a wider class of relaxation functions.

Let us start with Boltzmann theory [4, 5] for aging viscoelastic materials, where the stress σ is assumed to depend not only on the (instantaneous) strain ϵ but also on the strain history $\{\epsilon(s);\, 0 \leq s \leq t\}$. Thus, the stress-strain constitutive law reads

$$\begin{aligned}\sigma(\cdot,t) &= E\left\{\epsilon(\cdot,t) + \int_0^t \mu'(t-s)\epsilon(\cdot,s)ds\right\}\\ &= E\left\{\epsilon(\cdot,t) - \int_0^t g(t-s)\epsilon(\cdot,s)ds\right\},\end{aligned}$$

where the constant E stands for the Young modulus of elasticity and the function μ is known as the relaxation measure of the bar material. Also, we denote $g := -\mu' > 0$ for convenience in future statements.

In what follows, according to the theory developed for viscoelastic Timoshenko beams type, bending and shear deformations shall be considered for vibrations of prismatic bars, which extends somehow the Euler–Bernoulli assumptions for beams. Indeed, Timoshenko assumptions on beams allow for a rotation movement from the cross section and bending lines. This rotation angle comes from a shear deformation, which was not considered in Euler–Bernoulli assumptions, where the cross section is kept perpendicular to the bending line. Finally, having in mind the classical theories of Prüss [30, Chapter 9] and Drozdov and Kolmanovskii [12, Chapter 5], we make the following considerations.

Let us consider a beam $[0, L] \times \Omega$ of length $L > 0$ and uniform cross section $\Omega \subset \mathbb{R}^2$ made of homogeneous isotropic viscoelastic material. In the initial Timoshenko hypotheses it is assumed that

- $(0, 0)$ is the center of Ω, so that $\int_\Omega z dydz = \int_\Omega y dydz = 0$;
- the bending takes place only on the (x, z)-plane;
- $\mathrm{diam}\Omega << L$ (thin beams) and normal stresses are negligible in general;
- there are only two relevant stresses σ_{11} and σ_{13} in the stress tensor $\sigma = \{\sigma_{ij}\}$.

Thus, for viscoelastic Timoshenko beams, the stress-strain relations can be considered as follows:

$$\sigma_{11}(x,z,t) = E\left\{\epsilon_{11}(x,z,t) - \int_0^t g_1(t-s)\epsilon_{11}(x,z,s)ds\right\}, \tag{2.1}$$

$$\sigma_{13}(x,z,t) = 2kG\left\{\epsilon_{13}(x,z,t) - \int_0^t g_2(t-s)\epsilon_{13}(x,z,s)ds\right\}, \tag{2.2}$$

where G is the constant shear modulus, k is a shear correction coefficient, and g_1, g_2 are relaxation kernels. Accordingly, the displacements and the rotation angle are denoted as follows:

- $u = u(x,t)$, the longitudinal displacement of points lying on the horizontal axis;
- $\phi = \phi(x,t)$, the vertical (lateral) bar displacement;
- $\psi = \psi(x,t)$, the angle of rotation for the normal to the longitudinal axis;
- $w_1(x,z,t) = u(x,t) + z\psi(x,t)$, longitudinal displacement;
- $w_2(x,z,t) = \phi(x,t)$, vertical displacement.

Under this notation, the standard formulas for the components of the infinitesimal strain tensor (see, e.g., (2.4) on p. 339 in [12]) can be expressed by

$$\epsilon_{11}(x,z,t) := \frac{\partial w_1}{\partial x} = u_x(x,t) + z\psi_x(x,t), \tag{2.3}$$

$$\epsilon_{13}(x,z,t) := \frac{1}{2}\left(\frac{\partial w_1}{\partial z} + \frac{\partial w_2}{\partial x}\right) = \frac{1}{2}\left(\psi(x,t) + \phi_x(x,t)\right). \tag{2.4}$$

Additionally, concerning formulas to compute bending moment and shear force (see, e.g., (9.10)–(9.11) on p. 237 in [30]) we find

$$M(x,t) = \int_\Omega z\sigma_{11}(x,z,t)dydz, \tag{2.5}$$

$$S(x,t) = \int_\Omega \sigma_{13}(x,z,t)dydz, \tag{2.6}$$

respectively, which are normalized identities by the area A and inertial moment I of the cross section Ω, namely,

$$A = \int_\Omega dydz \qquad \text{and} \qquad I = \int_\Omega z^2 dydz.$$

Hence, using relations (2.1), (2.3), and (2.5), one can compute the classical (and well-known) viscoelastic law for bending moment,

$$\begin{aligned} M = E\overbrace{\left(\int_\Omega zdydz\right)}^{=0}\left(u_x - \int_0^t g_1(t-s)u_x(s)ds\right) \\ + E\underbrace{\left(\int_\Omega z^2dydz\right)}_{=I}\left(\psi_x - \int_0^t g_1(t-s)\psi_x(s)ds\right), \end{aligned}$$

that is,

$$M \;=\; EI\left(\psi_x - \int_0^t g_1(t-s)\psi_x(s)\,ds\right). \tag{2.7}$$

Moreover, from relations (2.2), (2.4), and (2.6), the following new viscoelastic law for shear force arises:

$$S = kGA\left((\phi_x+\psi) - \int_0^t g_2(t-s)(\phi_x+\psi)(s)ds\right). \tag{2.8}$$

Summarizing, the constitutive relations (2.7)–(2.8) provide bending and shear deformations in the context of Timoshenko beams over viscoelastic materials depending on strain history. It is worth mentioning that when neglecting viscoelastic effects (for example, if the memory kernels vanish $g_1 = g_2 = 0$), then (2.7) and (2.8) obviously become the classical elastic relations for bending moment and shear force, respectively:

$$M = EI\psi_x, \tag{2.9}$$

$$S = kGA(\phi_x+\psi). \tag{2.10}$$

Finally, in order to derive the desired viscoelastic Timoshenko systems, we consider the classical model in differential equations for vibrations of prismatic beams originated in Timoshenko's works [32, 33]:

$$\begin{cases} \rho A\phi_{tt} - S_x = 0, \\ \rho I\psi_{tt} - M_x + S = 0, \end{cases} \tag{2.11}$$

for $(x,t) \in (0,L)\times\mathbb{R}^+$, where ρ represents the mass density per area unit. Therefore, under the above considerations, we are able to provide a precise deduction of at least three different types of viscoelastic Timoshenko systems as follows.

2.1. Viscoelastic law acting only on the bending moment. Using (2.7) and (2.10), system (2.11) turns into the classical partially viscoelastic Timoshenko problem:

$$\begin{cases} \rho A\phi_{tt} - kGA(\phi_x+\psi)_x = 0, \\ \rho I\psi_{tt} - EI\left(\psi_{xx} - \int_0^t g_1(t-s)\psi_{xx}(s)\,ds\right) + kGA(\phi_x+\psi) = 0, \end{cases} \tag{2.12}$$

which is precisely the well-known problem (1.2) by denoting the memory kernel g_1 as g and the coefficients

$$\rho_1 = \rho A, \quad \rho_2 = \rho I, \quad \kappa = kGA, \quad b = EI. \tag{2.13}$$

As stated in section 1, problem (2.12) and its related version with past history were first introduced by Ammar-Khodja et al. [3] and Muñoz Rivera and Fernández Sare [27] and, subsequently, studied by several authors in the context of stability along the time.

2.2. Viscoelastic law acting only on the shear force. In this case, using (2.8) and (2.9) instead of (2.7) and (2.10), respectively, system (2.11) becomes the partially viscoelastic Timoshenko problem

$$\begin{cases} \rho A\phi_{tt} - kGA\left((\phi_x+\psi)_x - \int_0^t g_2(t-s)(\phi_x+\psi)_x(s)ds\right) = 0, \\ \rho I\psi_{tt} - EI\psi_{xx} + kGA\left((\phi_x+\psi) - \int_0^t g_2(t-s)(\phi_x+\psi)(s)ds\right) = 0, \end{cases} \tag{2.14}$$

which consists exactly of the new viscoelastic problem proposed in (1.1) with coefficients given by (2.13) and memory kernel denoted by $g_2 = g$, which is taken just to simplify the notation. As remarked in section 1, this system has not been considered in the literature and constitutes the main object of study in this paper.

2.3. Viscoelastic law applied to both deformations. To supplement the set of different viscoelastic Timoshenko systems arising from the above modeling, we can take into account both viscoelastic laws (2.7)–(2.8). In this case, system (2.11) is driven to the next fully viscoelastic Timoshenko system:

$$\begin{cases} \rho A\phi_{tt} - kGA\left((\phi_x+\psi)_x - \displaystyle\int_0^t g_2(t-s)(\phi_x+\psi)_x(s)ds\right) = 0, \\ \rho I\psi_{tt} - EI\left(\psi_{xx} - \displaystyle\int_0^t g_1(t-s)\psi_{xx}(s)\,ds\right) \\ \qquad\qquad + kGA\left((\phi_x+\psi) - \displaystyle\int_0^t g_2(t-s)(\phi_x+\psi)(s)ds\right) = 0. \end{cases} \tag{2.15}$$

A slightly changed version of (2.15) was considered by Grasselli, Pata, and Prouse [14], who presented the problem with past history, nonlinear source terms, and external forces. Thus, the asymptotic behavior of solutions was studied by assuming that both kernels g_1 and g_2 are of exponential type. In this case, since there are two viscoelastic damping mechanisms acting on the system, all results obtained for (2.15) do not depend on the relation between the wave speeds; see, for instance, [14, section 3].

3. Uniform stability: The case of equal wave speeds. In this section we shall prove, under assumption (1.3), the uniform stability of the viscoelastic Timoshenko model (1.1) with coefficients $\rho_1, \rho_2, \kappa, b > 0$ given in (2.13). Such a statement will be achieved as a consequence of a new observability inequality to the energy solution combined with recent (and general) results from [18, 21].

In order to simplify notation hereafter, let us start by fixing the standard convolution operator denoted as

$$(g * u)(t) := \int_0^t g(t-s)u(s)ds.$$

Thus, having in mind such a notation, we can rewrite system (1.1) as follows:

$$\rho_1\,\phi_{tt} - \kappa\,(\phi_x+\psi)_x + \kappa(g*(\phi_x+\psi)_x) = 0 \quad \text{in} \quad (0,L)\times\mathbb{R}^+, \tag{3.1}$$

$$\rho_2\,\psi_{tt} - b\,\psi_{xx} + \kappa\,(\phi_x+\psi) - \kappa(g*(\phi_x+\psi)) = 0 \quad \text{in} \quad (0,L)\times\mathbb{R}^+. \tag{3.2}$$

To the couple (ϕ,ψ) we consider the mixed Dirichlet–Neumann boundary condition

$$\phi(0,t) = \phi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0, \quad t \geq 0, \tag{3.3}$$

and initial conditions

$$\phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x),\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x),\ x \in (0,L). \tag{3.4}$$

3.1. Notation and preliminary results. Let us start by introducing the following standard functional spaces:

$$
\begin{aligned}
L^2 &:= L^2(0,L), \quad ||u||_2^2 = \int_0^L |u(x)|^2\, dx,\\
H^1 &:= H^1(0,L), \quad ||u||_{H^1}^2 = \|u_x\|_2^2 + \|u\|_2^2,\\
L^2_* &:= L^2_*(0,L) = \left\{ u \in L^2(0,L)\,;\, \frac{1}{L}\int_0^L u(x)\, dx = 0 \right\},\\
H^1_0 &:= H^1_0(0,L) = \Big\{ u \in H^1(0,L)\,;\, u(0) = u(L) = 0 \Big\},\\
H^1_* &:= H^1_*(0,L) = \left\{ u \in H^1(0,L)\,;\, \frac{1}{L}\int_0^L u(x)\, dx = 0 \right\}.
\end{aligned}
$$

Due to Poincaré's inequality, we can also consider the equivalent norms in H^1_0 and H^1_*,

$$
||u||_{H^1_0} = \|u_x\|_2 \quad \text{and} \quad ||u||_{H^1_*} = \|u_x\|_2,
$$

respectively. In this work, we will always denote by $c_p > 0$ the Poincaré constant.

Assumption 3.1. $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing differentiable function such that

$$
g(0) > 0 \quad \text{and} \quad l := 1 - \int_0^\infty g(s) ds > 0. \tag{3.5}
$$

Using the pattern of the Faedo–Galerkin method (see Lions' book [23]) as applied to wave equations with memory we get a result on existence and uniqueness for (3.1)–(3.4). Such a result is summarized as follows and, for commodity, its proof will be omitted.

THEOREM 3.2. *Under the Assumption* 3.1 *and taking* $(\phi_0, \phi_1, \psi_0, \psi_1) \in H^1_0 \times L^2 \times H^1_* \times L^2_*$*, there exists a unique weak solution* (ϕ, ψ) *of problem* (3.1)–(3.4) *in the class*

$$
(\phi, \psi) \in C(\mathbb{R}^+; H^1_0 \times H^1_*) \cap C^1(\mathbb{R}^+; L^2 \times L^2_*).
$$

Furthermore, if $(\phi_0, \phi_1, \psi_0, \psi_1) \in (H^2 \cap H^1_0) \times H^1_0 \times (H^2 \cap H^1_*) \times H^1_*$*, then there exists a unique strong solution* (ϕ, ψ) *of problem* (3.1)–(3.4) *in the class*

$$
(\phi, \psi) \in C\left(\mathbb{R}^+; (H^2 \cap H^1_0) \times (H^2 \cap H^1_*)\right) \cap C^1(\mathbb{R}^+; H^1_0 \times H^1_*).
$$

Now we introduce some useful notation as follows. Given $u \in L^2_{loc}(\mathbb{R}^+; L^2)$, we set

$$
h(t) := 1 - \int_0^t g(s)\, ds, \tag{3.6}
$$

$$
(g \diamond u)(t) := \int_0^t g(t-s)(u(t) - u(s))\, ds, \tag{3.7}
$$

$$
(g \Box u)(t) := \int_0^t g(t-s)\|u(t) - u(s)\|_2^2\, ds, \tag{3.8}
$$

$$
\widehat{u}(x,t) := \int_0^x u(y,t)\, dy \tag{3.9}
$$

for $t > 0$ and $x \in (0, L)$.

LEMMA 3.3. *Under the above notation we have the following:*

1. *If* $u \in L^2(0,T;L^2)$, $T>0$, *then*

$$u-(g*u)=h(t)\,u+g\diamond u \quad \text{and} \quad \|(g\diamond u)(t)\|_2^2 \le \|g\|_{L^1(\mathbb{R}^+)}(g\Box u)(t). \tag{3.10}$$

2. *If* $(\phi,\psi)\in L^2(0,T;H_0^1\times H_*^1)$, $T>0$, *then*

$$p(\cdot,t):=\phi(\cdot,t)+\widehat{\psi}(\cdot,t)\in H_0^1(0,L). \tag{3.11}$$

Proof. The proof follows from direct computations. ∎

In what follows, we are going to see that problem (3.1)–(3.4) is dissipative with only one damping mechanism given by the convolution term involving the shear force component. Indeed, under the above notation and given a weak solution (ϕ,ψ) of problem (3.1)–(3.4), we define the corresponding energy functional $\mathcal{E}(t)=\mathcal{E}(\phi(t),\psi(t),\phi_t(t),\psi_t(t))$, $t\ge 0$, by

$$\mathcal{E}(t):=\frac{\rho_1}{2}\|\phi_t(t)\|_2^2+\frac{\rho_2}{2}\|\psi_t(t)\|_2^2+\frac{b}{2}\|\psi_x(t)\|_2^2+\frac{\kappa}{2}\,h(t)\|p_x(t)\|_2^2+\frac{\kappa}{2}(g\Box p_x)(t), \tag{3.12}$$

where $h(t)$ and $p(\cdot,t)$ are given in (3.6) and (3.11), respectively.

LEMMA 3.4. *The energy* $\mathcal{E}(t)$ *satisfies the following identity:*

$$\frac{d}{dt}\mathcal{E}(t)=-D(t),\quad t>0, \tag{3.13}$$

where

$$D(t)=\frac{\kappa}{2}\,g(t)\|p_x(t)\|_2^2-\frac{\kappa}{2}(g'\Box p_x)(t).$$

Proof. Taking the multipliers ϕ_t and ψ_t in (3.1) and (3.2), respectively, a straightforward computation leads to (3.13). ∎

From relation (3.13) and Assumption 3.1, one sees that $\frac{d}{dt}\mathcal{E}(t)\le 0$, which implies that the energy is nonincreasing with $\mathcal{E}(t)\le\mathcal{E}(0)$ for all $t\ge 0$.

3.2. Observability inequality. To the next result we assume an additional hypothesis on g.

Assumption 3.5. The memory kernel $g\in L^1(\mathbb{R}^+)$ is assumed to satisfy

$$\int_0^\infty g(s)ds>\max\left\{\frac{31}{32},\frac{64\,\rho_1\,L^2}{64\,\rho_1\,L^2+\rho_2}\right\}. \tag{3.14}$$

Remark 3.6. At this point, let us give some comments on Assumption 3.5 as follows.

i. We first note that the condition (3.14) means that the area below the graphic of g must be bounded from below. It is, somehow, unusual for systems with memory but it does not contradict the relation (3.5) since

$$C_0:=\max\left\{\frac{31}{32},\frac{64\,\rho_1\,L^2}{64\,\rho_1\,L^2+\rho_2}\right\}<1.$$

We also observe that once C_0 is close to 1, then (3.14) is a restrictive condition because it confines the range of admissible coefficients and powers in kernel examples, which seems to be not very applied in physical situations. Nevertheless, as we are going to see below in Example 3.7, such an assumption does not prevent some (classic) examples of memory kernels featuring different behaviors, recalling again that it is possible under proper restrictions on the size of their coefficients and powers.

ii. Additionally, we would like to stress that assumption (3.14) is required due to technical computations used to handle estimates for the new functionals introduced in (3.18)–(3.20). More precisely, (3.14) is regarded to construct a time t_0 such that function $1 - h(t)$ (see (3.6)) is bounded from below by a positive constant for all $t \geq t_0$. Indeed, from (3.14) we infer that there exists a time $t_0 > 0$ large enough such that

$$1 - h(t) = \int_0^t g(s)\, ds \geq \int_0^{t_0} g(s)\, ds := g_0 > C_0 \quad \forall\, t \geq t_0. \tag{3.15}$$

With inequality (3.15) in mind, we still mention that it will be crucial in the compatibility of the inequalities to be presented in (3.38). Summarizing, Assumption 3.5 leads to (3.15), which in turn will be precisely used in (3.38) and, consequently, in the inequalities right after.

Example 3.7. We illustrate some permissible kernels satisfying Assumptions 3.1 and 3.5, under suitable conditions imposed on the coefficients or/and powers.

(a) $g(t) = b\, e^{-at}$ for any $b > 0$ and $b < a < \frac{b}{C_0}$.
(b) $g(t) = \frac{a}{(t+1)^\gamma}$ for any $\gamma > 1$ and $C_0(\gamma - 1) < a < \gamma - 1$.
(c) $g(t) = \frac{a}{(t+e)[\ln(t+e)]^2}$ for $C_0 < a < 1$.
(d) $g(t) = ae^{-t^q}$ for $0 < q < 1$ and $a > 0$ properly chosen so that $C_0 < \int_0^\infty g(s)ds < 1$. For instance, in case $C_0 = \frac{31}{32}$, then $g(t) = \frac{49}{100}e^{-t^{1/2}}$ is a concrete kernel.
(e) Again in case $C_0 = \frac{31}{32}$, then peculiar kernels of transcendental type are given, e.g., by

$$g(t) = \frac{1}{[1 + \ln(t+1)]^{\frac{27}{20}[1+\ln(t+1)]}} \quad \text{and} \quad g(t) = \frac{1}{[t + \sqrt{e}]^{\frac{31}{50}\left[\ln\left(t+\sqrt{e}\right)+1\right]}}.$$

Proposition 3.8 (observability inequality). *Let Assumptions* 3.1 *and* 3.5 *be in place. If we additionally assume the equal wave speeds condition*

$$\frac{\rho_1}{\kappa} = \frac{\rho_2}{b}, \tag{3.16}$$

then there exist a time $T_0 > 0$ and a constant $C > 0$ such that

$$\mathcal{E}((n+1)T) \leq C \int_{nT}^{(n+1)T} (g\Box p_x)(t)dt + C \int_{nT}^{(n+1)T} D(t)dt \tag{3.17}$$

for all $n \in \mathbb{N}$ and $T > T_0$, where $C > 0$ is independent of n.

Proof. The proof will be done for regular solutions (ϕ, ψ) of (3.1)–(3.4) and the same conclusion holds true for weak solutions by using density arguments.

We start by defining the functionals

$$\chi_1(t) := \rho_1(\phi(t), \phi_t(t)) + \rho_2(\psi(t), \psi_t(t)), \tag{3.18}$$

$$\chi_2(t) := -\rho_1((g \diamond p)(t), \phi_t(t)), \tag{3.19}$$

$$\chi_3(t) := -\rho_2(h(t)\, p_x(t) + (g \diamond p_x)(t), \psi_t(t)) - \rho_2(\psi_x(t), \phi_t(t)). \tag{3.20}$$

Next, we are going to estimate the time derivative of the functionals χ_i, $i = 1, 2, 3$.

Estimate for $\chi_1'(t)$. Taking the derivative of $\chi_1(t)$ defined in (3.18), using (3.1)–(3.2), and performing some integrations by parts, we have

$$\begin{aligned}\chi_1'(t) &= \rho_1\|\phi_t(t)\|_2^2 + \rho_1(\phi(t), \phi_{tt}(t)) + \rho_2\|\psi_t(t)\|_2^2 + \rho_2(\psi(t), \psi_{tt}(t))\\ &= \rho_1\|\phi_t(t)\|_2^2 + \rho_2\|\psi_t(t)\|_2^2 - b\|\psi_x(t)\|_2^2 - \kappa\, h(t)\|p_x(t)\|_2^2\\ &\quad + \kappa\,((g \diamond p_x)(t), p_x(t)).\end{aligned} \tag{3.21}$$

Applying Cauchy–Schwarz's and Young's inequalities in (3.21), we obtain

$$\begin{aligned}\chi_1'(t) &\le \rho_1\|\phi_t(t)\|_2^2 + \rho_2\|\psi_t(t)\|_2^2 - b\|\psi_x(t)\|_2^2 - \kappa\left(h(t) - \frac{l}{2}\right)\|p_x(t)\|_2^2\\ &\quad + \frac{\kappa}{2l}\|(g \diamond p_x)(t)\|_2^2.\end{aligned} \tag{3.22}$$

Noting that $h(t) - \frac{l}{2} \ge \frac{h(t)}{2}$ for all $t > 0$, we deduce from (3.22) that

$$\begin{aligned}\chi_1'(t) &\le \rho_1\|\phi_t(t)\|_2^2 + \rho_2\|\psi_t(t)\|_2^2 - b\|\psi_x(t)\|_2^2 - \frac{\kappa}{2}\, h(t)\|p_x(t)\|_2^2\\ &\quad + \frac{\kappa}{2l}\|(g \diamond p_x)(t)\|_2^2.\end{aligned} \tag{3.23}$$

Estimate for $\chi_2'(t)$. Deriving the functional $\chi_2(t)$ set in (3.19) and using (3.1), we get

$$\begin{aligned}\chi_2'(t) &= -\rho_1((g \diamond p)_t(t), \phi_t(t)) - \rho_1((g \diamond p)(t), \phi_{tt}(t))\\ &= -\rho_1(1 - h(t))\|\phi_t(t)\|_2^2 - \rho_1([(g' \diamond p) + (1 - h)\widehat{\psi}_t](t), \phi_t(t))\\ &\quad + \kappa\|(g \diamond p_x)(t)\|_2^2 + \kappa\, h(t)(p_x(t), (g \diamond p_x)(t)).\end{aligned} \tag{3.24}$$

Using again Cauchy–Schwarz's and Young's inequalities one has

$$|((g' \diamond p)(t), \phi_t(t))| \le \frac{g_0}{4}\,\|\phi_t(t)\|_2^2 + \frac{1}{g_0}\|(g' \diamond p)(t)\|_2^2, \tag{3.25}$$

$$\left|(\widehat{\psi}_t(t), \phi_t(t))\right| \le \left[\frac{1}{2}\|\phi_t(t)\|_2^2 + \frac{1}{2}\|\widehat{\psi}_t(t)\|_2^2\right], \tag{3.26}$$

$$|(p_x(t), (g \diamond p_x)(t))| \le (1 - g_0)\frac{\rho_1 L^2}{\rho_2}\|p_x(t)\|_2^2 + \frac{\rho_2}{4(1 - g_0)\rho_1\, L^2}\|(g \diamond p_x)(t)\|_2^2. \tag{3.27}$$

Replacing (3.25)–(3.27) in (3.24) we obtain

$$\begin{aligned}\chi_2'(t) &\le -\frac{\rho_1}{2}\left(1 - h(t) - \frac{g_0}{2}\right)\|\phi_t(t)\|_2^2 + \kappa\,(1 - g_0)\frac{\rho_1 L^2}{\rho_2}\, h(t)\|p_x(t)\|_2^2\\ &\quad + \frac{\rho_1}{2}(1 - h(t))\|\widehat{\psi}_t(t)\|_2^2 + \kappa\left(1 + \frac{\rho_2}{4(1 - g_0)\rho_1\, L^2}\right)\|(g \diamond p_x)(t)\|_2^2\\ &\quad + \frac{\rho_1}{g_0}\|(g' \diamond p)(t)\|_2^2.\end{aligned} \tag{3.28}$$

From (3.15) we recall that $1 - h(t) \ge g_0$, and using standard computations, we conclude from (3.28) that

$$\begin{aligned}\chi_2'(t) &\le -\frac{\rho_1\, g_0}{4}\|\phi_t(t)\|_2^2 + \kappa\,(1 - g_0)\frac{\rho_1 L^2}{\rho_2}\, h(t)\|p_x(t)\|_2^2 + \frac{\rho_1\, L^2}{2}\|\psi_t(t)\|_2^2\\ &\quad + \frac{\rho_1\, c_p^2}{g_0}\|(g' \diamond p_x)(t)\|_2^2 + \kappa\left(1 + \frac{\rho_2}{4(1 - g_0)\rho_1\, L^2}\right)\|(g \diamond p_x)(t)\|_2^2\end{aligned} \tag{3.29}$$

for all $t \ge t_0$, where $c_p > 0$ is the Poincaré constant.

Estimate for $\chi_3'(t)$. Deriving $\chi_3(t)$ given in (3.20) we obtain

$$\begin{aligned} \chi_3'(t) = &-\rho_2\|\psi_t(t)\|_2^2 + \kappa[h(t)]^2\|p_x(t)\|_2^2 + 2\kappa\, h(t)\,(p_x(t), (g \diamond p_x)(t)) \\ &+ \kappa\,\|(g \diamond p_x)(t)\|_2^2 + \rho_2\,(\psi_t(t), [g\,p_x - g' \diamond p_x](t)) \\ &- \rho_2(\psi_x(t), \phi_{tt}(t)) + b(\psi_{xx}(t), [h\,p_x + g \diamond p_x](t)). \end{aligned} \tag{3.30}$$

Integrating by parts the last term of (3.30) and using (3.1) we get

$$\begin{aligned} \chi_3'(t) = &-\rho_2\|\psi_t(t)\|_2^2 + \kappa[h(t)]^2\|p_x(t)\|_2^2 + 2\kappa\, h(t)\,(p_x(t), (g \diamond p_x)(t)) \\ &+ \kappa\,\|(g \diamond p_x)(t)\|_2^2 + \rho_2\,(\psi_t(t), [g\,p_x - g' \diamond p_x](t)) \\ &+ \left(\frac{b\,\rho_1}{\kappa} - \rho_2\right)(\psi_x(t), \phi_{tt}(t)). \end{aligned} \tag{3.31}$$

Using over again Cauchy–Schwarz's and Young's inequalities, and recalling that $g(t) \leq g(0)$ for all $t > 0$, we deduce

$$|(\psi_t(t), [g\,p_x - g' \diamond p_x](t))| \leq \frac{1}{2}\,\|\psi_t(t)\|_2^2 + g(0)\,g(t)\|p_x\|_2^2 + \|(g' \diamond p_x)(t)\|_2^2, \tag{3.32}$$

$$|h(t)(p_x(t), (g \diamond p_x)(t))| \leq \frac{[h(t)]^2}{2}\|p_x(t)\|_2^2 + \frac{1}{2}\|(g \diamond p_x)(t)\|_2^2. \tag{3.33}$$

Replacing (3.32)–(3.33) in (3.31) we have

$$\begin{aligned} \chi_3'(t) \leq &-\frac{\rho_2}{2}\|\psi_t(t)\|_2^2 + 2\kappa\,[h(t)]^2\|p_x(t)\|_2^2 + 2\kappa\,\|(g \diamond p_x)(t)\|_2^2 \\ &+ \rho_2\,g(0)\,g(t)\|p_x(t)\|_2^2 + \rho_2\,\|(g' \diamond p_x)(t)\|_2^2 \\ &+ \left(\frac{b\,\rho_1}{\kappa} - \rho_2\right)(\psi_x(t), \phi_{tt}(t)). \end{aligned} \tag{3.34}$$

Thus, regarding the equal wave speeds assumption (3.16), we conclude from (3.34) that

$$\begin{aligned} \chi_3'(t) \leq &-\frac{\rho_2}{2}\|\psi_t(t)\|_2^2 + 2\kappa\,[h(t)]^2\|p_x(t)\|_2^2 + 2\kappa\,\|(g \diamond p_x)(t)\|_2^2 \\ &+ \rho_2\,g(0)\,g(t)\|p_x(t)\|_2^2 + \rho_2\,\|(g' \diamond p_x)(t)\|_2^2. \end{aligned} \tag{3.35}$$

Conclusion of the proof. Let η_1, $\eta_2 > 0$ be constants (to be determined later) and

$$\chi(t) := \eta_1\,\chi_1(t) + \eta_2\,\chi_2(t) + \chi_3(t). \tag{3.36}$$

Taking the derivative of $\chi(t)$ and using the estimates (3.23), (3.29), and (3.35), we have

$$\begin{aligned} \chi'(t) \leq &-\rho_1\,\left(\eta_2\frac{g_0}{4} - \eta_1\right)\|\phi_t(t)\|_2^2 - \eta_1\,b\|\psi_x(t)\|_2^2 \\ &- \left(\frac{\rho_2}{2} - \eta_1\rho_2 - \eta_2\,\frac{\rho_1\,L^2}{2}\right)\|\psi_t(t)\|_2^2 \\ &- \kappa\,\left(\frac{\eta_1}{2} - \eta_2\,(1-g_0)\frac{\rho_1 L^2}{\rho_2} - 2h(t)\right)\,h(t)\,\|p_x(t)\|_2^2 \\ &+ \kappa\left[\frac{\eta_1}{2l} + \eta_2\left(1 + \frac{\rho_2}{4(1-g_0)\rho_1\,L^2}\right) + 2\right]\|(g \diamond p_x)(t)\|_2^2 \\ &+ \rho_2\,g(0)\,g(t)\|p_x(t)\|_2^2 + \left(\eta_2\frac{\rho_1\,c_p^2}{g_0} + \rho_2\right)\,\|(g' \diamond p_x)(t)\|_2^2 \end{aligned} \tag{3.37}$$

for all $t \geq t_0$.

Now it is the precise moment where we use the strength of Assumption 3.5, by applying its prompt consequence (3.15). In fact, from the inequality $g_0 > C_0$ in (3.15), it is possible to carefully choose η_1 and η_2 such that

$$\frac{32(1-g_0)}{g_0} < \eta_2 < \frac{\rho_2}{2\rho_1 L^2} \quad \text{and} \quad 8(1-g_0) < \eta_1 < \frac{1}{4}\min\{\eta_2\, g_0, 1\}. \tag{3.38}$$

From the choices in (3.38) we observe that

- $\eta_2 \frac{g_0}{4} - \eta_1 > 0$,
- $\frac{\rho_2}{2} - \eta_1 \rho_2 - \eta_2\, \frac{\rho_1 L^2}{2} > 0$,
- $\frac{\eta_1}{2} - \eta_2\,(1-g_0)\frac{\rho_1 L^2}{\rho_2} - 2h(t) > \frac{3}{2}(1-g_0) > 0 \quad$ for all $t \geq t_0$.

In this case, combining (3.38) and (3.37) we arrive at

$$\begin{aligned}\chi'(t) &\leq -C\,\mathcal{E}(t) + C_{g_0}\,\|(g \diamond p_x)(t)\|_2^2 \\ &\quad + \rho_2\, g(0)\, g(t) \|p_x(t)\|_2^2 + \left(\eta_2 \frac{\rho_1\, c_p^2}{g_0} + \rho_2\right) \|(g' \diamond p_x)(t)\|_2^2,\end{aligned} \tag{3.39}$$

for all $t \geq t_0$, and some constants $C > 0$ and

$$C_{g_0} := \kappa \left[\frac{C}{2} + \frac{\eta_1}{2l} + \eta_2\left(1 + \frac{\rho_2}{4(1-g_0)\rho_1 L^2}\right) + 2\right] > 0.$$

Therefore, from (3.10), (3.13), and (3.39), we conclude that

$$\chi'(t) \leq -C\,\mathcal{E}(t) + C_1\,(g\Box p_x)(t) + C_1\, D(t) \quad \forall\, t \geq t_0, \tag{3.40}$$

for some constants $C, C_1 > 0$.

Let us consider $n \in \mathbb{N}$ and $T \geq t_0$. Thus, integrating (3.40) on $(nT, (n+1)T)$, we get

$$C \int_{nT}^{(n+1)T} \mathcal{E}(t)\, dt \leq -\chi(t)\Big|_{nT}^{(n+1)T} + C_1 \int_{nT}^{(n+1)T} (g\Box p_x)(t)\, dt + C_1 \int_{nT}^{(n+1)T} D(t)\, dt. \tag{3.41}$$

In addition, from the definition of functional $\chi(t)$ in (3.36), we have

$$\begin{aligned}|\chi(t)| &\leq \eta_1\, |\chi_1(t)| + \eta_2\, |\chi_2(t)| + |\chi_3(t)| \\ &\leq \frac{1}{2}\left[\rho_1(\eta_1+\eta_2) + \rho_2\right] \|\phi_t(t)\|_2^2 + \frac{1}{2}\left[\rho_1\eta_1 + \rho_2\right] \|\psi_t(t)\|_2^2 \\ &\quad + \frac{1}{2}\left[2\rho_1\eta_1 c_p^2 + \rho_2\right] \|p_x(t)\|_2^2 + \frac{1}{2}\left[\rho_1\eta_1 c_p^2(1+2c_p^2) + \rho_2\right] \|\psi_x(t)\|_2^2 \\ &\quad + \frac{1}{2}\left[\rho_1\eta_2 c_p^2 + \rho_2\right] \|(g \diamond p_x)(t)\|_2^2 \\ &\leq C_2\left(\frac{\rho_1}{2}\|\phi_t(t)\|_2^2 + \frac{\rho_2}{2}\|\psi(t)\|_2^2 + \frac{b}{2}\|\psi_x(t)\|_2^2 + \frac{\kappa\, l}{2}\,\|p_x(t)\|_2^2 + \frac{\kappa}{2}\|(g \diamond p_x)(t)\|_2^2\right)\end{aligned}$$

for some $C_2 > 0$. From (3.6) and (3.10) we deduce

$$|\chi(t)| \leq C_2\,\mathcal{E}(t) \quad \forall\, t \geq 0,$$

and, consequently,

$$\left|\chi(t)\Big|_{nT}^{(n+1)T}\right| \leq 2\, C_2\,[\mathcal{E}((n+1)T) + \mathcal{E}(nT)]. \tag{3.42}$$

Thus, from (3.41), (3.42) and since $\mathcal{E}(t)$ is nonincreasing, we get

$$T\,\mathcal{E}((n+1)T) \leq \; 2\,C_2\,[\mathcal{E}((n+1)T) + \mathcal{E}(nT)] + C_1 \int_{nT}^{(n+1)T} (g\Box p_x)(t)\,dt \\ + C_1 \int_{nT}^{(n+1)T} D(t)\,dt. \tag{3.43}$$

Hence, using again (3.13) in (3.43), we conclude

$$(T-C)\,\mathcal{E}((n+1)T) \leq C \int_{nT}^{(n+1)T} (g\Box p_x)(t)\,dt + C \int_{nT}^{(n+1)T} D(t)\,dt,$$

from where inequality (3.17) follows for $T > T_0 := \max\{t_0, 2C\} > 0$. This completes the proof of Proposition 3.8. ◻

3.3. Uniform decay rates. Once we have obtained the observability inequality (3.17), our stability results for the energy $\mathcal{E}(t)$ set in (3.12) rely on the construction of a suitable function to estimate $\int_{nT}^{(n+1)T}(g\Box p_x)(t)\,dt$ in terms of the damping integral term $\int_{nT}^{(n+1)T} D(t)\,dt$. To do so, we state the same additional assumptions on the memory kernel g as in [18, 21] (see also [6]). Hence, our next arguments in the proof of stability are completely similar to those provided, e.g., in [21]. For the reader's convenience, we provide a short proof in each case.

Assumption 3.9. The memory kernel $g \in L^1(\mathbb{R}^+) \cap C^1(\mathbb{R}^+)$ satisfies the following nonlinear differential inequality:

$$g'(t) \leq -H(g(t)) \quad \forall\ t > 0, \tag{3.44}$$

where $H \in C^1([0,\infty))$ is a positive, strictly increasing, convex function with $H(0) = 0$. We also assume that there exists $\alpha_0 \in (0,1)$ such that

$$\int_0^\infty g^{1-\alpha_0}(s)ds < \infty. \tag{3.45}$$

THEOREM 3.10 (uniform decay rate I). *Under the assumptions of Proposition* 3.8*, if we also assume Assumption* 3.9*, then the energy $\mathcal{E}(t)$ decays uniformly to zero when t goes to infinity. More precisely, there exists $T_1 > 0$ such that*

$$\mathcal{E}(t) \leq S\left(\frac{t}{T_1} - 1\right) \quad \forall\ t > T_1, \tag{3.46}$$

where $S(t)$ satisfies the ODE

$$\frac{d}{dt}S(t) + q_{\alpha_0}(S(t)) = 0, \quad S(0) = \mathcal{E}(0), \tag{3.47}$$

with

$$q_{\alpha_0} \approx \widehat{H}_{\alpha_0} \quad \text{and} \quad \widehat{H}_{\alpha_0}(s) = c_1 H(c_2 s^{\frac{1}{\alpha_0}}),$$

for some constants $c_1, c_2 > 0$ which may depend on α_0.

Proof. As mentioned above, the statement of Theorem 3.10 follows by combining Assumption 3.9 with estimate (3.17) and then applying Lemma 3.3 in [19]. For the reader's convenience we present below a short proof.

According to [18, 21], conditions (3.44)–(3.45) are sufficient to guarantee the existence of a positive, increasing, convex function H_{α_0} given by

$$H_{\alpha_0}(s) = C_1 H(C_2 s^{\frac{1}{\alpha_0}}), \quad C_1, C_2 > 0.$$

From Jensen's inequality we have

$$(g\Box p_x)(t) \le H_{\alpha_0}^{-1}(D(t)) \quad \forall\ t > 0. \tag{3.48}$$

Integrating (3.48) on $(nT, (n+1)T)$ and applying again Jensen's inequality results in

$$\int_{nT}^{(n+1)T} (g\Box p_x)(t)\, dt \le \widehat{H}_{\alpha_0}^{-1}\left(\int_{nT}^{(n+1)T} D(t)\, dt\right), \tag{3.49}$$

where $\widehat{H}_{\alpha_0}$ is a rescaled version of H_{α_0} given by

$$\widehat{H}_{\alpha_0}(s) = T H_{\alpha_0}\left(T^{-1}s\right) = TC_1 H\left(C_2 T^{-\frac{1}{\alpha_0}} s^{\frac{1}{\alpha_0}}\right).$$

From (3.17) and (3.49) there exists $T_1 > 0$ such that

$$\mathcal{E}((n+1)T) \le \widetilde{H}_{\alpha_0}^{-1}\left(\int_{nT}^{(n+1)T} D(t)dt\right) \quad \forall\ T > T_1, \tag{3.50}$$

where we define

$$\widetilde{H}_{\alpha_0}^{-1}(s) = C(\widehat{H}_{\alpha_0}^{-1} + I_d)(s) = \left[c_1 H^{-1}(c_2 s)\right]^{\alpha_0} + Cs$$

with $c_1 = (CT)^{\frac{1}{\alpha_0}}/C_2$ and $c_2 = 1/TC_1$ independent of n. It is worth noting that $\widetilde{H}_{\alpha_0} \in C^1([0,\infty))$ is also a positive, increasing, convex function such that $\widetilde{H}_{\alpha_0}(0) = 0$.

Now, combining (3.13) and (3.50) we arrive at

$$\mathcal{E}((n+1)T) + \widetilde{H}_{\alpha_0}(\mathcal{E}((n+1)T)) \le \mathcal{E}(nT) \quad \forall\ T > T_1. \tag{3.51}$$

Hence, applying Lemma 3.3 in [19] with

$$s_n = \mathcal{E}(nT), \quad p = \widetilde{H}_{\alpha_0}, \quad S(0) = \mathcal{E}(0),$$

we conclude that $\mathcal{E}(t)$ satisfies (3.46), where $S(t)$ is a solution of (3.47) so that

$$q_{\alpha_0} = I_d - (I_d + \widetilde{H}_{\alpha_0})^{-1} \quad \text{and} \quad \lim_{t\to\infty} S(t) = 0. \tag{3.52}$$

Moreover, following [20, 21] one sees that $q_{\alpha_0} = \widetilde{H}_{\alpha_0}(I_d + \widetilde{H}_{\alpha_0})^{-1} \approx \widehat{H}_{\alpha_0}$, which finishes the proof of Theorem 3.10. ∎

Note that the achievement of Theorem 3.10 is not sharp since the decay rate depends on a parameter $\alpha_0 < 1$. To obtain such a sharp decay rate in the sense that the energy decays at the same memory kernels rate, we must impose (like in [21]) the following stronger technical assumption on g.

Assumption 3.11. Let y be a solution of the system

$$\frac{dy}{dt} + H(y) = 0, \quad y(0) = g(0).$$

Also, let us assume that there exists $\alpha_0 \in (0,1)$ such that $y^{1-\alpha_0} \in L^1(\mathbb{R}^+)$ and for some $r > 0$,

$$H \in C^1([0,\infty)) \cap C^2(0,r) \tag{3.53}$$

and

$$\liminf_{s\to 0^+} \{s^2 H''(s) - sH'(s) + H(s)\} \geq 0. \tag{3.54}$$

THEOREM 3.12 (uniform decay rate II). *Under the assumptions of Proposition* 3.8, *if we additionally assume Assumptions* 3.9 *and* 3.11, *then there exists* $T_2 > 0$ *such that the energy* $\mathcal{E}(t)$ *satisfies*

$$\mathcal{E}(t) \leq S\left(\frac{t}{T_2} - 1\right) \quad \forall\ t > T_2, \tag{3.55}$$

where $S(t)$ *satisfies the ODE*

$$\frac{d}{dt}S(t) + q_1(S(t)) = 0, \quad S(0) = \mathcal{E}(0), \tag{3.56}$$

with

$$q_1 \approx \widehat{H} \quad \text{and} \quad \widehat{H}(s) = c_3 H(c_4 s),$$

for some constants $c_3, c_4 > 0$ *which may depend on* H *and* α_0.

Proof. The statement of Theorem 3.12 is a consequence of the new Assumption 3.11 and estimate (3.17). Its proof relies on the same arguments as provided in [21] with several technical tools. For a complete and detailed proof we refer to [21, section 14.3]. For the sake of brevity we only present a sketch of the proof.

According to Lemmas 14.4 and 14.5 in [21], conditions (3.53)–(3.54) are enough to construct a function H_{1,α_0} given by

$$H_{1,\alpha_0}(s) = \alpha_0 s^{1-\frac{1}{\alpha_0}} H(s^{\frac{1}{\alpha_0}}),$$

which is increasing and convex on $(0,\delta)$, for some $\delta \in (0,r)$, $H_{1,\alpha}(0) = 0$, and such that

$$(g\Box p_x)(t) \leq \overline{H}_{1,\alpha_0}^{-1}(D(t)) \quad \forall\ t > 0, \tag{3.57}$$

where $\overline{H}_{1,\alpha_0}^{-1}(s) = C_3 H_{1,\alpha_0}^{-1}(C_4 s)$ with constants $C_3, C_4 > 0$.

Integrating (3.57) on $(nT,(n+1)T)$ and applying Jensen's inequality we get

$$\int_{nT}^{(n+1)T} (g\Box p_x)(t)\,dt \leq \widehat{H}_{1,\alpha_0}^{-1}\left(\int_{nT}^{(n+1)T} D(t)\,dt\right), \tag{3.58}$$

where

$$\widehat{H}_{1,\alpha_0}^{-1}(s) = T\overline{H}_{1,\alpha_0}^{-1}(T^{-1}s) = TC_3 H_{1,\alpha_0}^{-1}\left(C_4 T^{-1} s\right).$$

Combining (3.17) and (3.58), and proceeding similarly to (3.50)–(3.52), there exists $T_2 > 0$ such that $\mathcal{E}(t)$ satisfies (3.55) and $S(t)$ is the solution of (3.56) with

$$q_1 := q_{1,\alpha_0} = I_d - (I_d + \widetilde{H}_{1,\alpha_0})^{-1} \quad \text{and} \quad \widetilde{H}_{1,\alpha_0}^{-1} = c_3 \widehat{H}_{1,\alpha_0}^{-1}(c_4 s) + Cs$$

for some constants $c_3, c_4 > 0$. In addition, following Lemmas 14.7 and 14.8 in [21], then q_{1,α_0} has similar end behavior as $\widehat{H}_{1,\alpha_0}$ which is a rescaled version of H_{1,α_0}. Finally, the above process can be reiterated for H_{1,α_0} in finite steps with increasing values of α_0 to achieve a controlling function $H_{1,1} \approx H$ satisfying the conclusion of Theorem 3.12. The detailed proof of such iteration process is given in [21]. We also refer to [22, subsection 2.3] for a (summarized) step by step iteration for optimality. □

Remark 3.13. According to [18, 21], Assumptions 3.9 and 3.11 address (at least) exponential and polynomial memory kernels similar to those expressed in the first two items of Example 3.7.

4. Nonuniform stability: The case of different wave speeds. In this section we are going to conclude that problem (3.1)–(3.4) is not uniform stable on the weak phase space $H_0^1 \times L^2 \times H_*^1 \times L_*^2$ when the mathematical condition (1.3) is not taken into account, that is, the case which highlights the physical meaning of the system. Since problem (3.1)–(3.4) does not meet semigroup properties, we cannot use directly the theory in linear operators as applied, for instance, to autonomous problems. Instead, we shall use a constructive semigroup approximation along with known results on a spectrum of evolution operators; see [1, 2, 11, 28]. For this purpose, we follow similar lines as in [3, section 3], where the authors give a particular (and nice) treatment for the classical viscoelastic Timoshenko system (1.2).

Summarizing, we are going to show that condition (1.3) is necessary to reach uniform decay rates of the energy solution, even in case of exponential kernels. Thus, in what follows, we take (3.44) with $H(s) = \delta s$, $1 < \delta < \frac{1}{C_0}$, and then g satisfies

$$g(t) \leq g(0)e^{-\delta t}, \quad t > 0. \tag{4.1}$$

Also, to the next considerations, we are going to denote by $\mathcal{E}_g(\phi, \psi)(t)$ the energy functional set in (3.12) that corresponds to the solution (ϕ, ψ) of system (1.1). In this section, our main result reads as follows.

THEOREM 4.1. *Let us assume that g satisfies* (4.1). *If*

$$\frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2}, \tag{4.2}$$

then the energy functional $\mathcal{E}_g(\phi, \psi)(t)$ does not decay uniformly as t goes to infinity. In other words, there does not exist a positive function $d \in L^1(0, \infty) \cap L^2_{loc}([0, \infty))$ such that

I. $\lim_{t\to\infty} d(t) = 0$,
II. $\mathcal{E}_g(\phi, \psi)(t) \leq d^2(t)\mathcal{E}_g(\phi, \psi)(0),\ t > 0$.

The proof of Theorem 4.1 shall be completed later, in subsection 4.2, as a consequence of several proper (and technical) results for an approximate problem.

4.1. Approximate problem and technical results. First, we consider some notation and preliminary results. We start by fixing $\eta \in (0, \frac{\delta}{2})$ and defining function

$$g_n(t) = \eta e^{-\eta t} B_n\left(f\,,\, e^{-\eta t}\right), \quad n \geq 1, \tag{4.3}$$

where $f(x)=\frac{g\circ j^{-1}(x)}{\eta x}$, with function $j:[0,\infty)\to(0,1]$ being the bijection

$$j(t)=e^{-\eta t},$$

and $B_n(\cdot,\cdot)$ are the Bernstein polynomials given by

$$B_n(f,x):=\sum_{\nu=0}^{n}\begin{pmatrix} n \\ \nu \end{pmatrix} f(\nu/n)x^{\nu}(1-x)^{n-\nu} \quad \text{for } f \text{ and } x.$$

Under the above notation, let us also define

$$g_n=\sum_{\nu=1}^{n} f\left(\nu/n\right)\theta_{n,\nu},$$
$$\theta_{n,\nu}(t)=\eta\begin{pmatrix} n \\ \nu \end{pmatrix} e^{-(\nu+1)\eta t}(1-e^{-\eta t})^{n-\nu}.$$

LEMMA 4.2. *Under the above notation and g satisfying* (4.1), *we have that for any $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that*

$$\|g-g_n\|_{W^{1,1}(0,\infty)}<\varepsilon \quad \forall\ n\geq N.$$

Moreover, for all $n\in\mathbb{N}$, g_n satisfies
(a) $g_n\geq 0,\ g_n'\leq 0$;
(b) $\lim_{n\to\infty}\int_0^{\infty} g_n(t)dt=\int_0^{\infty} g(t)dt$.

Proof. The proof is analogous to the one in [3, Lemma 3.1]. See also [11, Theorem 2.1]. ∎

Now, let us consider the following approximate system:

$$\rho_1\overline{\phi}_{tt}-\kappa\overline{p}_{xx}+\kappa(g_n*\overline{p}_{xx})=0, \tag{4.4}$$
$$\rho_2\overline{\psi}_{tt}-b\overline{\psi}_{xx}+\kappa\overline{p}_{x}-\kappa(g_n*\overline{p}_{x})=0, \tag{4.5}$$

where $\overline{p}_x=\overline{\phi}_x+\overline{\psi}$, with initial data

$$(\phi_0,\phi_1,\psi_0,\psi_1)\in H_0^1\times L^2\times H_*^1\times L_*^2.$$

If $(\overline{\phi},\overline{\psi})$ is a solution of system (4.4)–(4.5), then the associated energy is

$$\mathcal{E}_{g_n}(\overline{\phi},\overline{\psi})(t)=\frac{1}{2}\Big\{\rho_1\|\overline{\phi}_t(t)\|_2^2+\rho_2\|\overline{\psi}_t(t)\|_2^2+b\|\overline{\psi}_x(t)\|_2^2+\kappa h_n(t)\|\overline{p}_x(t)\|_2^2+\kappa(g_n\Box\overline{p}_x)(t)\Big\},$$

where $h_n(t)=1-\int_0^t g_n(s)ds$, and satisfies

$$\frac{d}{dt}\mathcal{E}_{g_n}(\overline{\phi},\overline{\psi})(t)=\frac{\kappa}{2}(g_n'\Box\overline{p}_x)(t)-\frac{\kappa}{2}g_n(t)\|\overline{p}_x(t)\|_2^2\leq 0 \quad \forall\ t>0.$$

LEMMA 4.3. *Assume that g satisfies* (4.1). *Also, assume that there exists a function $d\in L^1(\mathbb{R}_+)$ such that*

$$\mathcal{E}_g(\phi,\psi)(t)\leq d^2(t)\mathcal{E}_g(\phi,\psi)(0) \quad \forall\ t>0.$$

Thus, there are constants $\tilde{C}_0,\tilde{C}_1>0$ such that

(i) $\mathcal{E}_{g_n}(\phi-\overline{\phi},\psi-\overline{\psi})(t)\leq\varepsilon\tilde{C}_0\mathcal{E}_{g_n}(\overline{\phi},\overline{\psi})(0)$,
(ii) $|\mathcal{E}_g(\phi,\psi)(t)-\mathcal{E}_{g_n}(\overline{\phi},\overline{\psi})(t)|\leq\varepsilon^{1/2}\tilde{C}_1\mathcal{E}_g(\phi,\psi)(0)$,

for some $\varepsilon>0$ sufficiently small.

Proof. Let $\varepsilon>0$ be a positive to be chosen later and (ϕ,ψ) and $(\overline{\phi},\overline{\psi})$ be the solutions of problems (3.1)–(3.2) and (4.4)–(4.5), respectively. If $z=\phi-\overline{\phi}$ and $w=\psi-\overline{\psi}$, then (z,w) is the solution of problem

$$\rho_1 z_{tt}-\kappa\tilde{p}_{xx}+\kappa g_n*\tilde{p}_{xx}=\kappa(g_n-g)*p_{xx}, \tag{4.6}$$
$$\rho_2 w_{tt}-bw_{xx}+\kappa\tilde{p}_x-\kappa g_n*\tilde{p}_x=\kappa(g-g_n)*p_x, \tag{4.7}$$

where $\tilde{p}_x=z_x+w$, with Dirichlet–Neumann boundary conditions like (3.3) and null initial data. The associated energy functional is now given by

$$\begin{aligned}\mathcal{E}_{g_n}(z,w)(t)=&\frac{1}{2}\Big\{\rho_1\|z_t\|_2^2+\rho_2\|w_t\|_2^2+b\|w_x\|_2^2\\&+\kappa\left(1-\int_0^t g_s(t)ds\right)\|\tilde{p}_x\|_2^2+\kappa g_n\Box\tilde{p}_x(t)\Big\}\end{aligned}$$

and satisfies

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{g_n}(z,w)(t)\leq\ &\kappa\underbrace{\int_0^t\frac{d}{ds}\left((g-g_n)*p_x\ ,\ \tilde{p}_x\right)_2 ds}_{:=I_1}\\&-\ \kappa\underbrace{\int_0^t\left(\frac{d}{ds}[(g-g_n)*p_x]\ ,\ \tilde{p}_x\right)_2 ds}_{:=I_2}.\end{aligned}\tag{4.8}$$

Using Lemma 4.2 and hypothesis on d, estimates for I_1 and I_2 are given below:

$$I_1\leq\frac{1}{\kappa\, l}\varepsilon\left[\mathcal{E}_g(\phi,\psi)(0)+\mathcal{E}_{g_n}(z,w)(t)\right], \tag{4.9}$$
$$I_2\leq\frac{1}{\kappa\, l}\left\{2\varepsilon\|d\|_1\mathcal{E}_g(\phi,\psi)(0)+(\varepsilon+1)\int_0^t\left[d(s)+\xi(s)\right]\mathcal{E}_{g_n}(z,w)(s)ds\right\}, \tag{4.10}$$

where $\xi(s)=|(g-g_n)'|*d(s)$. Replacing (4.9) and (4.10) in (4.8), we have

$$\left[1-\frac{\varepsilon}{l}\right]\mathcal{E}_{g_n}(z,w)(t)\leq\frac{\varepsilon}{l}(2\|d\|_1+1)\mathcal{E}_g(\phi,\psi)(0)+(\varepsilon+1)\int_0^t\left[d(s)+\xi(s)\right]\mathcal{E}_{g_n}(z,w)(s)ds.$$

Choosing $\varepsilon<\frac{l}{2}$ and using Gronwall's inequality we have

$$\mathcal{E}_{g_n}(z,w)(t)\leq\varepsilon\frac{(4\|d\|_1+2)}{l}\exp\left(2(\varepsilon+1)\int_0^t[d(s)+\xi(s)]ds\right)\mathcal{E}_g(\phi,\psi)(0).$$

Using again Lemma 4.2, we have $\int_0^t d(s)+\xi(s)ds\leq C(\varepsilon+1)$ for some $C>0$. Thus, keeping in mind that $\varepsilon<\frac{l}{2}$, we arrive at

$$\mathcal{E}_{g_n}(z,w)(t)\leq\varepsilon\frac{(4\|d\|_1+2)}{l}e^{2C(l/2+1)^2}\mathcal{E}_g(\phi,\psi)(0). \tag{4.11}$$

Item (i). Inequality in (i) follows directly from (4.11) by putting the constant $\tilde{C}_0=\frac{(4\|d\|_1+2)}{l}e^{2C(l/2+1)^2}$ and from the fact that $\mathcal{E}_g(\phi,\psi)(0)=\mathcal{E}_{g_n}(\overline{\phi},\overline{\psi})(0)$.

Item (ii). In order to prove (ii), we first observe that straightforward computations lead us to the next inequality for any $\varepsilon > 0$ and n sufficiently large:

$$|\mathcal{E}_g(\phi,\psi)(t) - \mathcal{E}_{g_n}(\phi,\psi)(t)| \leq \frac{4+l}{l^2}\, \varepsilon\, \mathcal{E}_g(\phi,\psi)(0). \tag{4.12}$$

On the other hand, we also have

$$|\mathcal{E}_{g_n}(\phi,\psi)(t) - \mathcal{E}_{g_n}(\overline{\phi},\overline{\psi})(t)| \leq 10\mathcal{E}_g^{1/2}(\phi,\psi)(0)\mathcal{E}_{g_n}^{1/2}(z,w)(t). \tag{4.13}$$

Combining (4.12) and (4.13) and using item (i), we obtain

$$|\mathcal{E}_g(\phi,\psi)(t) - \mathcal{E}_{g_n}(\overline{\phi},\overline{\psi})(t)| \leq \left[\frac{4+l}{l^{3/2}} + 10\sqrt{\tilde{C}_0}\right]\ \varepsilon^{1/2}\ \mathcal{E}_g(\phi,\psi)(0).$$

Therefore, inequality in (ii) follows by taking $\tilde{C}_1 = \frac{4+l}{l^{3/2}} + 10\sqrt{\tilde{C}_0}$. ◻

Taking advantage of the notation introduced above, we have

$$g_n * \overline{p}_{xx} = \sum_{\nu=1}^{n} f\left(\nu/n\right) y_{n,\nu_x},$$
$$y_{n,\nu}(t) := \theta_{n,\nu} * \overline{p}_{xx}(t) = \int_0^t \theta_{n,\nu}(t-s)\overline{p}_{xx}(s)ds.$$

Now, let us also define the real vector-valued function $Y_n : \mathbb{R}_+ \to \mathbb{R}^n$ by

$$Y_n = \begin{pmatrix} y_{n,1} \\ \vdots \\ y_{n,n} \end{pmatrix}.$$

Thus, Y_n is the solution of the system

$$\begin{cases} Y_n'(t) = A_n Y_n + D_n \overline{p}_x, & t > 0, \\ Y_n(0) = 0, \end{cases}$$

where $D_n = (0, \ldots, 0, \eta)'$ and

$$A_n = (a_{i,j}) \in \mathbb{M}_n(\mathbb{R}), \quad a_{i,j} = \begin{cases} -\eta(i+1), & j = i; \\ \eta(i+1), & j = i+1; \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

Under this notation, we have the following result.

Lemma 4.4. *The operator* A_n *defined* (4.14) *is the infinitesimal generator of a* C_0*-semigroup* $\{e^{tA_n}\}$ *on* $\left[L^2(0,L)\right]^n = L^2(0,L) \times \cdots \times L^2(0,L)$. *Moreover,* $\{e^{tA_n}\}$ *satisfies the following uniform estimate:*

$$\|e^{tA_n}\|_{[L^2(0,L)]^n} \leq e^{-\frac{\eta}{2}t} \quad \forall t > 0.$$

Proof. At this point, the proof can be found in [3, Lemma 3.3]. ◻

With the above notation, system (4.4)–(4.5) can be rewritten as

$$\rho_1\overline{\phi}_{tt} - \kappa\overline{p}_{xx} + \kappa B_n Y_{nx} = 0, \tag{4.15}$$

$$\rho_2\overline{\psi}_{tt} - b\overline{\psi}_{xx} + \kappa\overline{p}_x - \kappa B_n Y_n = 0, \tag{4.16}$$

$$Y_n' - A_n Y_n = D_n\overline{p}_x, \tag{4.17}$$

with boundary conditions

$$\overline{\phi}(0,t) = \overline{\phi}(L,t) = \overline{\psi}_x(0,t) = \overline{\psi}_x(L,t) = 0 \tag{4.18}$$

and initial conditions

$$\overline{\phi}(x,0) = \phi_0(x),\ \overline{\phi}_t(x,0) = \phi_1(x),\ \overline{\psi}(x,0) = \psi_0(x),\ \overline{\psi}_t(x,0) = \psi_1(x),\ Y_n(0) = 0, \tag{4.19}$$

where $B_n := (f(1/n), \ldots, f(n/n))$.

In addition, the energy functional associated with problem (4.15)–(4.19) is

$$\mathcal{E}_n(\overline{\phi}, \overline{\psi}, Y_n)(t) = \frac{1}{2}\left\{\rho_1\|\overline{\phi}_t\|_2^2 + \rho_2\|\overline{\psi}_t\|_2^2 + b\|\overline{\psi}_x\|_2^2 + \kappa\|\overline{p}_x\|_2^2 + \|Y_n\|^2_{[L^2(0,L)]^n}\right\}$$

for $t \geq 0$. Let us now introduce the energy space $\mathcal{H} = L^2 \times L^2 \times L^2_* \times L^2_* \times \left[L^2(0,L)\right]^n$. If

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} := \begin{pmatrix} \sqrt{\rho_1}\overline{\phi}_t - \sqrt{\kappa}\overline{p}_x + \sqrt{\kappa}B_n Y_n \\ \sqrt{\rho_2}\overline{\psi}_t - \sqrt{b}\overline{\psi}_x \\ \sqrt{\rho_1}\overline{\phi}_t + \sqrt{\kappa}\overline{p}_x - \sqrt{\kappa}B_n Y_n \\ \sqrt{\rho_2}\overline{\psi}_t + \sqrt{b}\overline{\psi}_x \\ Y_n \end{pmatrix},$$

then Z satisfies the equation

$$Z_t = \Lambda Z_x + MZ \tag{4.20}$$

with boundary condition

$$(z_i + (-1)^{i+1}z_{i+2})(0,t) = (z_i + (-1)^{i+1}z_{i+2})(L,t) = 0, \quad i = 1, 2, \tag{4.21}$$

where

$$\Lambda = \operatorname{diag}\left(-\sqrt{\frac{\kappa}{\rho_1}}, -\sqrt{\frac{b}{\rho_2}}, \sqrt{\frac{\kappa}{\rho_1}}, \sqrt{\frac{b}{\rho_2}}, 0_n\right) \quad \text{and} \quad M = \left(\begin{array}{c|c} M_{n4} & M_{4n} \\ \hline N_{n4} & A_n + D_n B_n \end{array}\right)$$

with

$$M_{n4} = \begin{pmatrix} -\frac{\eta}{2}f\left(\frac{n}{n}\right) & -\frac{1}{2}\sqrt{\frac{\kappa}{\rho_2}} & \frac{\eta}{2}f\left(\frac{n}{n}\right) & -\frac{1}{2}\sqrt{\frac{\kappa}{\rho_2}} \\ \frac{1}{2}\sqrt{\frac{\kappa}{\rho_2}} & 0 & -\frac{1}{2}\sqrt{\frac{\kappa}{\rho_2}} & 0 \\ \frac{\eta}{2}f\left(\frac{n}{n}\right) & \frac{1}{2}\sqrt{\frac{\kappa}{\rho_2}} & -\frac{\eta}{2}f\left(\frac{n}{n}\right) & \frac{1}{2}\sqrt{\frac{\kappa}{\rho_2}} \\ \frac{1}{2}\sqrt{\frac{\kappa}{\rho_2}} & 0 & -\frac{1}{2}\sqrt{\frac{\kappa}{\rho_2}} & 0 \end{pmatrix},$$

$$M_{4n} = \begin{pmatrix} \sqrt{\kappa}B_nA_n + \eta\sqrt{\kappa}B_nD_nB_n \\ 0 \\ -\sqrt{\kappa}B_nA_n - \eta\sqrt{\kappa}B_nD_nB_n \\ 0 \end{pmatrix},$$

$$N_{n4} = \begin{pmatrix} -\frac{1}{2\sqrt{\kappa}}D_n & 0 & \frac{1}{2\sqrt{\kappa}}D_n & 0 \end{pmatrix}.$$

Also, let us consider the matrices that allow us to express Z in another way,

$$P_n = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 & \sqrt{\kappa}B_n \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & I_n \end{pmatrix},$$

$$P_n^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 & \sqrt{\kappa}B_n \\ 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 & -\sqrt{\kappa}B_n \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_n \end{pmatrix},$$

and let $e^{(\Lambda\partial_x + M_0)t}$ be the semigroup associated with problem

$$\begin{aligned} &Z_t = \Lambda Z_x + M_0 Z, \\ &(z_i + (-1)^{i+1}z_{i+2})(0,t) = (z_i + (-1)^{i+1}z_{i+2})(L,t) = 0, \quad i = 1,2, \\ &M_0 := \operatorname{diag}\left(-\frac{\eta}{2}f(n/n)\ ,\ 0\ ,\ -\frac{\eta}{2}f(n/n)\ ,\ 0\ ,\ A_n + D_nB_n\right). \end{aligned}$$

Therefore, following the ideas of [3, section 3], which in turn relies on the results proved in [1, 2], one can prove that if condition (4.2) holds, then the $e^{(\Lambda\partial_x+M)t} - e^{(\Lambda\partial_x+M_0)t}$ is a compact operator. Furthermore, the eigenvalues of $\Lambda\partial_x + M_0$ are given by

(4.22)

$$\begin{aligned} &\sigma(\Lambda\partial_x + M_0) \\ &= \sigma(A_n + D_nB_n) \cup \left\{\sqrt{\frac{b}{\rho_2}}\frac{m\pi}{L}i,\ m \in \mathbb{Z}\right\} \cup \left\{-\frac{\eta}{2}f(n/n) + \sqrt{\frac{b}{\rho_2}}\frac{m\pi}{L}i,\ m \in \mathbb{Z}\right\}. \end{aligned}$$

The next result compares the energy solution of system (4.20)–(4.21) with the energy $\mathcal{E}_{g_n}(\overline{\phi},\overline{\psi})$ related to problem (4.4)–(4.5).

Lemma 4.5. *Under the above notation and assumptions of Theorem 4.1, if we consider any $n \in \mathbb{N}$ and $(\phi_0,\phi_1,\psi_0,\psi_1) \in H_0^1 \times L^2 \times H_*^1 \times L_*^2$, then there exists a constant $C > 0$ such that*

$$\|Z(t)\|^2 = \|e^{(\Lambda\partial_x+M)t}Z_0\|^2 \le C\left(\mathcal{E}_{g_n}(\overline{\phi},\overline{\psi})(t) + \int_0^t e^{-\frac{\eta(t-s)}{2}}\mathcal{E}_{g_n}(\overline{\phi},\overline{\psi})(s)ds\right), \tag{4.23}$$

where

$$Z_0 = P_n^{-1}\left(\sqrt{\rho_1}\phi_1\ ,\ \sqrt{\kappa}(\phi_{0x} + \psi_0)\ ,\ \sqrt{\rho_2}\psi_1\ ,\ \sqrt{b}\psi_{0x}\ ,\ 0^n\right)^T.$$

Proof. Let $(\phi_0, \phi_1, \psi_0, \psi_1)$ and Z_0 as in the statement of Lemma 4.5. If $Z(t)$ is the respective solution of (4.20)–(4.21), then it is not difficult to see that

$$Z(t) = P_n^{-1} \left(\sqrt{\rho_1}\phi_t \ , \ \sqrt{\kappa}(\phi_x + \psi) \ , \ \sqrt{\rho_2}\psi_t \ , \ \sqrt{b}\psi_x \ , \ Y_n \right)^T .$$

Performing straightforward calculations we have

$$\|Z(t)\|^2 \leq \frac{16}{l} \ \mathcal{E}_{g_n}(\overline{\phi}, \overline{\psi})(t) + 8\kappa\|B_n Y_n\|^2 + \|Y_n\|^2. \tag{4.24}$$

Using Lemma 4.2(b), an estimate for $B_n Y_n$ is gotten as

$$\|B_n Y_n\|^2 \leq \frac{2(1-l)}{\kappa \ l} \ \mathcal{E}_{g_n}(\overline{\phi}, \overline{\psi})(t). \tag{4.25}$$

Besides, using Lemma 4.4, an estimate for Y_n goes as follows:

$$\|Y_n\|^2 \leq -\frac{4[e^{\frac{\eta t}{2}} - 1]}{l} \int_0^t e^{-\frac{\eta(t-s)}{2}} \mathcal{E}_{g_n}(\overline{\phi}, \overline{\psi})(s) ds. \tag{4.26}$$

Combining (4.24), (4.25), and (4.26) we obtain

$$\|Z(t)\|^2 \leq \max\left\{ \frac{8(3-l)}{l}, \frac{4}{l} \right\} \left[\mathcal{E}_{g_n}(\overline{\phi}, \overline{\psi})(t) + \int_0^t e^{-\frac{\eta(t-s)}{2}} \mathcal{E}_{g_n}(\overline{\phi}, \overline{\psi})(s) ds \right].$$

Hence, inequality (4.23) follows by taking $C = \max\{\frac{8(3-l)}{l}, \frac{4\eta}{l}\}$. ∎

In light of the above results, we are now in position to complete the proof of Theorem 4.1 as follows.

4.2. Proof of Theorem 4.1: Completion. Let us suppose that there exists a function $d \in L^1(0, \infty) \cap L^2_{loc}(0, \infty)$ such that items I and II of Theorem 4.1 hold true, namely, $\lim_{t \to \infty} d(t) = 0$ and

$$\mathcal{E}_g(\phi, \psi)(t) \leq d^2(t)\mathcal{E}_g(\phi, \psi)(0), \quad t > 0. \tag{4.27}$$

Thus, from Lemma 4.3(ii) and inequality (4.27) it follows that

$$\mathcal{E}_{g_n}(\overline{\phi}, \overline{\psi})(t) \leq \left(\varepsilon^{1/2}\tilde{C}_1 + d(t)^2 \right) \mathcal{E}_g(\phi, \psi)(0). \tag{4.28}$$

In addition, for $Y_n(t) = \int_0^t e^{-(t-s)A_n} D_n \overline{p}_x(s) ds$, it follows from Lemma 4.5 that

$$\begin{aligned} \|Z(t)\|^2 &= \|e^{(\Lambda\partial_x + M)t} Z_0\|^2 \\ &\leq C\varepsilon^{1/2}\tilde{C}_1 \left(1 + \frac{2}{\kappa} \right) \mathcal{E}_g(\phi, \psi)(0) \\ &\quad + C \left[d(t)^2 + \int_0^t d(s)^2 e^{-\frac{\eta(t-s)}{2}} ds \right] \mathcal{E}_g(\phi, \psi)(0). \end{aligned} \tag{4.29}$$

where $Z(t)$ is the solution of (4.20)–(4.21). In particular, let initial data Z_0^m be given as the eigenfunctions of the operator $\Lambda\partial_x + M_0$, associated with the eigenvalues $\lambda_m = \sqrt{\frac{\rho_2}{b}} \frac{m\pi}{L} i$, $m \in \mathbb{Z}$. In this case, $Z_0^m = \left(0 \ , \ z_2(0)e^{-\frac{m\pi}{L}i} \ , \ 0 \ , \ z_4(0)e^{\frac{m\pi}{L}i} \ , \ 0^n \right)^T$, $m \in \mathbb{Z}$, and

$$\|Z_0^m\|^2 = |z_2(0)|^2 2L \quad \text{for} \quad m \in \mathbb{Z}.$$

Thus, for $|z_2(0)| = \frac{1}{\sqrt{2L}}$ we have $\|Z_0^m\| = 1$ for every $m \in \mathbb{Z}$. Since $\{Z_0^m\}_{m\in\mathbb{Z}}$ are the eigenfunctions of $\Lambda\partial_x + M_0$, we have

$$\|e^{(\Lambda\partial_x+M_0)t}Z_0^m\| = \left|e^{\sqrt{\frac{\rho_2}{b}}\frac{m\pi}{L}i}\right| \|Z_0^m\| = 1 \quad \text{for} \quad m \in \mathbb{Z}. \tag{4.30}$$

On the other hand, using (4.29) we have

$$\begin{aligned}\|e^{(\Lambda\partial_x+M_0)t}Z_0^m\|^2 &\leq 2\| \left(e^{(\Lambda\partial_x+M_0)t} - e^{(\Lambda\partial_x+M)t}\right) Z_0^m\|^2 + 2\|e^{(\Lambda\partial_x+M)t}Z_0^m\|^2 \\ &\leq 2\| \left(e^{(\Lambda\partial_x+M_0)t} - e^{(\Lambda\partial_x+M)t}\right) Z_0^m\|^2 + 2C\varepsilon^{1/2}\tilde{C}_1\left(1+\frac{2}{\kappa}\right)\mathcal{E}_g(\phi,\psi)(0) \\ &\quad + 2C\left[d(t)^2 + \int_0^t d(s)^2 e^{-\frac{\eta(t-s)}{2}}ds\right]\mathcal{E}_g(\phi,\psi)(0).\end{aligned} \tag{4.31}$$

Since $\lim_{t\to\infty} d(t) = 0$ we have

$$\lim_{t\to\infty}\left[d(t)^2 + \int_0^t d(s)^2 e^{-\frac{\eta(t-s)}{2}}ds\right] = 0. \tag{4.32}$$

Also, since $e^{(\Lambda\partial_x+M_0)t-e^{(\Lambda\partial_x+M)t}}$ is a compact operator, reducing to a subsequence if necessary, we have

$$\left\|\left(e^{(\Lambda\partial_x+M_0)} - e^{(\Lambda\partial_x+M)t}\right) Z_0^m\right\| \to 0 \quad \text{as} \quad |m| \to \infty. \tag{4.33}$$

Therefore, using (4.31), (4.32), and (4.33) for $t > 0$ and $|m|$ sufficiently large, and also taking $\varepsilon > 0$ sufficiently small, we conclude

$$\|e^{(\Lambda\partial_x+M_0)t}Z_0^m\|^2 < 1,$$

which is in contradiction with (4.30). This concludes the proof of Theorem 4.1. ◻

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